4.4 Functions of a r.v.

X discrete $\Rightarrow$ g(X) discrete

$$E[g(X)] = \sum_{x \in R(X)} g(x) P(X = x)$$

Consequence:
$$E[aX + b] = a E[X] + b$$
$$E[aX] = a E[X]$$
$$E[b] = b$$

Example 4b A product that is sold seasonally yields a net profit of b dollars for each unit sold and a net loss of ℓ dollars for each unit left unsold when the season ends. The number of units of the product that are ordered at a specific department store during any season is a random variable having probability mass function $p(i), i \geq 0$. If the store must stock this product in advance, determine the number of units the store should stock so as to maximize its expected profit.

X = "units ordered"

X is a r.v. w/ mass func. $p(i)$

$R(X) = \{0, 1, 2, \ldots\}$

s = "units stocked"

s is a nonnegative integer that we choose

R = "profit in dollars"

$$R = \begin{cases} bX - (s-X)\ell & \text{if } X \le s, \\ sb & \text{if } X > s. \end{cases}$$

Goal: find s that maximizes ER.

R is a function of X

So $R = g(X)$ for some func. g.

What is g?

$$g(x) = \begin{cases} bx - (s-x)\ell & \text{if } x \le s, \\ sb & \text{if } x > s. \end{cases}$$

This is the whole key to this problem. To compute ER, we need to use the theorem about a func. of a r.v. To use that thm, we need to be able to write down that function.

$$E[R] = E[g(X)]$$

$$= \sum_{i \in R(X)} g(i) P(X=i)$$

$$= \sum_{i=0}^{\infty} g(i) p(i)$$

$$= \sum_{i=0}^{s} (bi - (s-i)\ell) p(i) + \sum_{i=s+1}^{\infty} sb\, p(i)$$

$$= \dots$$

lots of algebra (see the book for details)

$$= \underbrace{sb + (b+\ell)\sum_{i=0}^{s}(i-s)p(i)}_{\text{call this } f(s)}$$

we want to maximize $f(s)$

The domain of f is $\{0,1,2,\dots\}$

Cannot use f' to maximize.

Look at differences instead

$$f(s+1) - f(s) = \dots$$

lots of algebra (see book)

$$= b - (b+\ell)\sum_{i=0}^{s} p(i)$$

So...

$$f(s+1) > f(s) \iff \frac{b}{b+\ell} > \underbrace{\sum_{i=0}^{s} p(i)}_{= P(X \le s) = F_X(s)}$$

Find the largest s with $F_X(s) < \frac{b}{b+\ell}$ and call it s^*.

$$\boxed{\text{We should stock } s^* + 1 \text{ units.}}$$

4.5 Variance

$\mu = E[X]$

$$\mathrm{Var}(X) = E[(X-\mu)^2] \quad \text{(definition)}$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 \quad \text{(alternate formula)}$$

$$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$$

read the pfs/ derivations in book

$$SD(X) = \sqrt{\mathrm{Var}(X)}$$

standard deviation of X

Expl 5a

Roll a fair die.

X = "the number rolled"

$\mathrm{Var}(X) = ?$

$$E[X] = 3.5 = \frac{7}{2} \quad \text{(computed earlier)}$$

$$E[X^2] = \sum_{n=1}^{6} n^2 P(X=n) = \frac{1}{6}(1^2+2^2+\cdots+6^2)$$

$$= \cdots = \frac{91}{6}$$

$$\mathrm{Var}(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12}$$

$$= \boxed{\frac{25}{12}}$$

4.6 Bernoulli/Binomial r.v.s

$0 \leq p \leq 1$

$X \sim \text{Bernoulli}(p)$ means $P(X=1) = p$
$P(X=0) = 1-p$

$n \in \mathbb{N}$

$X \sim \text{Binom}(n,p)$ means

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

Example 6b It is known that screws produced by a certain company will be defective with probability .01, independently of one another. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

A = "The package must be replaced."

N = the # of defective screws in the package.

$N \sim \text{Binom}(10, 0.01)$

$A = \{N > 1\}$

$P(A) = ?$

$N \sim \text{Binom}(10, 0.01)$

$$P(N=k) = \binom{10}{k} 0.01^k \, 0.99^{10-k}, \quad k = 0, 1, \ldots, 10$$

$$P(A) = P(N>1) = \underbrace{\sum_{k=2}^{10}\binom{10}{k}0.01^k\,0.99^{10-k}}_{\text{too much to calculate}}$$

$$P(N>1) = 1 - P(N \le 1)$$

$$= 1 - \left(P(N=0) + P(N=1)\right)$$

$$= 1 - \underbrace{\cancel{\binom{10}{0}}}_{1}\,\underbrace{\cancel{0.01^0}}_{1}\,0.99^{10} - \underbrace{\cancel{\binom{10}{1}}}_{10}\,0.01^1\,0.99^9$$

$$= 1 - 0.99^{10} - 0.1(0.99^9)$$

$$= 1 - 0.99^9(0.99 + 0.1)$$

$$= \boxed{1 - 1.09(0.99^9)} \approx 0.004266$$

Example 4j **The problem of the points**

Independent trials resulting in a success with probability p and a failure with probability $1 - p$ are performed. What is the probability that n successes occur before m failures? If we think of A and B as playing a game such that A gains 1 point when a success occurs and B gains 1 point when a failure occurs, then the desired probability is the probability that A would win if the game were to be continued in a position where A needed n and B needed m more points to win.

partial SOLN 1

(this example is from Section 3.4)

$A_{n,m}$ = "n successes occur before m failures."

