5.6.1 Gamma distribution

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx, \qquad \alpha > 0$$

gamma function

- $\Gamma(1) = 1$
- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$
- $\Gamma(n) = (n-1)!$

$\alpha > 0, \ \lambda > 0$

$X \sim \text{Gamma}(\alpha, \lambda)$ means X has density

$$f(x) = \begin{cases} C x^{\alpha-1} e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{o.w.} \end{cases}$$

$$C = \frac{\lambda^\alpha}{\Gamma(\alpha)}$$

$\text{Gamma}(1, \lambda) = \text{Exp}(\lambda)$

$\text{Gamma}(n, \lambda) =$ sum of n indep. $\text{Exp}(\lambda)$
(Section 6.3.2)

Expl 6a

$X \sim \text{Gamma}(\alpha, \lambda)$, $E[X] = ?$. $Var(X) = ?$

$$E[X] = \int_0^\infty x \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\lambda x} dx$$

density of $\text{Gamma}(\alpha+1, \lambda) = \underbrace{\frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^\alpha e^{-\lambda x}}_{\text{integrates to } 1}$

$$\therefore \int_0^\infty x^\alpha e^{-\lambda x} dx = \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}$$

$$E[X] = \frac{\cancel{\lambda^\alpha}}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\lambda^{\cancel{\alpha}+1}} = \frac{\alpha \cancel{\Gamma(\alpha)}}{\lambda \cancel{\Gamma(\alpha)}} = \frac{\alpha}{\lambda}$$

$$Var(X) = \underset{\text{similar}}{\cdots} = \frac{\alpha}{\lambda^2}$$

5.7 Dist. of a func. of a r.v.

Example 7e **The Lognormal Distribution** If X is a normal random variable with mean μ and variance σ^2, then the random variable

$$Y = e^X$$

is said to be a *lognormal* random variable with parameters μ and σ^2.

Find the density f_Y of Y.

$R(Y) = (0, \infty)$, so $f_Y(y)$ will be 0 for $y \le 0$.

$y > 0$:

default base for log is e in mathematics, 10 in engineering, and 2 in computer science.

$$F_Y(y) = P(Y \le y) = P(e^X \le y)$$

$$= P(X \le \log y)$$

$$= P(\mu + \sigma Z \le \log y) \quad , \; Z \sim N(0,1)$$

$$= P\left(Z \le \frac{\log y - \mu}{\sigma}\right)$$

$$= \Phi\left(\frac{\log y - \mu}{\sigma}\right)$$

$$f_Y(y) = F_Y'(y) = \frac{d}{dy}\left[\Phi\left(\frac{\log y - \mu}{\sigma}\right)\right]$$

$$= \Phi'\left(\frac{\log y - \mu}{\sigma}\right) \cdot \frac{1}{\sigma} \cdot \frac{1}{y}$$

$$= \frac{1}{\sigma y} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\log y - \mu}{\sigma}\right)^2\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}\, y} e^{-(\log y - \mu)^2 / 2\sigma^2}$$

So

$$\boxed{f(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi\sigma^2}\, y} e^{-(\log y - \mu)^2/2\sigma^2} & \text{if } y > 0, \\ 0 & \text{if } y \le 0. \end{cases}}$$

HW: Ch. 5: ~~42-44~~

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