The Principles of Probability: From Formal Logic to Measure Theory to the Principle of Indifference

Jason Swanson

Department of Mathematics
University of Central Florida

Probability and Related Fields Seminar
Indiana University, Bloomington, IN, October 31, 2024

With the exception of mathematics, everything we know is conjecture.

—George Polya, 1954 (paraphrased)

Examples

- Laws of physics
- Guilt of a defendant
- Historical facts
- Economic principles
- The sun will rise tomorrow.

None of these can be shown with complete certainty.

But some are more certain than others.

Deductive reasoning: Used to establish mathematical facts.

Inductive reasoning: Used to establish everything else.

Rules of deductive reasoning

- Formalized in mathematical logic
- Detailed and precise
- Well-understood
- Universally accepted
- Highly successful

Rules of inductive reasoning

- There *are* rules. Example:

  The Fundamental Inductive Pattern (Polya, 1954):

$$\frac{\begin{array}{c} A \text{ implies } B \\ B \text{ is true} \end{array}}{A \text{ becomes more plausible}}$$

  This is the foundation of empirical science.
- There is no successful, agreed-upon formalization
- Obvious tool to use: probability

Mathematicians, philosophers, and physicists have made efforts toward formalizing inductive reasoning with probability.

Incomplete list:

- Leibniz (1670)
- Jacob Bernoulli (1713)
- Bayes (1763)
- Laplace (1774)
- Bolzano (1837)
- De Morgan (1837)
- Boole (1854)
- Keynes (1921)
- Wittgenstein (1922)
- Reichenbach (1949)
- Carnap (1950)
- Scott and Krauss (1966)
- Nilsson (1986)
- Jaynes (2003)

$\mathcal{L} = \mathcal{L}_{\omega_1,\omega}$: a predicate language that allows countable conjunctions and disjunctions

$\mathcal{L}^0 \subseteq \mathcal{L}$: the set of sentences in $\mathcal{L}$

An *inductive statement* is a triple, (X, φ, p), where $X \subseteq \mathcal{L}^0$, $\varphi \in \mathcal{L}^0$, and $p \in [0, 1]$.

$\mathcal{L}^{\mathrm{IS}}$: the set of inductive statements

(R1) The rule of logical equivalence
Let $P \subseteq \mathcal{L}^{\mathrm{IS}}$. If $(X, \varphi, p) \in P$, $X' \equiv X$, and $\varphi' \equiv_X \varphi$, then $(X', \varphi', p) \in P$ and there is no other p' with $(X', \varphi', p') \in P$.

If P satisfies (R1), then P is a function from a subset of $\mathfrak{P}\mathcal{L}^0 \times \mathcal{L}^0$ to $[0, 1]$. Write $P(\varphi \mid X) = p$ for $(X, \varphi, p) \in P$.

$\mathrm{ante}\, P$: set of all X such that $P(\varphi \mid X) = p$ for some φ and p

(R2) The rule of logical implication

$X \in \text{ante}\, P, X \vdash \varphi \Rightarrow P(\varphi \mid X) = 1$

(R3) The rule of material implication

$X \in \text{ante}\, P, P(\varphi \mid X, \psi) = 1 \Rightarrow P(\psi \to \varphi \mid X) = 1$

(R4) The rule of deductive transitivity

(a) $P(\varphi \mid X) = 1, \varphi \vdash \psi \Rightarrow P(\psi \mid X) = 1$

(b) $X' \in \text{ante}\, P, X' \vdash X, P(\varphi \mid X) = 1 \Rightarrow P(\varphi \mid X') = 1$

(R5) The addition rule

$X \vdash \neg(\varphi \wedge \psi) \Rightarrow P(\varphi \vee \psi \mid X) = P(\varphi \mid X) + P(\psi \mid X)$

(R6) The multiplication rule

$P(\varphi \wedge \psi \mid X) = P(\varphi \mid X) P(\psi \mid X, \varphi)$

(R7) The continuity rule

$X, \varphi_n \vdash \varphi_{n+1} \Rightarrow P(\bigvee_n \varphi_n \mid X) = \lim_n P(\varphi_n \mid X)$