B = "First trial is a success."

$$P(A_{n,m}) = P(B)P(A_{n,m}|B) + P(B^c)P(A_{n,m}|B^c)$$

$$= p\,P(A_{n-1,m}) + (1-p)P(A_{n,m-1})$$

$$q_{n,m} := P(A_{n,m})$$

$$\begin{cases} q_{n,m} = p\,q_{n-1,m} + (1-p)q_{n,m-1} \\ q_{n,0} = 0 \\ q_{0,m} = 1 \end{cases}$$

can solve this recursion, but it's a mess...

<u>SOLN 2</u> N = trial on which n^{th} success occurs

M = " m^{th} failure "

$$P(N<M) = ?$$

X = # of successes in 1^{st} $n+m-1$ trials

Y = # of failures "

$$X+Y = n+m-1$$

$X \sim \mathrm{Binom}(n+m-1, p)$, $\quad Y = m-1+n-X$

$$X \geq n \iff N \leq n+m-1$$

$$\Updownarrow$$

$$Y \leq m-1 \iff M > n+m-1$$

$$X \geq n \Rightarrow \begin{matrix} N \leq n+m-1 \\ \text{and} \\ M > n+m-1 \end{matrix} \Rightarrow N < M$$

$$X < n \Rightarrow \begin{matrix} N > n+m-1 \\ \text{and} \\ M \leq n+m-1 \end{matrix} \Rightarrow N > M \Rightarrow N \geq M$$

Therefore,

$$X \geq n \iff N < M$$

$$\{X \geq n\} = \{N < M\}$$

$$P(N<M) = P(X \geq n)$$

$$= \boxed{\sum_{k=n}^{n+m-1} \binom{n+m-1}{k} p^k (1-p)^{n+m-1-k}}$$

Properties of binom. dist.

$X \sim \mathrm{Binom}(n,p) \Rightarrow E[X] = np$

$\mathrm{Var}(X) = np(1-p)$

Stirling's approx. : $k! \sim k^{k+\frac{1}{2}} e^{-k} \sqrt{2\pi}$

"is asymptotic to"

$a_k \sim b_k$ means $\frac{a_k}{b_k} \xrightarrow{k\to\infty} 1$.

Expl 6g

n = the population of my state

cn = the # of electoral votes my state awards

N = the # of electoral votes I personally award

(Imagine I vote last. If the election is tied before I vote, I award cn electoral votes. Otherwise, I award 0.)

my "power" $\overset{\text{defn}}{=} E[N]$

(This is a weird definition of "power", but we're just following the example.)

Assumptions:

- $n = 2k+1$ (so it's possible for the election to be tied when I go to vote)
- all the other voters choose from among the two candidates by flipping a coin. (kind of a weird assumption, but again, we're just following along)

my "power" $\approx$?

$E[N] \approx$?

A = "Not counting my vote, the election is tied."

$N = cn\mathbb{1}_A$

X = the # of other people that voted for the first candidate.

$X \sim \text{Binom}(2k, \frac{1}{2})$

$A = \{X = k\}$

$$E[N] = E[cn\,\mathbb{1}_A] = cn\,E[\mathbb{1}_A]$$

$$= cn\,P(A) = cn\,P(X=k)$$

$$= cn \binom{2k}{k} \cancel{\left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{2}\right)^{2k-k}}$$

$$\left(\tfrac{1}{2}\right)^{2k} = \frac{1}{2^{2k}}$$

$$= cn \cdot \frac{(2k)!}{k!\,k!\,2^{2k}}$$

Stirling's approx.

$$\sim cn \cdot \frac{(2k)^{2k+\frac{1}{2}} e^{-2k} \sqrt{2\pi}}{\left(k^{k+\frac{1}{2}} e^{-k} \sqrt{2\pi}\right)^2 2^{2k}}$$

$$= cn \cdot \frac{2^{\cancel{2k}+\frac{1}{2}}\ \cancel{k^{2k+\frac{1}{2}}}\ \cancel{e^{-2k}}\ \cancel{\sqrt{2\pi}}}{k^{\cancel{2k+1}\ \frac{1}{2}}\ \cancel{e^{-2k}} \sqrt{(2\pi)}\ \cancel{2^{2k}}}$$

$$= cn \cdot \frac{\sqrt{2}}{\sqrt{k}\,\sqrt{2\pi}} = \frac{cn}{\sqrt{\pi k}}.$$

But $k \sim \frac{n}{2}$, so...

$$E[N] \sim \frac{cn}{\sqrt{\frac{\pi n}{2}}} = \boxed{c\sqrt{\frac{2}{\pi}}\sqrt{n}}$$

This example is supposed to illustrate that voters in larger states have more "power". But it only works if you accept their definition of "power" and you accept their assumptions.

HW: Ch. 4: 1, 3, 4, 10, 13, 17–19, 20, 23, 27, 30, 31, 32,
35, 39, 42, 44, 48, 50
38, 41, 43, 47, 49