6.4 Conditional distributions (discrete case)

X, Y discrete:

$$p_{X|Y}(x|y) = P(X = x \mid Y = y)$$ only defined if $P(Y=y) > 0$

conditional mass function of X given Y

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \quad (\text{if } p_Y(y) > 0)$$

$$F_{X|Y}(x|y) = P(X \le x \mid Y = y)$$

conditional distribution function

Example 4a Suppose that $p(x, y)$, the joint probability mass function of X and Y, is given by

$$p(0,0) = .4 \quad p(0,1) = .2 \quad p(1,0) = .1 \quad p(1,1) = .3$$

Calculate the conditional probability mass function of X given that $Y = 1$.

$p_{X|Y}(x|1) = ?$

Is $p_{X|Y}(x|1)$ defined? Is $p_Y(1) > 0$?

$$p_Y(1) = \sum_{x \in R(X)} p(x, 1)$$

$$= p(0,1) + p(1,1) = 0.2 + 0.3 = 0.5 \checkmark$$

$$p_{X|Y}(x|1) = \frac{p(x,1)}{0.5}$$

$$p(x,1) = \begin{cases} 0.2 & \text{if } x=0, \\ 0.3 & \text{if } x=1, \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore \boxed{p_{X|Y}(x|1) = \begin{cases} \frac{2}{5} & \text{if } x=0, \\ \frac{3}{5} & \text{if } x=1, \\ 0 & \text{o.w.} \end{cases}}$$

Example 4b If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, calculate the conditional distribution of X given that $X + Y = n$.

$Z = X + Y$

$p_{X|Z}(k|n) = ?$

$$p_{X|Z}(k|n) = \frac{p_{X,Z}(k,n)}{p_Z(n)} \quad \text{if } p_Z(n) > 0,$$

$p_Z(n) =$

$$P(Z=n) = \begin{cases} e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^n}{n!} & \text{if } n=0,1,2,\ldots \\ 0 & \text{o.w.} \end{cases}$$

$Z \sim \text{Poisson}(\lambda_1+\lambda_2)$

$$p_{X,Z}(k,n) = P(X=k,\ X+Y=n) \leftarrow \begin{matrix} n \geq 0 \\ 0 \leq k \leq n \end{matrix}$$

$$= P(X=k,\ Y=n-k)$$

$$= P(X=k)\,P(Y=n-k)$$

$$= e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{n-k}}{(n-k)!}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{1}{k!\,(n-k)!}\,\lambda_1^k \lambda_2^{n-k}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{1}{n!}\binom{n}{k}\lambda_1^k \lambda_2^{n-k}.$$

undefined o.w.

$$\boxed{\begin{array}{l} \text{If } n=0,1,2,\ldots \text{ then} \\[1em] p_{X|Z}(k|n) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } 0 \leq k \leq n \\ 0 & \text{o.w.} \end{cases} \\[1em] \qquad\qquad \left(\text{Here, } p = \dfrac{\lambda_1}{\lambda_1+\lambda_2}\right) \end{array}}$$

If $n=0,1,2,\ldots,$

then $X \mid Y=n \sim \mathrm{Binom}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$

6.5 Conditional distributions (continuous case)

X, Y jointly continuous:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

↑ conditional density of X given Y

$$P(X \in [x, x+\Delta x] \mid Y \in [y, y+\Delta y]) \approx f_{X|Y}(x|y)\Delta x$$

$$P(X \in A \mid Y = y) \underset{\uparrow}{=} \int_A f_{X|Y}(x|y)\, dx$$

definition (convention)

$$F_{X|Y}(x|y) = P(X \leq x \mid Y = y) = \int_{-\infty}^{x} f_{X|Y}(t|y)\, dt$$

↑ conditional distribution function

Example 5a The joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{12}{5}x(2-x-y) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute the conditional density of X given that $Y = y$, ~~where $0 < y < 1$~~.