A set $\overline{P} \subseteq \mathcal{L}^{\text{IS}}$ is *complete* if it satisfies (R1)–(R7) and
(i) $\overline{P}(\varphi \mid X), \overline{P}(\psi \mid X)$ exist $\Rightarrow \overline{P}(\varphi \wedge \psi \mid X)$ exists
(ii) $X, X \cup \{\psi\} \in \operatorname{ante} \overline{P} \Rightarrow \overline{P}(\psi \mid X)$ exists

---

(R8) The rule of inductive extension
If $\overline{P}(\varphi \mid X) = p$ for all complete $\overline{P} \supseteq P$, then $P(\varphi \mid X) = p$.

(R9) The rule of deductive extension
If $S \subseteq \mathcal{L}^0$ is nonempty and $P(\theta \mid X) = 1$ for all $\theta \in S$, then $X \cup S \in \operatorname{ante} P$ and $P(\,\cdot \mid X, S) = P(\,\cdot \mid X)$.

An *inductive theory* is a set $P \subseteq \mathcal{L}^{\text{IS}}$ that satisfies (R1)–(R9).

Can use this to define $Q \vdash (X, \varphi, p)$ for $Q \subseteq \mathcal{L}^{\text{IS}}$.

An *(inductive) model* is a probability space $\mathscr{P} = (\Omega, \Sigma, \mathbb{P})$ where Ω is a set of $\mathcal{L}$-structures.

$\varphi_\Omega = \{\omega \in \Omega \mid \varphi \text{ is true in } \omega\}$ $\quad \mathscr{P} \vDash \varphi \Leftrightarrow \overline{\mathbb{P}}\, \varphi_\Omega = 1$

$$\mathscr{P} \vDash (X, \varphi, p) \Leftrightarrow X \equiv Y \cup \{\psi\}, \text{ where } \mathscr{P} \vDash Y \;\&\; \frac{\overline{\mathbb{P}}\, \varphi_\Omega \cap \psi_\Omega}{\overline{\mathbb{P}}\, \psi_\Omega} = p.$$

Can use this to define $Q \vDash (X, \varphi, p)$ for $Q \subseteq \mathcal{L}^{\mathrm{IS}}$.

Theorem (σ-compactness, Thm 5.3.19)

X is satisfiable (i.e. $\mathscr{P} \vDash X$ for some $\mathscr{P}$) iff every countable subset of X is satisfiable.

Theorem (Soundness and completeness)

- $X \vdash \varphi \Leftrightarrow X \vDash \varphi :\Leftrightarrow (\mathscr{P} \vDash X \Rightarrow \mathscr{P} \vDash \varphi)$
- $Q \vdash (X, \varphi, p) \Leftrightarrow Q \vDash (X, \varphi, p)$

$Th\,\mathscr{P} = \{\varphi \in \mathcal{L}^0 \mid \mathscr{P} \vDash \varphi\}$ (a deductive theory)
$[T_0, Th\,\mathscr{P}]$: set of deductive theories T with $T_0 \subseteq T \subseteq Th\,\mathscr{P}$

$X \hookrightarrow Y$: $X \equiv Y \cup \Phi$ for some countable $\Phi \subseteq \mathcal{L}^0$
$X \hookrightarrow [T_0, Th\,\mathscr{P}]$: $X \hookrightarrow T$ for some $T \in [T_0, Th\,\mathscr{P}]$

$\mathbf{Th}\,\mathscr{P} = \{(X, \varphi, p) \in \mathcal{L}^{\mathrm{IS}} \mid \mathscr{P} \vDash (X, \varphi, p)\}$
$\mathbf{Th}\,\mathscr{P}\rfloor_{[T_0, Th\,\mathscr{P}]} = \{(X, \varphi, p) \in \mathbf{Th}\,\mathscr{P} \mid X \hookrightarrow [T_0, Th\,\mathscr{P}]\}$

Models Determine Theories (Thm 4.2.4 + Prop 3.5.10)

If $\mathscr{P}$ is a model with $\mathscr{P} \vDash T_0$, then $P = \mathbf{Th}\,\mathscr{P}\rfloor_{[T_0, Th\,\mathscr{P}]}$ is a complete inductive theory with root T_0.

