3.5 $P(\cdot | F)$ is a probability

The function $P(\cdot | F)$ satisfies the axioms of probability.

$\therefore$ every theorem that's true for $P(\cdot)$ is true for $P(\cdot | F)$

i.e. we can add conditioning to any formula.

e.g. $P(A) = P(B)P(A|B) + P(B^c)P(A|B^c)$

$$P(A|C) = P(B|C)\,P(A|B\cap C) + P(B^c|C)\,P(A|B^c\cap C)$$

A and B are <u>conditionally indep. given C</u> if

$$P(A\cap B|C) = P(A|C)\,P(B|C),$$

or

$$P(A|B\cap C) = P(A|C)$$
$$P(B|A\cap C) = P(B|C)$$

Extends as you expect to multiple events

Example 4m

The gambler's ruin problem

Two gamblers, A and B, bet on the outcomes of successive flips of a coin. On each flip, if the coin comes up heads, A collects 1 unit from B, whereas if it comes up tails, A pays 1 unit to B. They continue to do this until one of them runs out of money. If it is assumed that the successive flips of the coin are independent and each flip results in a head with probability p, what is the probability that A ends up with all the money if he starts with i units and B starts with $N - i$ units?

(This example is from Section 3.4.)

E = "B goes broke."

E_i = "A starts with \$$i$."

$P(E|E_i) = ?$

$P_i := P(E|E_i)$

$P_N = 1$

$P_0 = 0$

$P_i = ?$ for $1 \leq i \leq N-1$

H = "A wins the 1st flip."

$P(H) = p$

$q := 1-p$

$1 \leq i \leq N-1$

H and E_i are indep.

starting w/ \$$i$ then winning the 1st flip is the same as starting w/ \$$(i+1)$

$$P_i = P(E|E_i) = \overset{p}{\cancel{P(H|E_i)}}\,P(E|\overset{E_{i+1}}{\cancel{H\cap E_i}})$$

$$+\underset{q}{\cancel{P(H^c|E_i)}}\,P(E|\underset{E_{i-1}}{\cancel{H^c\cap E_i}})$$

$p+q=1$

$$\left.\begin{aligned}
P_i &= p P_{i+1} + q P_{i-1} \\
\overset{\downarrow}{(p+q)} P_i &= p P_{i+1} + q P_{i-1} \\
p P_i + q P_i &= p P_{i+1} + q P_{i-1} \\
q(P_i - P_{i-1}) &= p(P_{i+1} - P_i) \\
P_{i+1} - P_i &= \frac{q}{p}(P_i - P_{i-1})
\end{aligned}\right\} \text{trick / clever algebra}$$

$$P_0 = 0$$

$$P_2 - P_1 = \frac{q}{p}(P_1 - \overset{0}{\cancel{P_0}}) = \frac{q}{p} P_1$$

$$P_3 - P_2 = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1$$

$$\vdots$$

$$P_N - P_{N-1} = \left(\frac{q}{p}\right)^{N-1} P_1$$

So... for any k,

$$P_k = P_k - P_0 = \overset{\text{telescoping sum}}{\sum_{j=1}^{k}}(P_j - P_{j-1}) = \sum_{j=1}^{k}\left(\frac{q}{p}\right)^{j-1} P_1$$

$$r := \frac{q}{p}$$

$$P_k = P_1 \sum_{j=1}^{k} r^{j-1}$$

but $P_1 = ?$ (will need to use $P_N = 1$ to figure that out)

Case 1: $p \neq \frac{1}{2}$

In this case, $r \neq 1$, so

$$P_k = P_1 \sum_{j=1}^{k} r^{j-1} = P_1 \cdot \frac{1-r^k}{1-r}$$

partial sum of geom. series

We know that $P_N = 1$, so . . .

$$1 = P_1 \cdot \frac{1-r^N}{1-r} \Rightarrow P_1 = \frac{1-r}{1-r^N}$$

$$\therefore \quad P_k = \frac{1-r}{1-r^N} \cdot \frac{1-r^k}{1-r} = \frac{1-r^k}{1-r^N}$$

Sub in $r = \frac{q}{p}$:

$$P_k = \frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N} \cdot \frac{p^N}{p^N} = \frac{p^N - p^{N-k}q^k}{p^N - q^N}$$

Case 2: $p = \frac{1}{2}$

In this case, $r = 1$, so

$$P_k = P_1 \sum_{j=1}^{k} r^{j-1} = P_1 k$$

We know that $P_N = 1$, so...

$$1 = P_1 N \Rightarrow P_1 = \frac{1}{N}$$

$$\therefore \; P_k = \frac{k}{N}$$

Final answer:

$$\boxed{P_k = P(E \mid E_k) = \begin{cases} \dfrac{p^N - p^{N-k} q^k}{p^N - q^N} & \text{if } p \neq \frac{1}{2}, \\[2ex] \dfrac{k}{N} & \text{if } p = \frac{1}{2}. \end{cases}}$$

Expl 4m continued

Let F = "A goes broke."

