4.1 Random Variables

Informal

A random variable is a specific, well-defined quantity whose exact value we might be uncertain about.

e.g. $\quad \underbrace{X = \text{"The number of water molecules in my body."}}_{\text{a noun phrase denoting a real number}}$

Formal

A random variable is a function $X : \Omega \to \mathbb{R}$.

Expl

Flip a fair coin 2 times.

Y = "the number of heads we flip"

$$\Omega = \{ (H,H), (H,T), (T,H), (T,T) \}$$

Y is the function from Ω to $\mathbb{R}$ given by this table:

ω	$Y(\omega)$
(H,H)	2
(H,T)	1
(T,H)	1
(T,T)	0

$A =$ "We flip at least one head."

$A =$ "$Y \geq 1$."

$A = \{\omega : Y(\omega) \geq 1\}$

$A = \{Y \geq 1\} \longleftarrow$ shorthand notation

$P(A) = P(Y \geq 1)$ drop the curly brackets inside $P(\cdot)$

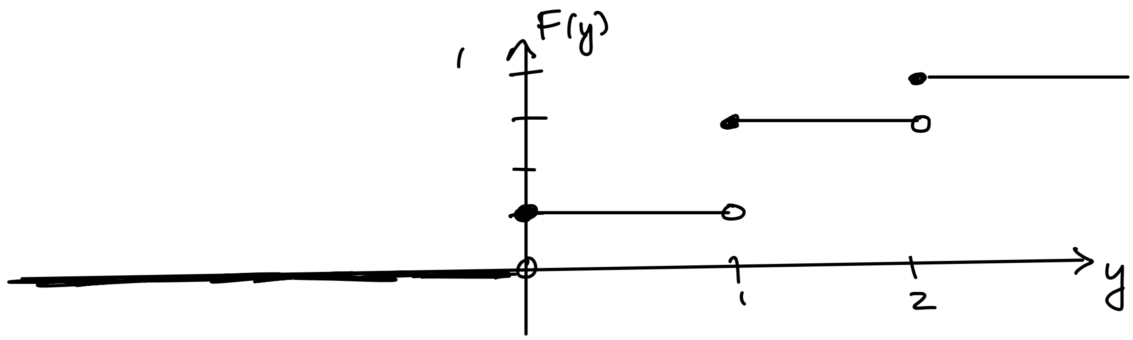
$$P(Y \geq 1) = P(Y=1) + P(Y=2)$$
$$= P(\{(H,T),(T,H)\}) + P(\{H,H\})$$
$$= \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$$

<u>distribution function</u>: $F_X(x) = P(X \leq x)$

In our example,

$$F_Y(y) = P(Y \le y) = \begin{cases} 0 & \text{if } y < 0, \\ P(Y=0) & \text{if } 0 \le y < 1, \\ P(Y \le 1) & \text{if } 1 \le y < 2, \\ 1 & \text{if } y \ge 2 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0, \\ \frac{1}{4} & \text{if } 0 \le y < 1, \\ \frac{3}{4} & \text{if } 1 \le y < 2, \\ 1 & \text{if } y \ge 2 \end{cases}$$



dist. funcs. are nondecreasing

4.2 Discrete Random Variables

A r.v. whose range is <u>countable</u> is a <u>discrete</u> r.v.

can be listed as elements in a sequence

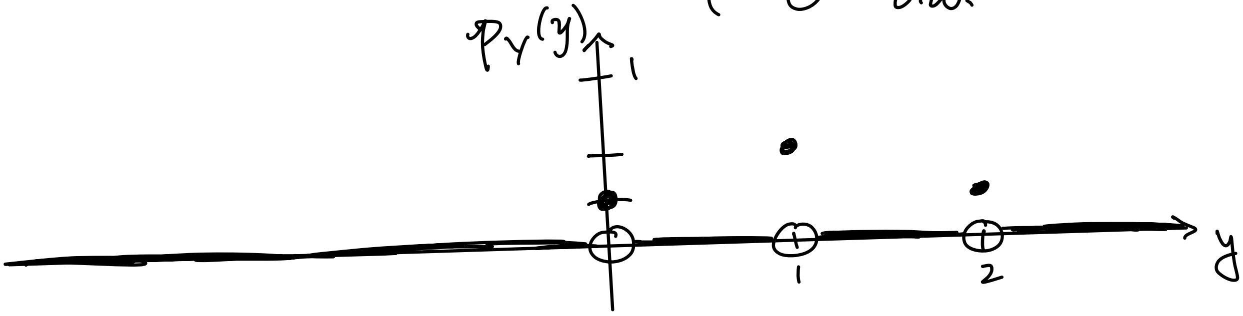
our expl. : $\underbrace{R(Y)}_{\text{range of } Y} = \{0, 1, 2\}$, so Y is discrete