Theories Determine Models (Thm 4.2.6)

If P is a complete inductive theory with root T_0, then there exists a model $\mathscr{P}$ such that $P = \mathbf{Th}\,\mathscr{P}\rfloor_{[T_0, Th\,\mathscr{P}]}$.

(S, Γ, ν): probability space

$\langle X_i \mid i \in I \rangle$: family of r.v.s

X_i takes values in (R_i, Γ_i); Γ is generated by the X_i's

We call $(S, \Gamma, \nu, \langle X_i \mid i \in I \rangle)$ a **modern probability model**

$\mathcal{L}_R$: predicate language with constant symbols $\{\underline{r} \mid r \in \bigcup_i R_i\}$ and unary relation symbols $\{\underline{V_i} \mid i \in I, V_i \in \Gamma_i\}$

$x \in \underline{V_i}$ is shorthand for $\underline{V_i}\, x$

$\mathcal{R}$: $\mathcal{L}_R$-structure with domain $R = \bigcup_i R_i$ and $\underline{r}^{\mathcal{R}} = r$, $\underline{V}^{\mathcal{R}} = V$

$T_R = \{\varphi \in \mathcal{L}_R^0 \mid \varphi \text{ is true in } \mathcal{R}\}$

$\mathcal{L}$: language obtained from $\mathcal{L}_R$ by adding a constant symbol $\underline{X}_i$ for each $i \in I$

Embedding Theorem I (Thm 5.4.2)

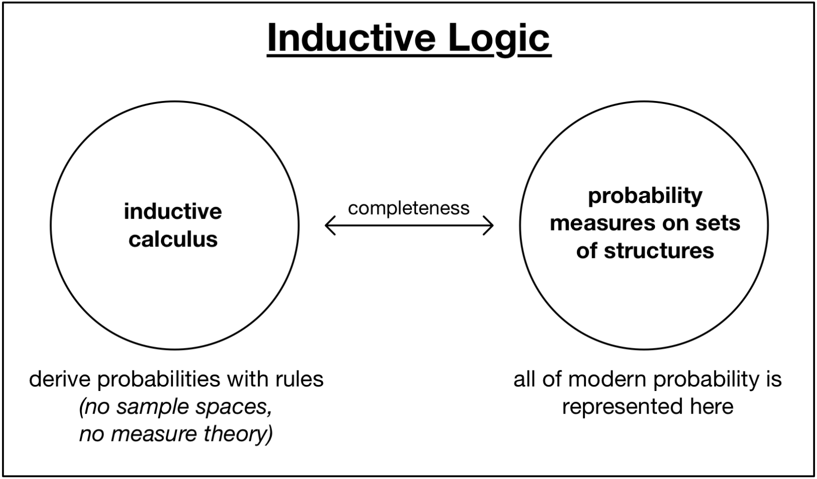
There exists an $\mathcal{L}$-model $\mathscr{P} = (\Omega, \Sigma, \mathbb{P})$ with $\mathscr{P} \vDash T_R$ and a function $h : S \to \Omega$ mapping $x \in S$ to $\omega \in \Omega$ such that

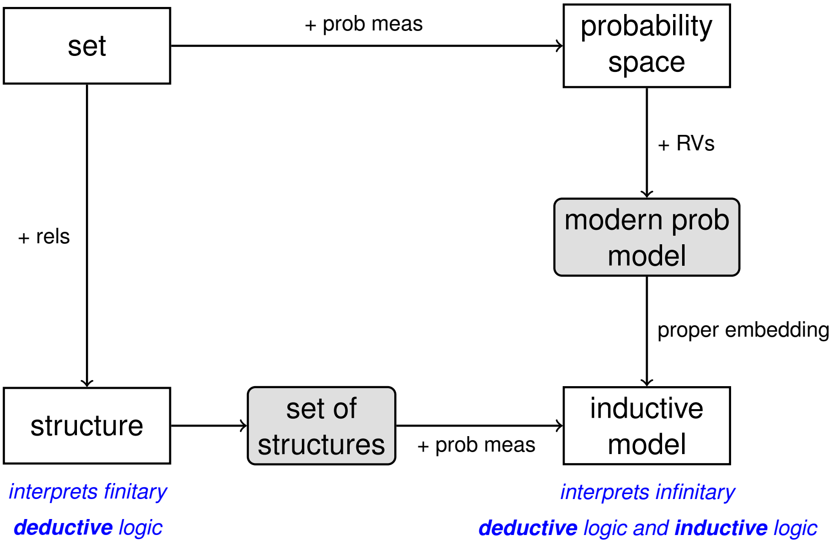
- $x \in \{X_i \in V\} \Leftrightarrow \underline{X}_i \in \underline{V_i}$ is true in ω,
- $U \in \Gamma \Rightarrow U = h^{-1}\varphi_\Omega$ for some $\varphi \in \mathcal{L}^0$, and
- h is a measure space isomorphism,