(going to compute the whole thing)

$$f_{X|Y}(x|y) = ?$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \quad (\text{if } f_Y(y) > 0)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \begin{cases} \int_0^1 \frac{12}{5} x(2-x-y)\,dx & \text{if } 0<y<1, \\ 0 & \text{o.w.} \end{cases}$$

For $0<y<1$:

$$f_Y(y) = \frac{12}{5}\int_0^1 (2x - x^2 - xy)\,dx$$

$$= \frac{12}{5}\left(x^2 - \frac{x^3}{3} - \frac{1}{2}x^2 y\right)\Big|_{x=0}^{x=1}$$

$$= \frac{12}{5}\left(\cancel{1 - \frac{1}{3}}^{\,\frac{2}{3}} - \frac{1}{2}y\right)$$

$$= \frac{8}{5} - \frac{6}{5}y$$

So...

$$f_Y(y) = \begin{cases} \frac{8}{5} - \frac{6}{5}y & \text{if } 0<y<1, \\ 0 & \text{o.w.} \end{cases}$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{f(x,y)}{\frac{8}{5} - \frac{6}{5}y} \quad \text{if } 0 < y < 1 \quad \left(\begin{array}{c}\text{o.w.}\\ \text{undefined}\end{array}\right)$$

Looking at the piecewise defn of $f(x,y)$, for a given $y \in (0,1)$, there are two possibilities:

<u>Case 1:</u> $x \notin (0,1)$

$$f(x,y) = 0 \Rightarrow f_{X|Y}(x|y) = 0$$

<u>Case 2:</u> $x \in (0,1)$

$$f_{X|Y}(x|y) = \frac{\frac{12}{5}x(2-x-y)}{\frac{8}{5} - \frac{6}{5}y} \cdot \frac{5}{5}$$

$$= \frac{\overset{6}{\cancel{12}}x(2-x-y)}{{}_{4}\cancel{8} - \underset{3}{\cancel{6}}y}$$

$$= \frac{6x(2-x-y)}{4-3y}.$$

$$\boxed{f_{X|Y}(x|y) = \begin{cases} \dfrac{6x(2-x-y)}{4-3y} & \text{if } 0<x,y<1, \\ 0 & \text{if } 0<y<1 \text{ and } x \notin (0,1) \end{cases}}$$

undefined o.w.

Example 5b Suppose that the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \dfrac{e^{-x/y}e^{-y}}{y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find $P\{X > 1|Y = y\}$.

$$P(X>1|Y=y) = \int_1^\infty f_{X|Y}(x|y)\,dx$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \quad \text{if} \quad f_Y(y) > 0,$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \begin{cases} \displaystyle\int_0^\infty \frac{e^{-x/y} e^{-y}}{y}\, dx & \text{if } y > 0, \\ 0 & \text{o.w.} \end{cases}$$

For $y > 0$,

$$\int_0^\infty \frac{e^{-x/y} e^{-y}}{y}\, dx = e^{-y} \int_0^\infty \frac{1}{y} e^{-x/y}\, dx$$

$$= e^{-y} \cdot \left(-e^{-x/y} \Big|_{x=0}^{x=\infty} \right)$$

$$= e^{-y} \left(0 - (-1) \right)$$

$$= e^{-y}$$

So

$$f_Y(y) = \begin{cases} e^{-y} & \text{if } y > 0, \\ 0 & \text{o.w.} \end{cases}$$

(So $Y \sim \mathrm{Exp}(1)$)

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{f(x,y)}{e^{-y}} \quad \text{if } y>0$$

$y>0$:

$x \le 0 \Rightarrow f(x,y)=0 \Rightarrow f_{X|Y}(x|y)=0$

$x>0$:

$$f_{X|Y}(x|y) = \frac{\left(\frac{e^{-x/y}e^{-y}}{y}\right)}{e^{-y}}$$

$$= \frac{1}{y}e^{-x/y}$$

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}e^{-x/y} & \text{if } y>0 \text{ and } x>0, \\ 0 & \text{if } y>0 \text{ and } x\le 0, \end{cases}$$

(if $y>0$, then $X|Y=y \sim Exp(\frac{1}{y})$)

Now...

$$P(X>1 \mid Y=y) = \int_1^\infty f_{X|Y}(x|y)\,dx$$

If $y \le 0$, the integrand is undefined, so the integral is undefined.

If $y > 0$, then

$$\int_1^\infty f_{X|Y}(x|y)\,dx = \int_1^\infty \frac{1}{y} e^{-x/y}\,dx$$

$$= -e^{-x/y} \Big|_{x=1}^{x=\infty}$$

$$= 0 - (-e^{-1/y}) = e^{-1/y}$$

So...

$$\boxed{P(X>1 \mid Y=y) = e^{-1/y} \quad \text{if } y>0}$$

undefined o.w.

