# 1.1 Systems of linear equations

$$\underset{\uparrow}{a_1}x_1 + \underset{\uparrow}{a_2}x_2 + \cdots + \underset{\uparrow}{a_n}x_n = b \quad \longleftarrow \quad \underline{\text{linear equation}}$$

<u>coefficients</u>

<u>Expl</u> Is $4x_1 - 5x_3 + 2 = x_1$ linear?

rearrange: $3x_1 - 5x_3 = -2$

$$3x_1 + 0x_2 + (-5)x_3 = -2 \qquad \boxed{\text{YES}}$$

<u>Expl</u> Is $4x_1 - 5x_2 = x_1x_2$ linear?

No, b/c of $x_1x_2$ term

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} (*)$$

$(*)$ is a <u>system of linear equations</u>

or a <u>linear system</u>

($m$ equations in $n$ unknowns)

A solution to (*) is an ordered n-tuple $(x_1, x_2, \ldots, x_n)$ that makes all equations in (*) true. The solution set of (*) is the set of all solutions.

**Thm** There are only 3 possibilities:

(1) (*) has no solutions

(2) (*) has one solution

(3) (*) has infinitely many solutions

(*) is inconsistent if (1) is true.

(*) is consistent if (2) or (3) are true.

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An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

The coefficient matrix of (*) is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \longleftarrow m \times n$$

The augmented matrix of (*) is

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \longleftarrow m \times (n+1)$$

Elementary row operations

1. Replacement $(R_i + cR_j \rightarrow R_i)$
2. Interchange $(R_i \leftrightarrow R_j)$
3. Scaling $(cR_i \rightarrow R_i\,,\ c \neq 0)$

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$$\begin{tikzcd} \text{system} \arrow[r] & \text{augmented matrix} \arrow[d, squiggly, "\text{any sequence of row operations}"] \\ \text{new system} & \text{new augmented matrix} \arrow[l] \end{tikzcd}$$

Thm The new system has the same solution set as the original system.

## Expl 1

Find the solution set of the system

$x_1 - 2x_2 + x_3 = 0, \ 2x_2 - 8x_3 = 8, \ 5x_1 - 5x_3 = 10.$

$$
\begin{aligned}
1x_1 - 2x_2 + 1x_3 &= 0 \\
0x_1 + 2x_2 - 8x_3 &= 8 \\
5x_1 + 0x_2 - 5x_3 &= 10
\end{aligned}
$$

Augmented matrix

$$
\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right)
\xrightarrow{R_3 - 5R_1 \to R_3}
\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{array}\right)
$$

$$
\xrightarrow{R_3 - 5R_2 \to R_3}
\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{array}\right)
\xrightarrow{\frac{1}{30}R_3 \to R_3}
\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right)
$$

$$
\xrightarrow{R_2 + 8R_3 \to R_2}
\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)
\xrightarrow{R_1 - R_3 \to R_1}
\left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)
$$

$$
\xrightarrow{\frac{1}{2}R_2 \to R_2}
\left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)
\xrightarrow{R_1 + 2R_2 \to R_1}
\underbrace{\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)}_{\text{new augmented matrix}}
$$

new system: $\left.\begin{aligned} x_1 \qquad\qquad &= 1 \\ x_2 \qquad &= 0 \\ x_3 &= -1 \end{aligned}\right\}$ same solution set as original

$$\boxed{\text{One solution: } (1,0,-1)}$$

- Elementary row operations are reversible.

$cR_i \to R_i$ reversed by $\frac{1}{c}R_i \to R_i$

$R_i \leftrightarrow R_j$ " " $R_i \leftrightarrow R_j$

$R_i + cR_j \longrightarrow R_i$ " " $R_i - cR_j \to R_i$

- Two matrices A and B are <u>row equivalent</u> if A can be transformed into B using elementary row operations.

- If two linear systems have augmented matrices that are row equivalent, then they have the same solution set.
(rewording of earlier thm)

## Expl 3

Is the following system consistent?

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned}$$

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Consistent means it has at least one soln.

$$\begin{pmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{pmatrix} \xrightarrow{R_1 \Leftrightarrow R_2} \begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{pmatrix}$$

augmented matrix

$$\xrightarrow{R_3 - 2R_1 \to R_3} \begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{pmatrix} \xrightarrow{R_3 + 2R_2 \to R_3} \begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{pmatrix}$$

new augmented matrix

new system: $$\left.\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 15 \end{aligned}\right\} \text{same soln set as original}$$

new system has no solutions

original system is $\boxed{\text{inconsistent}}$