and $P = \mathbf{Th}\,\mathscr{P}\rfloor_{[T_R, Th\,\mathscr{P}]}$ satisfies

$$P(\textstyle\bigwedge_{k=1}^n \underline{X}_{i(k)} \in \underline{V_k} \mid T_R) = \nu \textstyle\bigcap_{k=1}^n \{X_{i(k)} \in V_k\}.$$

Measure Theory	*Inductive Logic*
outcome	structure
event	sentence
set membership	satisfiability in a structure
random variable	constant symbol


Inductive Logic
inductive calculus
completeness
probability measures on sets of structures
derive probabilities with rules
(no sample spaces, no measure theory)
all of modern probability is represented here


set
+ prob meas
probability space
+ RVs
modern prob model
+ rels
proper embedding
structure
set of structures
+ prob meas
inductive model
interprets finitary
deductive logic
interprets infinitary
deductive logic and **inductive** logic

Example 1

$\mathcal{L}$: a language with constants $\{\underline{X}_n \mid n \in \mathbb{N}\} \cup \{h, t\}$

For $\mathbf{s} = \langle s_n \rangle \in \{h, t\}^{\mathbb{N}}$, let $\psi_{\mathbf{s}} = \neg \bigwedge_{n\in\mathbb{N}} \underline{X}_n = s_n$.

$$X = \{\textstyle\bigwedge_{n\in\mathbb{N}}(\underline{X}_n = h \vee \underline{X}_n = t)\} \cup \{\psi_{\mathbf{s}} \mid \mathbf{s} \in \{h, t\}^{\mathbb{N}}\}$$

Every countable subset of X is satisfiable.

By σ-compactness, X is satisfiable. Therefore, there exists $\mathscr{P}$ such that $\mathscr{P} \vDash X$.

(S, Γ, ν): prob. sp.; $\langle X_n \mid n \in \mathbb{N}\rangle$: i.i.d. coin flips

Build $\mathscr{P}$ as in Embedding Theorem I. Then $\mathscr{P} \vDash X$.

Example 2

I: an uncountable set

$\mathcal{L}$: a language with constants $\{\underline{X}_t \mid t \in I\} \cup \{\underline{n} \mid n \in \mathbb{N}\}$

$$X = \{\underline{m} \neq \underline{n} \mid m, n \in \mathbb{N}, m \neq n\} \cup \{\textstyle\bigvee_{n\in\mathbb{N}} \underline{X}_t = \underline{n} \mid t \in I\} \cup \{\underline{X}_s \neq \underline{X}_t \mid s, t \in I, s \neq t\}$$

Every countable subset of X is satisfiable.

By σ-compactness, X is satisfiable. Choose an inductive model $\mathscr{P}$ such that $\mathscr{P} \vDash T_0 = T(X)$. Let $P = \mathbf{Th}\,\mathscr{P}\!\downharpoonright_{[T_0, Th\,\mathscr{P}]}$.

Then $P(\underline{X}_s \neq \underline{X}_t \mid T_0) = 1$ for all $s \neq t$.