X and Y are jointly normal if,

(bivariate) (Gaussian)

(multivariate)

whenever a and b are real #'s, not both 0,

$aX + bY$ has a normal distribution

- X and Y jointly norm. $\Rightarrow \left\{ \begin{array}{l} X \text{ is normal and} \\ Y \text{ is normal} \end{array} \right.$

(take $a=1, b=0$ and $a=0, b=1$)

- indep. normals are jointly normal

Covariance and correlation

$$\mathrm{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])] \quad \text{(defn)}$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] \quad \text{(alt. formula)}$$

$$\mathrm{Cov}(X,X) = \mathrm{Var}(X) \quad \text{(special case)}$$

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

correlation of X and Y

$$-1 \le \rho(X,Y) \le 1$$

X and Y are <u>uncorrelated</u> if $\rho(X,Y) = 0$.

indep. $\Rightarrow$ uncorrelated
$\not\Leftarrow$

<u>Joint density for jointly normal r.v.s</u>

Suppose X, Y jointly normal. There are 5 relevant constants: $\mu_X, \sigma_X, \mu_Y, \sigma_Y, \rho$

$E[X]$ $SD(X)$ $E[Y]$ $SD(Y)$ $\rho(X,Y)$

Notation: $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mu = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$, $Q = \begin{pmatrix} \sigma_X^2 & \sigma_X\sigma_Y\rho \\ \sigma_X\sigma_Y\rho & \sigma_Y^2 \end{pmatrix}$

<u>covariance matrix of X and Y</u>

joint density of X and Y:

$$f(x_1, x_2) = \frac{1}{2\pi\sqrt{\det Q}} \exp\left(-\frac{1}{2}(x-\mu)^T Q^{-1}(x-\mu)\right)$$

(a form of this w/o matrices is in the book)

Conditional densities for joint normals

X, Y jointly normal

$X \sim N(\mu_X, \sigma_X^2)$

$Y \sim N(\mu_Y, \sigma_Y^2)$

ρ: corr. coeff.

Define

$$\mu = \mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y - \mu_Y)$$

$$\sigma^2 = \sigma_X(1-\rho^2)$$

Then $X | Y = y \sim N(\mu, \sigma^2)$ or

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

6.5 $\frac{1}{2}$

Conditional distributions : mixed case (not in book)

X : discrete

$R(X) = \{x_1, x_2, x_3, \ldots\}$

Y: continuous

joint mass/density function

$$P(X = x_i, Y \in A) = \int_A f(x_i, y)\, dy$$

$$f \geq 0 \quad \text{and} \quad \sum_i \int_{-\infty}^{\infty} f(x_i, y)\, dy = 1$$

$$p_X(x_i) = \int_{-\infty}^{\infty} f(x_i, y)\, dy \,, \quad f_Y(y) = \sum_i f(x_i, y)$$

<u>conditional density of Y given X:</u>

$$f_{Y|X}(y \mid x_i) = \frac{f(x_i, y)}{p_X(x_i)}$$

$$P(Y \in A \mid X = x_i) = \int_A f_{Y|X}(y \mid x_i)\, dy$$

<u>conditional mass function of X given Y:</u>

$$p_{X|Y}(x_i \mid y) = P(X = x_i \mid Y = y) = \frac{f(x_i, y)}{f_Y(y)}$$

defn/convention

<u>Expl</u>

Imagine we generate a r.v. $Y \sim \text{Exp}(1)$ on our computer. It produces a number $Y = y$. We then set $\lambda = y$ and generate $X \sim \text{Poisson}(\lambda)$. (a) Find $P(X \geq 1, Y \leq 2)$. (b) What is the distribution of X?

(a)

Given:

$$Y \sim Exp(1), \quad X \mid Y=y \sim Poisson(y)$$

$$f_Y(y) = \begin{cases} e^{-y} & \text{if } y>0, \\ 0 & \text{o.w.} \end{cases}$$

$y>0$:

$$p_{X|Y}(k|y) = P(X=k \mid Y=y) = \begin{cases} e^{-y} \cdot \frac{y^k}{k!} & \text{if } k=0,1,2,\ldots \\ 0 & \text{o.w.} \end{cases}$$

$$p_{X|Y}(k|y) = \frac{f(k,y)}{f_Y(y)} \quad \text{if } f_Y(y) > 0$$

$\hookrightarrow y>0$

Want to find $f(k,y)$.

