2.4 Some simple propositions

Prop. 2.4.2 (monotonicity)

$$E \subseteq F \Rightarrow P(E) \leq P(F)$$

Pf: Assume $E \subseteq F$.

Then $F = E \cup (F \cap E^c)$

$$P(F) = P(\underbrace{E}_{\text{mut. excl.}} \cup \underbrace{(F \cap E^c)}) \underset{\text{add. rule.}}{=} P(E) + P(F \cap E^c)$$

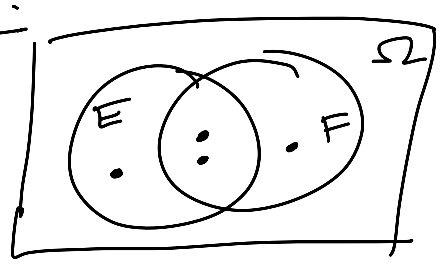
$P(F \cap E^c) \geq 0$ (Axiom 1)

$\therefore P(F) \geq P(E) + 0 = P(E)$. $\square$

Prop 2.4.3 (inclusion-exclusion)

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Idea:



Pf:

$$\left.\begin{aligned} A &= E \cap F^c \\ B &= E \cap F \\ C &= F \cap E^c \end{aligned}\right\} \text{mut. excl.}$$

$E = A \cup B$, $F = B \cup C$, $E \cup F = A \cup B \cup C$

$$\begin{aligned} P(E \cup F) &= P(A \cup B \cup C) = P(A) + P(B) + P(C) \\ &= P(A) + P(B) + P(B) + P(C) - P(B) \\ &= P(A \cup B) + P(B \cup C) - P(B) \\ &= P(E) + P(F) - P(E \cap F). \quad \square \end{aligned}$$

Expl 2.4a

A = "She likes the 1st book."

B = " " 2nd " ."

Given

$P(A) = 0.5$

$P(B) = 0.4$

$P(A \cap B) = 0.3$

Goal

$P(A^c \cap B^c) = ?$

$$A^c \cap B^c = (A \cup B)^c$$

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B)$$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.5 + 0.4 - 0.3 = 0.6 \end{aligned}$$

$$\therefore\ P(A^c \cap B^c) = 1 - 0.6 = \boxed{0.4}$$

Inclusion - Exclusion, general form:

See Prop. 4.4 for the general pattern.

Expl:

$$P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D)$$
$$- P(A \cap B) - P(A \cap C) - P(A \cap D) - P(B \cap C) - P(B \cap D) - P(C \cap D)$$
$$+ P(A \cap B \cap C) + P(A \cap B \cap D) + P(A \cap C \cap D) + P(B \cap C \cap D)$$
$$- P(A \cap B \cap C \cap D)$$

Stopping early gives alternating inequalities.

E.g.

$$P(A \cup B \cup C) \le P(A) + P(B) + P(C)$$

but

$$P(A \cup B \cup C) \ge P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$$

2.5 Spaces w/ equally likely outcomes

Ω = finite set

$$P(A) = \frac{|A|}{|\Omega|}$$

Example 2.5a If two dice are rolled, what is the probability that the sum of the upturned faces will equal 7?

$$\Omega = \{(1,1),(1,2),(1,3),\dots,(1,6),$$
$$(2,1),(2,2),(2,3),\dots,(2,6),$$
$$\vdots$$
$$(6,1),(6,2),(6,3),\dots,(6,6)\}$$

P = equally likely

A = "The sum is 7."

$A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$P(A) = ?$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6}{36} = \boxed{\frac{1}{6}}$$

Visualizing Ω with sums written inside:

die #1 \ die #2	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

e.g. prob. that the sum is 10 is $\frac{3}{36} = \frac{1}{12}$

Example 2.5b If 3 balls are "randomly drawn" from a bowl containing 6 white and 5 black balls, what is the probability that one of the balls is white and the other two black?

SOLN 1

Ω = {all ways of choosing 3 balls, where order doesn't matter}

$\mathbb{P}$ = equally likely

A = "1 ball is white and 2 are black."