Connections to other areas

- Computer science
  - Quantum computing
  - Artificial intelligence and machine learning
- Philosophy
  - Philosophy of science
  - Interpretations of probability *(principle of indifference)*
  - Epistemology
  - Probabilistic arguments
    (doomsday argument, simulation hypothesis,
    self-sampling hypotheses, superintelligence/singularity)

Connections to other areas

- Mathematics
  - Foundations and set theory
  - Probabilistic methods
- Statistics
  - Bayesian methods
- Physics
  - Quantum physics
    (interpretations, many-worlds, Bohmian mechanics)
  - Statistical mechanics *(based on principle of indifference)*
  - Dynamical systems with noise

The Principle of Indifference

The Principle of Indifference originates with Laplace. It is the core of the classical interpretation of probability.

> *The theory of chance consists in reducing all the events of the same kind to a certain number of cases equally possible, that is to say, to such as we may be equally undecided about in regard to their existence, and in determining the number of cases favorable to the event whose probability is sought. The ratio of this number to that of all the cases possible is the measure of this probability.*
>
> *—Laplace, 1814*

The most famous description of it is by Keynes:

> *The Principle of Indifference asserts that if there is no* known *reason for predicating of our subject one rather than another of several alternatives, then relatively to such knowledge the assertions of each of these alternatives have an* equal *probability. Thus* equal *probabilities must be assigned to each of several arguments, if there is an absence of positive ground for assigning* unequal *ones.*
>
> *This rule, as it stands, may lead to paradoxical and even contradictory conclusions.*
>
> *—Keynes, 1921*

L: extralogical signature (set of symbols for constants, n-ary relations, and n-ary functions)

A bijection $\pi : L \to L$ is a *signature permutation* if s^π has the same type and arity as s.

φ^π: the formula obtained by replacing s with s^π
$X^\pi = \{\varphi^\pi \mid \varphi \in X\}$; X is π*-invariant* if $X^\pi \equiv X$

Deductive Indifference (Thm 7.1.3)

$$X \vdash \varphi \quad \text{iff} \quad X^\pi \vdash \varphi^\pi$$

Without Loss of Generality (Cor 7.1.4)

Suppose $\Phi \subseteq \mathcal{L}$ is countable and $X \vdash \bigvee \Phi$. Fix $\theta_0 \in \Phi$. Assume for all $\theta \in \Phi$, there is a signature permutation π such that $\theta_0^\pi = \theta$ and X and φ are π-invariant. Then $X, \theta_0 \vdash \varphi$ implies $X \vdash \varphi$.

(R10) The principle of indifference

If $P(\varphi \mid X) = p$ and $X^\pi \in \operatorname{ante} P$, then $P(\varphi^\pi \mid X^\pi) = p$.

If P is an inductive theory that satisfies the principle of indifference and X is π-invariant, then $P(\varphi^\pi \mid X) = P(\varphi \mid X)$, by the rule of logical equivalence.

Example

b_1, b_2: constant symbols; C_0, C_1: unary relation symbols
$L = \{b_1, b_2, C_0, C_1\}$

$\zeta_1 : b_1 \neq b_2$
$\zeta_2 : \forall x((C_0 x \vee C_1 x) \wedge \neg(C_0 x \wedge C_1 x))$
$T_0 = T(\zeta_1 \wedge \zeta_2)$

$$\varphi_0 = C_0 b_1 \wedge C_0 b_2 \qquad \varphi_2 = C_0 b_1 \wedge C_1 b_2$$
$$\varphi_1 = C_1 b_1 \wedge C_0 b_2 \qquad \varphi_3 = C_1 b_1 \wedge C_1 b_2$$

$\mathcal{C}$: set of all inductive theories with root T_0 such that

- $P(\varphi_n \mid T_0)$ exists for all $n \in \{0, 1, 2, 3\}$, and
- P satisfies the principle of indifference

Theorem (Prop 7.3.2)

(a) *If $P \in \mathcal{C}$, then $P(\varphi_0 \mid T_0) = P(\varphi_3 \mid T_0)$ and $P(\varphi_1 \mid T_0) = P(\varphi_2 \mid T_0)$.*

(b) *For any $p \in (0, 1)$, there exists $P \in \mathcal{C}$ such that $P(\varphi_0 \mid T_0) = P(\varphi_3 \mid T_0) = p/2$ and $P(\varphi_1 \mid T_0) = P(\varphi_2 \mid T_0) = (1 - p)/2$.*

Proof sketch

(a) T_0 is invariant under $\pi = (b_1\ b_2)$ and $\varphi_1^\pi = \varphi_2$.
Therefore, $P(\varphi_1 \mid T_0) = P(\varphi_2 \mid T_0)$.