$y>0$: $f(k,y) = f_Y(y)\, p_{X|Y}(k|y)$

$$= \begin{cases} e^{-y} \cdot e^{-y} \cdot \frac{y^k}{k!} & \text{if } k=0,1,2,\ldots \\ 0 & \text{o.w.} \end{cases}$$

$y<0$: $f(k,y) = 0$

$$f(k,y) = \begin{cases} e^{-2y} \cdot \dfrac{y^k}{k!} & \text{if } y>0,\ k=0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

$$P(X \geq 1, Y \leq 2) = \sum_{k=1}^{\infty} \int_{-\infty}^{2} f(k,y)\,dy$$

$$= \sum_{k=1}^{\infty} \int_0^2 e^{-2y} \cdot \frac{y^k}{k!}\,dy = \int_0^2 \sum_{k=1}^{\infty} e^{-2y} \cdot \frac{y^k}{k!}\,dy$$

$$= \int_0^2 e^{-2y} \sum_{k=1}^{\infty} \frac{y^k}{k!}\,dy = \int_0^2 e^{-2y}(e^{y}-1)\,dy$$

$$= \int_0^2 (e^{-y} - e^{-2y})\,dy = \left(-e^{-y} + \frac{1}{2}e^{-2y}\right)\Big|_{y=0}^{y=2}$$

$$= -e^{-2} + \frac{1}{2}e^{-4} + 1 - \frac{1}{2} = \boxed{\frac{1}{2} - e^{-2} + \frac{1}{2}e^{-4}}$$

OR:

$$P(X \geq 1, Y \leq 2) = P(Y \leq 2) - P(X=0, Y \leq 2)$$

$$= \int_{-\infty}^{2} f_Y(y)\,dy - \int_{-\infty}^{2} f(0,y)\,dy$$

$$= \int_0^2 e^{-y}\,dy - \int_0^2 e^{-2y}\,dy = \cdots \quad \text{(same)}$$

(b) $p_X(k) = \int_{-\infty}^{\infty} f(k,y)\,dy$

$$= \int_0^\infty e^{-2y} \cdot \frac{y^k}{k!}\,dy = \frac{1}{k!} \int_0^\infty \underbrace{y^k e^{-2y}}_{}\,dy$$

$k = 0, 1, 2, \ldots$

density of Gamma $(\underbrace{k+1}_{\alpha}, \underset{\lambda}{2})$

$$= \frac{1}{k!} \cdot \frac{\Gamma(\alpha)}{\lambda^\alpha} = \frac{1}{k!} \cdot \frac{k!}{2^{k+1}} = 2^{-k-1}$$

$$\boxed{p_X(k) = \begin{cases} 2^{-k-1} & \text{if } k = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}}$$

<u>Expl</u> Generate $U \sim \text{Unif}(0,1)$ and obtain a number p. Build a coin w/ prob. of heads p. Flip it 3 times. Given that we flip two heads, what is the prob. the coin is <u>biased toward tails?</u>

N = # of heads in 1st 3 flips

$P(U < 0.5 \mid N = 2) = ?$

Given: $U \sim \text{Unif}(0,1)$, $N \mid U = p \sim \text{Binom}(3, p)$

$$f_U(p) = \begin{cases} 1 & \text{if } 0<p<1, \\ 0 & \text{o.w.} \end{cases}$$

$0<p<1$:

$$p_{N|U}(k|p) = \begin{cases} \binom{3}{k} p^k (1-p)^{3-k} & \text{if } k=0,1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

$$p_{N|U}(k|p) = \frac{f(k,p)}{f_U(p)} \Rightarrow f(k,p) = \binom{3}{k} p^k (1-p)^{3-k}$$

$$f(k,p) = \begin{cases} \binom{3}{k} p^k (1-p)^{3-k} & \text{if } 0<p<1,\ k=0,1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

$$P(U<0.5 \,|\, N=2) = \frac{P(U<0.5, N=2)}{P(N=2)}$$

$$P(U<0.5, N=2) = \int_{-\infty}^{0.5} f(2,p)\,dp$$

$$= \int_0^{\frac{1}{2}} \binom{3}{2} p^2 (1-p)\,dp = \cdots = \frac{5}{64}$$

$$P(N=2) = p_N(2) = \int_{-\infty}^{\infty} f(2,p)\,dp$$

$$= \int_0^1 \binom{3}{2} p^2 (1-p)\,dp = \cdots = \frac{1}{4}$$

$\therefore\ P(U<0.5 \mid N=2) = \frac{5}{64} \cdot \frac{4}{1} = \boxed{\frac{5}{16}}$