What is the relationship between E and F?

It's tempting to think they're negations of one another. But they're not!

$$\xcancel{F = E^c}$$

The reason is because there's a third possibility:

G = "The game goes on forever (i.e. no one ever wins all the money)." ←

E, F, and G form a partition (exactly one of them is true).

$$P(G \mid E_i) = ?$$

$$\underset{\substack{\uparrow \\ \text{we know this} \\ \text{already}}}{P(E \mid E_i)} + \underset{\substack{\uparrow \\ \text{if we can find this, we} \\ \text{can solve for } P(G \mid E_i)}}{P(F \mid E_i)} + P(G \mid E_i) = 1$$

$$P(G \mid E_i) = 1 - \left(P(E \mid E_i) + P(F \mid E_i)\right)$$

Let F_i = "B starts with \$$i$."

By symmetry, $P(F|F_i)$ is the same as $P(E|E_i)$, but with the roles of p and q reversed:

$$P(F|F_i) = \begin{cases} \dfrac{q^N - q^{N-i} p^i}{q^N - p^N} & \text{if } p \neq \frac{1}{2}, \\ \dfrac{i}{N} & \text{if } p = \frac{1}{2}. \end{cases}$$

Also, $F_i = E_{N-i}$. So...

$$P(F|E_i) = P(F|F_{N-i})$$

$$= \begin{cases} \dfrac{q^N - q^i p^{N-i}}{q^N - p^N} & \text{if } p \neq \frac{1}{2}, \\ \dfrac{N-i}{N} & \text{if } p = \frac{1}{2}. \end{cases}$$

Now we can compute $P(G|E_i)$:

Case 1: $p = \frac{1}{2}$

$$P(G|E_i) = 1 - \left(P(E|E_i) + P(F|E_i)\right)$$

$$= 1 - \left(\frac{i}{N} + \frac{N-i}{N}\right)$$

$$= 1 - 1$$

$$= 0$$

Case 2: $p \neq \frac{1}{2}$

$$P(G|E_i) = 1 - \left(P(E|E_i) + P(F|E_i)\right)$$

$$= 1 - \left(\frac{p^N - p^{N-i}q^i}{p^N - q^N} + \frac{q^N - q^i p^{N-i}}{q^N - p^N}\right)$$

$$= 1 - \left(\frac{p^N - \cancel{p^{N-i}q^i} - q^N + \cancel{q^i p^{N-i}}}{p^N - q^N}\right)$$