T_0 is invariant under $\pi = (C_0\ C_1)$ and $\varphi_0^\pi = \varphi_3$.
Therefore, $P(\varphi_0 \mid T_0) = P(\varphi_3 \mid T_0)$.

(b) $\omega_n = (\{1, 2\}, L^{\omega_n})$, $n \in \{0, 1, 2, 3\}$

$$
\begin{array}{llll}
b_1^{\omega_n} = 1 & b_2^{\omega_n} = 2 & & \\
C_0^{\omega_0} = \{1,2\} & C_0^{\omega_1} = \{2\} & C_0^{\omega_2} = \{1\} & C_0^{\omega_3} = \emptyset \\
C_1^{\omega_0} = \emptyset & C_1^{\omega_1} = \{1\} & C_1^{\omega_2} = \{2\} & C_1^{\omega_3} = \{1,2\}
\end{array}
$$

Then φ_m is true in ω_n iff $m = n$.

$\Omega = \{\omega_0, \omega_1, \omega_2, \omega_3\}$, $\Sigma = \mathfrak{P}\Omega$,
$\mathbb{P}\{\omega_0\} = \mathbb{P}\{\omega_3\} = p/2$, $\mathbb{P}\{\omega_1\} = \mathbb{P}\{\omega_2\} = (1-p)/2$
$\mathscr{P} = (\Omega, \Sigma, \mathbb{P}) \vDash T_0$

Let $P = \mathbf{Th}\,\mathscr{P}\!\downharpoonleft_{[T_0, Th\,\mathscr{P}]}$. Then $P(\varphi_n \mid T_0) = \mathbb{P}(\varphi_n)_\Omega = \mathbb{P}\{\omega_n\}$. Only remains to show that P satisfies the principle of indifference.

π: signature permutation, not the identity
Assume $P(\varphi \mid X) = p$ and $X^\pi \in \operatorname{ante} P$
Want to show that $P(\varphi^\pi \mid X^\pi) = p$

Suppose $C_0^\pi = C_0$. Then $\pi = (b_1\ b_2)$

ω_n^π: in ω_n, everywhere swap 1 and 2

$\Omega^\pi = \{\omega_0^\pi, \omega_1^\pi, \omega_2^\pi, \omega_3^\pi\}$, $\Sigma^\pi = \mathfrak{P}\Omega^\pi$,
$\mathbb{P}^\pi\{\omega_0^\pi\} = \mathbb{P}^\pi\{\omega_3^\pi\} = p/2$, $\mathbb{P}^\pi\{\omega_1^\pi\} = \mathbb{P}^\pi\{\omega_2^\pi\} = (1-p)/2$
$\mathscr{P}^\pi = (\Omega^\pi, \Sigma^\pi, \mathbb{P}^\pi)$

Theorem (Thm 7.1.7)

$$\mathscr{P} \vDash (X, \varphi, p) \quad \textit{iff} \quad \mathscr{P}^\pi \vDash (X^\pi, \varphi^\pi, p)$$

$\omega_n \mapsto \omega_n^\pi$ is a pointwise isomorphism between measure spaces
$\omega_n \simeq \omega_n^\pi$ as structures; therefore $\mathscr{P} \simeq \mathscr{P}^\pi$

Inductive Isomorphism Theorem (Thm 5.3.24)

If $\mathscr{P} \simeq \mathscr{Q}$, then $\mathscr{P} \vDash (X, \varphi, p)$ iff $\mathscr{Q} \vDash (X, \varphi, p)$.

We therefore have $\mathscr{P} \vDash (X^\pi, \varphi^\pi, p)$, so that $P(\varphi^\pi \mid X^\pi) = p$
A similar proof covers the case $C_0^\pi = C_1$