---

7. Properties of Expectations

7.2 Expectations of Sums

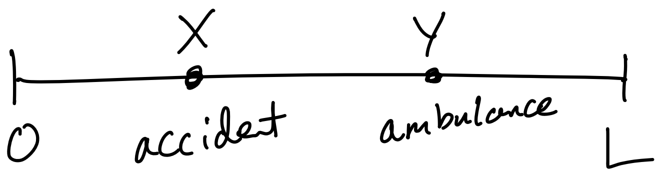
$$E[g(X,Y)] = \begin{cases} \displaystyle\sum_{x\in R(X)} \sum_{y \in R(Y)} g(x,y)\,p(x,y) & \text{(both discrete)} \\[2ex] \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,f(x,y)\,dx\,dy & \text{(jointly continuous)} \\[2ex] \displaystyle\sum_{x\in R(X)} \int_{-\infty}^{\infty} g(x,y)\,f(x,y)\,dy & \text{(mixed)} \end{cases}$$

other related facts:

- $X \le Y \Rightarrow E[X] \le E[Y]$
- $X \le a \Rightarrow E[X] \le a$

Used to prove that $E[X+Y] = E[X] + E[Y]$.

Example 2a An accident occurs at a point X that is uniformly distributed on a road of length L. At the time of the accident, an ambulance is at a location Y that is also uniformly distributed on the road. Assuming that X and Y are independent, find the expected distance between the ambulance and the point of the accident.



$X \sim Unif(0, L)$

$Y \sim Unif(0, L)$

X, Y indep.

$E[|X-Y|] = ?$

$$E[|X-Y|] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y| f_{X,Y}(x,y)\,dx\,dy$$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

b/c X and Y indep.

$$f_X(x) = \begin{cases} \frac{1}{L} & \text{if } 0 < x < L, \\ 0 & \text{o.w.} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{L} & \text{if } 0 < y < L, \\ 0 & \text{o.w.} \end{cases}$$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{L^2} & \text{if } 0<x<L \text{ and } 0<y<L, \\ 0 & \text{o.w.} \end{cases}$$

$$E[|X-Y|] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y| f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^L \int_0^L |x-y| \cdot \frac{1}{L^2}\,dx\,dy$$

$$= \frac{1}{L^2}\int_0^L \left(\int_0^L |x-y|\,dx \right) dy$$

$$= \frac{1}{L^2}\int_0^L \left(\int_0^y (y-x)\,dx + \int_y^L (x-y)\,dx \right) dy$$

$$= \frac{1}{L^2}\int_0^L \left(\left(xy - \frac{1}{2}x^2\right)\Big|_{x=0}^{x=y} + \left(\frac{1}{2}x^2 - xy\right)\Big|_{x=y}^{x=L} \right) dy$$

$$= \frac{1}{L^2}\int_0^L \left(y^2 - \frac{1}{2}y^2 - 0 + \frac{1}{2}L^2 - Ly - \frac{1}{2}y^2 + y^2 \right) dy$$

$$= \frac{1}{L^2}\int_0^L \left(y^2 - Ly + \frac{1}{2}L^2 \right) dy$$

$$= \frac{1}{L^2}\left(\frac{1}{3}y^3 - \frac{1}{2}Ly^2 + \frac{1}{2}L^2y\right)\Bigg|_{y=0}^{y=L}$$

$$= \frac{1}{L^2}\left(\frac{1}{3}L^3 - \cancel{\frac{1}{2}L^3} + \cancel{\frac{1}{2}L^3} - 0\right)$$

$$= \boxed{\frac{L}{3}}$$

Example 2c **The sample mean**

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables having distribution function F and expected value μ. Such a sequence of random variables is said to constitute a sample from the distribution F. The quantity

$$\overline{X} = \sum_{i=1}^{n} \frac{X_i}{n}$$

means they all have the same distribution

is called the *sample mean*. Compute $E[\overline{X}]$.

$$E[\bar{X}] = E\left[\sum_{i=1}^{n} \frac{X_i}{n}\right] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} \underbrace{\cancel{E[X_i]}}_{\mu}$$

$$= \frac{1}{n}\cdot n\mu = \boxed{\mu}$$

Example 2h **Expected number of matches**

Suppose that N people throw their hats into the center of a room. The hats are mixed up, and each person randomly selects one. Find the expected number of people who select their own hat.