discrete r.v.s have <u>mass functions</u>:

$$p_X(x) = P(X = x)$$

Our expl:

$$p_Y(y) = P(Y=y) = \begin{cases} \frac{1}{4} & \text{if } y=0 \\ \frac{1}{2} & \text{if } y=1 \\ \frac{1}{4} & \text{if } y=2 \\ 0 & \text{o.w.} \end{cases}$$



If X is discrete, then

$$\underbrace{E[X]}_{\text{expected value of } X} = \sum_{x \in R(X)} x P(X=x)$$

(don't get fooled by language. There's nothing "expected" about EV)

Our expl: $$E[Y] = \underbrace{\cancel{0 \cdot P(Y=0)}}_{0} + 1 \cdot \underbrace{\cancel{P(Y=1)}}_{\frac{1}{2}} + 2 \cdot \underbrace{\cancel{P(Y=2)}}_{\frac{1}{4}}$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Expl 4.3a

Roll a fair 6-sided die.

X = "The outcome of the roll."

$E[X] = ?$

$R(X) = \{1, 2, 3, 4, 5, 6\}$

X is discrete

$$E[X] = \sum_{n=1}^{6} n \overset{\frac{1}{6}}{\cancel{P(X=n)}} = \frac{1}{6} \sum_{n=1}^{6} n = \frac{1}{\cancel{6}} \cdot \frac{\cancel{6} \cdot 7}{2} = \boxed{3.5}$$

(don't get fooled by language. There's nothing "expected" about EV)

If A is an event, then

$$\underbrace{1_A}_{\text{indicator r.v.}}(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

This notation is standard in prob. theory, though it differs from Ross.

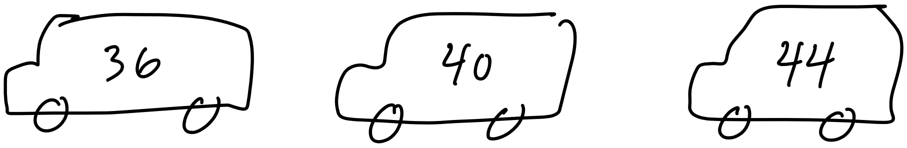
or

$$1_A = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false} \end{cases}$$

$$E[1_A] = 0 \cdot P(1_A = 0) + 1 \cdot P(1_A = 1)$$
$$= 0 \cdot P(A^c) + 1 \cdot P(A)$$

$$\boxed{E[1_A] = P(A)}$$

Example 3d A school class of 120 students is driven in 3 buses to a symphonic performance. There are 36 students in one of the buses, 40 in another, and 44 in the third bus. When the buses arrive, one of the 120 students is randomly chosen. Let X denote the number of students on the bus of that randomly chosen student, and find $E[X]$.



120 total

randomly choose a student

X = "number of students on chosen student's bus"

$E[X] = ?$

$R(X) = \{36, 40, 44\}$

X is discrete

$$E[X] = 36\,P(X=36) + 40\,P(X=40) + 44\,P(X=44)$$

$$= \overset{18}{\cancel{36}} \cdot \frac{\overset{3}{\cancel{36}}}{\underset{5}{\cancel{120}}} + 40 \cdot \frac{\overset{1}{\cancel{40}}}{\underset{3}{\cancel{120}}} + \overset{22}{\cancel{44}} \cdot \frac{\overset{11}{\cancel{44}}}{\underset{15}{\cancel{120}}}$$

$$= \frac{18 \cdot 9}{15} + \frac{200}{15} + \frac{242}{15} = \frac{180 - \cancel{18} + 200 + \overset{224}{\cancel{242}}}{15}$$

$$= \boxed{\frac{604}{15}} \approx 40.27$$