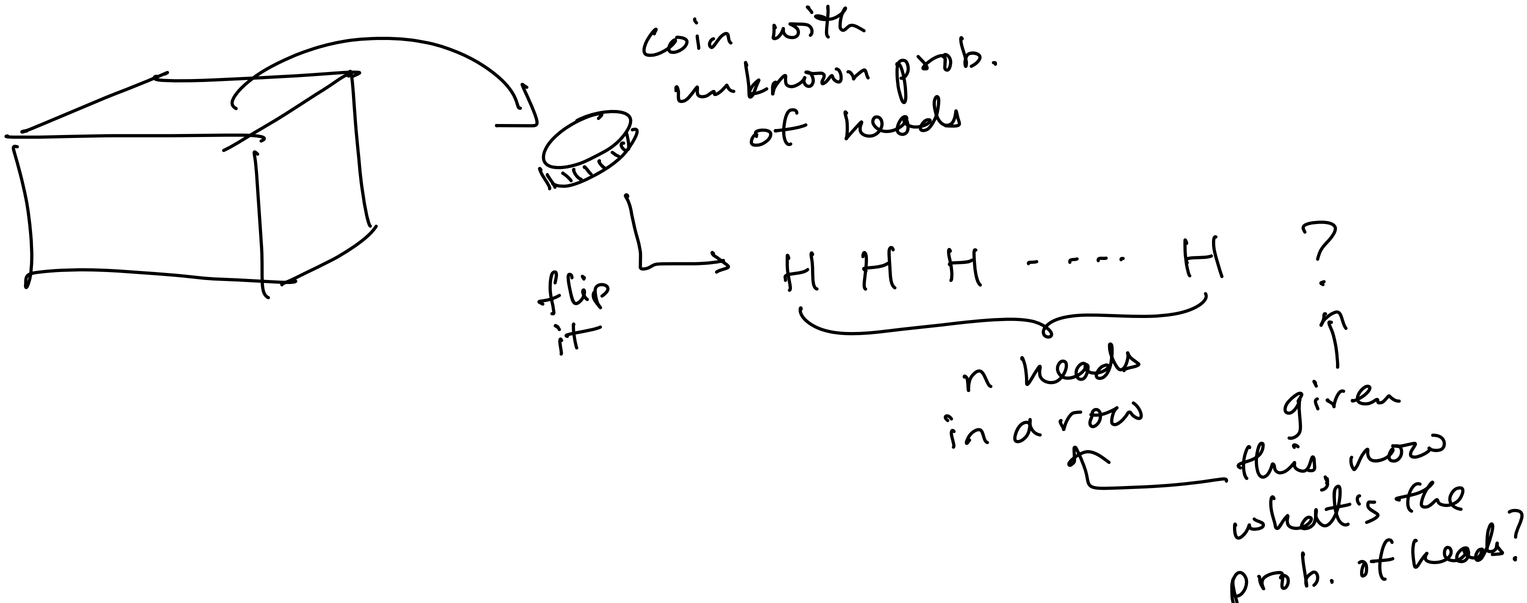
$$= 1 - 1$$

$$= 0.$$

In either case, $\boxed{P(G|E_i) = 0}$

Example 5e **Laplace's rule of succession**

There are $k + 1$ coins in a box. When flipped, the ith coin will turn up heads with probability $i/k, i = 0, 1, \ldots, k$. A coin is randomly selected from the box and is then repeatedly flipped. If the first n flips all result in heads, what is the conditional probability that the $(n + 1)$ flip will do likewise?



C_i = "The i^{th} coin is chosen." $(i = 0, 1, \ldots, k)$

A_j = "The j^{th} flip lands on heads."

$P(C_i) = \frac{1}{k+1}$

$P(A_j | C_i) = \frac{i}{k}$

$A_1, A_2, \ldots$ are conditionally independent, given C_i.

A = "The first n flips land on heads."

$P(A_{n+1} | A) = ?$

$C_0, C_1, \ldots, C_k$ are a partition. So...

$$P(A_{n+1} | A) = \sum_{i=0}^{k} P(C_i | A)\, P(A_{n+1} | C_i \cap A)$$

because A and A_{n+1} are conditionally indep., given C_i

$$= \sum_{i=0}^{k} P(C_i | A)\, P(A_{n+1} | C_i)$$

$$P(C_i | A) = \frac{P(C_i)\, P(A | C_i)}{\sum_{j=0}^{k} P(C_j)\, P(A | C_j)}$$

$$P(A | C_j) = P(A_1 \cap A_2 \cap \cdots \cap A_n | C_j)$$

$$\overset{\text{cond. indep.}}{=} P(A_1 | C_j)\, P(A_2 | C_j) \cdots P(A_n | C_j)$$

$$= \left(\frac{j}{k}\right)^n$$

Plugging back in:

$$P(C_i | A) = \frac{\frac{1}{k+1}\left(\frac{i}{k}\right)^n}{\sum_{j=0}^{k} \frac{1}{k+1}\left(\frac{j}{k}\right)^n}$$

cannot simplify this very much

$$P(A_{n+1}|A) = \sum_{i=0}^{k} P(C_i|A)\, \underset{\frac{i}{k}}{\cancel{P(A_{n+1}|C_i)}}$$

$$= \sum_{i=0}^{k} \left(\frac{\frac{1}{k+1}\left(\frac{i}{k}\right)^n}{\sum_{j=0}^{k} \frac{1}{k+1}\left(\frac{j}{k}\right)^n} \right) \cdot \frac{i}{k}$$

$$= \boxed{\frac{\sum_{i=0}^{k} \frac{1}{k+1}\left(\frac{i}{k}\right)^{n+1}}{\sum_{j=0}^{k} \frac{1}{k+1}\left(\frac{j}{k}\right)^n}}$$

The above sums are Riemann sums.

So if k is large:

$$P(A_{n+1}|A) \approx \frac{\int_0^1 x^{n+1}\,dx}{\int_0^1 x^n\,dx} = \frac{\frac{1}{n+2}\overset{1}{\cancel{x^{n+2}\Big|_0^1}}}{\frac{1}{n+1}\underset{1}{\cancel{x^{n+1}\Big|_0^1}}}$$

$$= \frac{\left(\frac{1}{n+2}\right)}{\left(\frac{1}{n+1}\right)} = \boxed{\frac{n+1}{n+2}}$$

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