A_j = "The j^{th} person selects their own hat."

$$1 \le j \le N$$

X = # of people that select their own hat.

$$X = \sum_{j=1}^{N} 1_{A_j}$$

$$E[X] = ?$$

$$E[X] = E\left[\sum_{j=1}^{N} 1_{A_j}\right] = \sum_{j=1}^{N} E[1_{A_j}]$$

$$= \sum_{j=1}^{N} P(A_j)$$

By symmetry, $P(A_j)$ is the same for all j.

$$P(A_j) = \frac{1}{N}$$

$$E[X] = \sum_{j=1}^{N} \frac{1}{N} = N \cdot \frac{1}{N} = \boxed{1}$$

Example 2i **Coupon-collecting problems**

Suppose that there are N types of coupons, and each time one obtains a coupon, it is equally likely to be any one of the N types. Find the expected number of coupons one needs to amass before obtaining a complete set of at least one of each type.

coupon = collectible card

how many cards do you have to buy to get the whole collection?

X = # of cards you buy in order to get the whole collection

$E[X] = ?$

X_n = given that you've already collected n distinct types, this is how many additional cards you buy in order to get your $(n+1)^{th}$ distinct type. $(n = 0, 1, 2, \ldots, N-1)$

E.g. $N=3$, cards $\boxed{A}$ $\boxed{B}$ $\boxed{C}$

My purchases, in order:

I have 1 card

I have 2 cards

I have 3 cards

$$\underbrace{\boxed{B}}_{X_0=1}\ \underbrace{\boxed{B}\ \boxed{C}}_{X_1=2}\ \underbrace{\boxed{B}\ \boxed{C}\ \boxed{C}\ \boxed{B}\ \boxed{A}}_{X_2=5}$$

when I had 0 cards, I got a new one after buying 1 card

when I had 1 card, I got a new one after buying 2 cards

when I had 2 cards I got a new one after buying 5 cards.

$$X = X_0 + X_1 + X_2 + \cdots + X_{N-1}$$

$$= \sum_{j=0}^{N-1} X_j$$

$$E[X] = \sum_{j=0}^{N-1} E[X_j]$$

$$X_j \sim ?$$

When you have j cards, how many trials (new card purchases) before first success (success means getting a new type)

$$\text{prob. of success} = \frac{N-j \leftarrow \text{\# of new types}}{N \leftarrow \text{\# of types}}$$

So $X_j \sim \text{Geom}\left(\frac{N-j}{N}\right)$

$$E[X_j] = \frac{1}{\left(\frac{N-j}{N}\right)} = \frac{N}{N-j}$$

$$E[X] = \sum_{j=0}^{N-1} \frac{N}{N-j}$$

$$= N \sum_{j=0}^{N-1} \frac{1}{N-j} \qquad (\text{sub } i = N-j)$$

$$= \boxed{N \sum_{i=1}^{N} \frac{1}{i}}$$

Extra stuff:

$$= N\Big(\underbrace{1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{N}}\Big)$$

partial sum of harmonic series, it's called the n^{th} harmonic number and is denoted by H_n.

$$\lim_{n\to\infty} (H_n - \log n) = \gamma \approx 0.5772156649$$

γ: the Euler-Mascheroni constant

for large n, $H_n \approx \gamma + \log n$

for large N, $E[X] \approx N(\gamma + \log N)$

Example 2l **A random walk in the plane**

Consider a particle initially located at a given point in the plane, and suppose that it undergoes a sequence of steps of fixed length, but in a completely random direction. Specifically, suppose that the new position after each step is one unit of distance from the previous position and at an angle of orientation from the previous position that is uniformly distributed over $(0, 2\pi)$. (See Figure 1.) Compute the expected square of the distance from the origin after n steps.