$P(A) = ?$

$$P(A) = \frac{|A|}{|\Omega|}$$

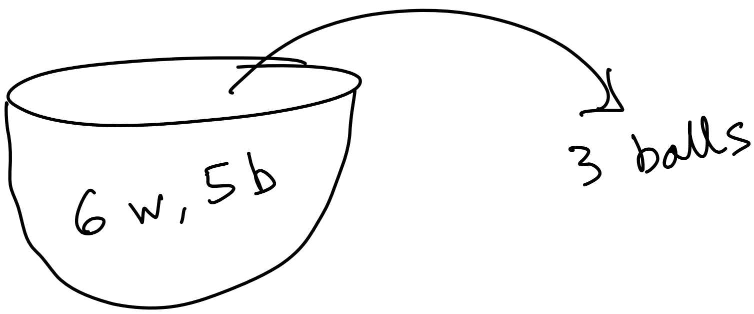
$$|\Omega| = \binom{11}{3}$$ [11 balls, choose 3, use binom. coeff. b/c order doesn't matter]

$$|A| = \binom{6}{1}\binom{5}{2}$$ [1st choose white ball, then choose black balls]

$$P(A) = \frac{\binom{6}{1}\binom{5}{2}}{\binom{11}{3}} = \frac{6 \cdot \frac{5 \cdot 4}{2 \cdot 1}}{\frac{11 \cdot 10 \cdot 9}{3 \cdot 2 \cdot 1}}$$

$$= \frac{\cancel{6}\cdot\cancel{5}\cdot 4}{\cancel{2}\cdot\cancel{1}} \cdot \frac{\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{11\cdot\underset{2}{\cancel{10}}\cdot\underset{3}{\cancel{9}}} = \boxed{\frac{4}{11}}$$

SOLN 2



Ω = {all ways of choosing 3 balls, where order <u>does</u> matter}

$\mathbb{P}$ = equally likely

A = "1 ball is white and 2 are black."

$P(A) = ?$

$$P(A) = \frac{|A|}{|\Omega|}$$

$|\Omega| = 11\cdot 10\cdot 9$ [Use permutation b/c order matters]

$|A| = ?$

mutually exclusive
A_1 = "Balls are picked in order wbb."
A_2 = " " bwb."
A_3 = " " bbw."

$$|A| = |A_1| + |A_2| + |A_3|$$

$$|A_1| = 6 \cdot 5 \cdot 4$$

pick white ball

pick 1st black ball

pick 2nd black ball (only 4 left to choose from at this point)

$$|A_2| = 5 \cdot 6 \cdot 4$$

$$|A_3| = 5 \cdot 4 \cdot 6$$

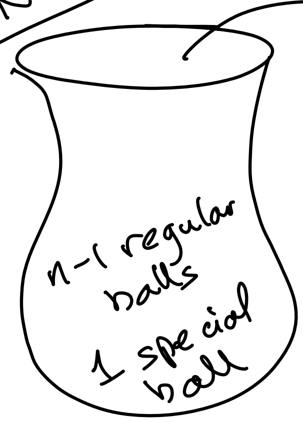
All are the same: $|A_1| = |A_2| = |A_3| = 120$

So $|A| = 120 + 120 + 120 = 360$

$$P(A) = \frac{\cancel{360}^{\,4}}{11 \cdot \cancel{10 \cdot 9}} = \boxed{\frac{4}{11}}$$

Example 2.5d An urn contains n balls, one of which is special. If k of these balls are withdrawn one at a time, with each selection being equally likely to be any of the balls that remain at the time, what is the probability that the special ball is chosen?

SOLN 1



k balls

$\Omega = \{$all ways of choosing k balls, order doesn't matter$\}$

P = equally likely

A = "The special ball is chosen."

$P(A) = ?$

$$|\Omega| = \binom{n}{k}$$

$$|A| = 1 \cdot \binom{n-1}{k-1}$$

choose the special ball

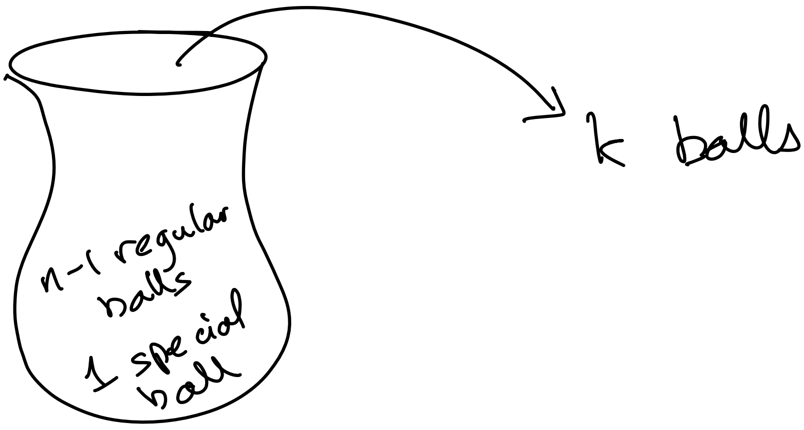
choose $k-1$ balls from among the regular balls

$$P(A) = \frac{|A|}{|\Omega|} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\left(\frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!}\right)}{\left(\frac{n!}{k!\,(n-k)!}\right)}$$

$$= \frac{(n-1)!}{(k-1)!\,\cancel{(n-k)!}} \cdot \frac{k!\,\cancel{(n-k)!}}{n!}$$

$$= \frac{(n-1)!}{n!} \cdot \frac{k!}{(k-1)!} = \frac{1}{n} \cdot \frac{k}{1} = \boxed{\frac{k}{n}}$$

SOLN 2



$A =$ "The special ball is chosen."

