7.5.4 Conditional Variance

$$\mathrm{Var}(X|Y) = E\left[(X - E[X|Y])^2 \mid Y\right] \quad \text{(defn)}$$

$$\mathrm{Var}(X|Y) = E[X^2|Y] - (E[X|Y])^2 \quad \left(\begin{matrix}\text{alt.}\\ \text{formula}\end{matrix}\right)$$

Theorem (see book for derivation):

$$\boxed{\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y])}$$

Example 5o Suppose that by any time t the number of people who have arrived at a train depot is a Poisson random variable with mean λt. If the initial train arrives at the depot at a time (independent of when the passengers arrive) that is uniformly distributed over $(0, T)$, what are the mean and variance of the number of passengers who enter the train?

$N(t)$: # of passengers that arrive by time t.

$N(t) \sim \mathrm{Poisson}(\lambda t)$, $\lambda > 0$ some fixed number

Y : time when train arrives

$Y \sim \mathrm{Unif}(0, T)$, $T > 0$ some fixed number

N and Y are indep.

X : # of passengers who enter the train

$X = N(Y)$

(a) $E[X] = ?$, (b) $\mathrm{Var}(X) = ?$

The idea is to condition on Y.

(a) $E[X] = E[N(Y)]$

$= E[E[N(Y) \mid Y]]$

$E[N(Y) \mid Y=y] = E[\underbrace{N(y)}_{\text{Poisson}(\lambda y)}] = \lambda y$

b/c N and Y are indep.

So... $E[N(Y) \mid Y] = \lambda Y$

$E[X] = E[E[N(Y) \mid Y]]$

$= E[\lambda Y]$

unif (0, T)

$= \lambda E[Y] = \lambda \cdot \frac{T}{2} = \boxed{\frac{\lambda T}{2}}$

(b) $\text{Var}(X) = E[\text{Var}(X \mid Y)] + \text{Var}(E[X \mid Y])$

$\text{Var}(X \mid Y) = \text{Var}(N(Y) \mid Y)$

$\text{Var}(N(Y) \mid Y=y) = \text{Var}(\underbrace{N(y)}_{\text{Poisson}(\lambda y)}) = \lambda y$

b/c N and Y are indep.

So... $\text{Var}(N(Y) \mid Y) = \lambda Y$ ✓

$$E[X|Y] = E[N(Y)|Y] \underset{\substack{\uparrow \\ \text{from part (a)}}}{=} \lambda Y$$

$$\begin{aligned} Var(X) &= E[\lambda Y] + Var(\lambda Y) \\ &= \lambda E[Y] + \lambda^2 Var(\underset{Unif(0,T)}{Y}) \\ &= \lambda \cdot \frac{T}{2} + \lambda^2 \cdot \frac{T^2}{12} = \boxed{\frac{\lambda^2 T^2 + 6\lambda T}{12}} \end{aligned}$$

Notice that I didn't need any integrals or series. If you understand conditional exp. conceptually, you often don't.

7.7 Moment generating functions

$M_X(t) = E[e^{tX}]$ moment generating func. (MGF)

domain of M_X is all t's such that the expected value exists (it fails to exist when the series or integral is divergent)

Always exists for $t=0$ and $M_X(0) = 1$.

Book derives $\boxed{\underbrace{E[X^n]}_{\text{the "moments" of } X} = M_X^{(n)}(0)}$ n^{th} derivative of M_X

If X and Y are indep., then

$$\boxed{M_{X+Y}(t) = M_X(t) M_Y(t)}$$

check: $M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX+tY}]$

$= E[e^{tX} e^{tY}] \underset{\text{indep.}}{=} E[e^{tX}] E[e^{tY}] = M_X(t) M_Y(t)$

Look at and get familiar w/ the tables of common MGFs. (Book derives a few of them.)

Table 7.1, p. 364 for discrete

Table 7.2, p. 365 for continuous

<u>Key thm</u>: If X and Y have the same MGF, then they have the same distribution

$\therefore$ Can use MGF to figure out distribution. (compute MGF and look it up on the table)

Example 7g **Sums of independent Poisson random variables**

Calculate the distribution of $X + Y$ when X and Y are independent Poisson random variables with means respectively λ_1 and λ_2.

$X \sim \text{Poisson}(\lambda_1)$, $Y \sim \text{Poisson}(\lambda_2)$

X, Y indep. $\Rightarrow M_{X+Y}(t) = M_X(t) M_Y(t)$

$$\left.\begin{aligned} M_X(t) &= e^{\lambda_1(e^t - 1)} \\ M_Y(t) &= e^{\lambda_2(e^t - 1)} \end{aligned}\right\} \text{from the tables}$$

$$M_{X+Y}(t) = e^{\lambda_1(e^t-1)} e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)}$$

From the table, we must have

$$X + Y \sim \boxed{\text{Poisson}(\lambda_1 + \lambda_2)}$$

Example 7h **Sums of independent normal random variables**

Show that if X and Y are independent normal random variables with respective parameters (μ_1, σ_1^2) and (μ_2, σ_2^2), then $X + Y$ is normal with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

$X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$

X, Y indep.

$$\left.\begin{aligned} M_X(t) &= e^{\mu_1 t + \sigma_1^2 t^2/2} \\ M_Y(t) &= e^{\mu_2 t + \sigma_2^2 t^2/2} \end{aligned}\right\} \text{from the tables}$$

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

$$= e^{\mu_1 t + \sigma_1^2 t^2/2} e^{\mu_2 t + \sigma_2^2 t^2/2}$$

$$= e^{(\mu_1+\mu_2)t + (\sigma_1^2+\sigma_2^2)t^2/2}$$

From the table, we must have

$$\boxed{X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)}$$

HW: Ch. 7 : Problems 70, 80