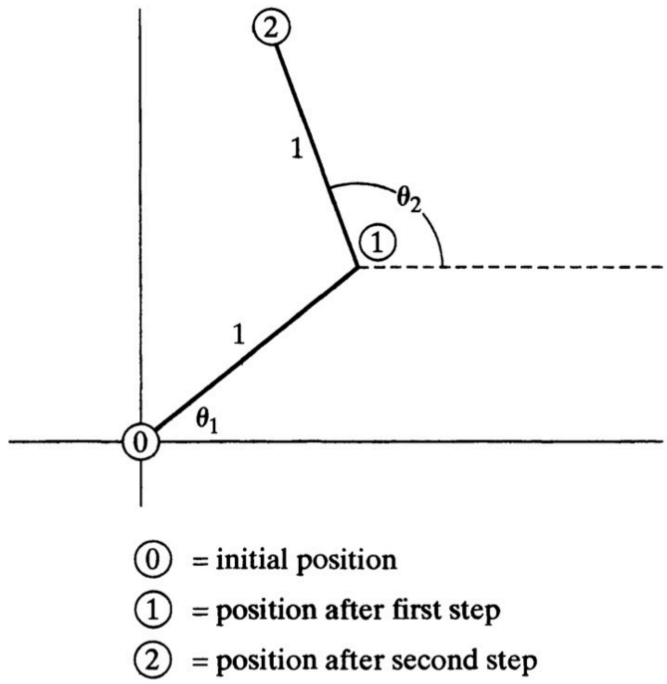


Figure 1

(U_n, V_n) = location of the particle after n steps

$(U_0, V_0) = (0,0)$

(X_n, Y_n) = vector pointing from (U_{n-1}, V_{n-1}) to (U_n, V_n)

θ_n = angle that (X_n, Y_n) makes with pos. x-axis.

$\theta_1, \theta_2, \ldots$ indep.

$\theta_n \sim \text{Unif}(0, 2\pi)$

D_n = distance from $(0,0)$ to (U_n, V_n)

$E[D_n^2] = ?$

$$X_n = \cos\theta_n$$
$$Y_n = \sin\theta_n$$

$$(U_n, V_n) = \sum_{j=1}^{n} (X_n, Y_n)$$
$$U_n = \sum_{j=1}^{n} X_n$$
$$V_n = \sum_{j=1}^{n} Y_n$$

$$\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{j=1}^{n} X_j\right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j$$

$$D_n^2 = U_n^2 + V_n^2$$
$$= \left(\sum_{i=1}^{n} X_i\right)^2 + \left(\sum_{i=1}^{n} Y_i\right)^2$$
$$= \left(\sum_{i=1}^{n} X_i^2 + \sum\sum_{\substack{1 \le i,j \le n \\ i \ne j}} X_i X_j\right)$$
$$+ \left(\sum_{i=1}^{n} Y_i^2 + \sum\sum_{\substack{1 \le i,j \le n \\ i \ne j}} Y_i Y_j\right)$$
$$= \sum_{i=1}^{n} \underbrace{\cancel{(X_i^2 + Y_i^2)}}_{1} + \sum\sum_{\substack{1 \le i,j \le n \\ i \ne j}} (X_i X_j + Y_i Y_j)$$

b/c (X_i, Y_i) is a unit vector

$$= n + \sum_{\substack{1 \le i,j \le n \\ i \ne j}}\sum (\cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j)$$

$$E[D_n^2] = n + \sum_{\substack{1 \le i,j \le n \\ i \ne j}}\sum E[\cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j]$$

θ_i and θ_j are indep.

$$= n + \sum_{\substack{1 \le i,j \le n \\ i \ne j}}\sum \Big(E[\cos\theta_i]E[\cos\theta_j] + E[\sin\theta_i]E[\sin\theta_j] \Big)$$

$$E[\cos\theta_i] = \int_0^{2\pi} (\cos x) \cdot \frac{1}{2\pi}\, dx = \frac{1}{2\pi}\int_0^{2\pi} \cos x\, dx$$

$\theta_i \sim \text{Unif}(0, 2\pi)$

$$= 0$$

$$E[\sin\theta_i] = \int_0^{2\pi} (\sin x) \cdot \frac{1}{2\pi}\, dx = \frac{1}{2\pi}\int_0^{2\pi} \sin x\, dx$$

$$= 0$$

So $E[D_n^2] = \boxed{n}$

HW: Ch.6: 42, 43-45, 46, 47, 54
38, 39-41, 42, 43, 50

Ch.7: 4, 5-7, 13, 16