$P(A) = ?$

Number the balls arbitrarily from 1 to n.

$A_{ij} =$ "Ball $\#i$ is the j^{th} one chosen."

These are mutually exclusive.

Some ball must be the j^{th} one chosen, so

$$\sum_{i=1}^{n} P(A_{ij}) = 1$$

By the arbitrariness of the numbering, $P(A_{ij})$ doesn't actually depend on i.

So $P(A_{ij}) = \frac{1}{n}$ for all i and j.

Let i_0 be the number of the special ball. Then

$$A = \bigcup_{j=1}^{k} A_{i_0 j}.$$

Again, b/c these are mutually exclusive,

$$P(A) = \sum_{j=1}^{k} P(A_{i_0 j}) = \sum_{j=1}^{k} \frac{1}{n} = \boxed{\frac{k}{n}}.$$

Example 2.5f A poker hand consists of 5 cards. If the cards have distinct consecutive values and are not all of the same suit, we say that the hand is a straight. For instance, a hand consisting of the five of spades, six of spades, seven of spades, eight of spades, and nine of hearts is a straight. What is the probability that one is dealt a straight?

Ω = {all ways to choose 5 cards from the 52-card deck}

P = equally likely

A = "A straight was chosen."

$P(A) = ?$

$$P(A) = \frac{|A|}{|\Omega|}, \quad |\Omega| = \binom{52}{5}$$

B = "5 consecutive cards were chosen."

C = "A straight flush was chosen."

$A = B \setminus C$ and $C \subseteq B$,

so $|A| = |B| - |C|$

$$|B| = \underbrace{10}_{\text{choose what rank to start on (ace-10)}} \cdot \underbrace{4}_{\text{choose the 1st card}} \cdot \underbrace{4}_{\text{choose the 2nd}} \cdot 4 \cdot 4 \cdot 4$$

choose what rank to start on (ace-10)

choose the 1st card

choose the 2nd ... etc.

$$|B| = 10 \cdot 4^5$$

$$|C| = 10 \cdot 4 \cdot 1$$

choose the rank to start on

choose the suit

take the determined cards

$$|C| = 10 \cdot 4$$

$$|A| = 10 \cdot 4^5 - 10 \cdot 4 = 10(4^5 - 4)$$

$$P(A) = \boxed{\frac{10(4^5-4)}{\binom{52}{5}}} \approx 0.0039$$

(not reasonable to simplify by hand)

Example 2.5h In the game of bridge, the entire deck of 52 cards is dealt out to 4 players. What is the probability that

(a) one of the players receives all 13 spades;
(b) each player receives 1 ace?

(a) $\Omega = \{$all ways of dealing 52 cards, 13 each, to North, South, East, and West$\}$

P = equally likely

A = "One player receives all spades."

$P(A) = ?$

$$P(A) = \frac{|A|}{|\Omega|}, \quad |\Omega| = \binom{52}{13,13,13,13} = \frac{52!}{13!\,13!\,13!\,13!}$$

$$|A| = \underset{\substack{\uparrow \\ \text{choose the player to get the spades}}}{4} \cdot \underset{\substack{\uparrow \\ \text{give them the spades}}}{1} \cdot \underset{\substack{\uparrow \\ \text{deal the remaining cards to the remaining players}}}{\binom{39}{13,13,13}}$$

$$|A| = 4 \cdot \frac{39!}{13!\,13!\,13!}$$

$$P(A) = \frac{4(39!)}{\cancel{(13!)^3}} \cdot \frac{(13!)^{\cancel{4}}}{(52!)}$$

$$= \boxed{4 \cdot \frac{39!\,13!}{52!}} \approx 6.3 \times 10^{-12}$$

(b) $\Omega = \{$all ways of dealing 52 cards, 13 each, to North, South, East, and West$\}$

P = equally likely

A = "Each player receives 1 ace."

$P(A) = ?$

$$P(A) = \frac{|A|}{|\Omega|}, \quad |\Omega| = \binom{52}{13,13,13,13} = \frac{52!}{(13!)^4}$$

$$|A| = \underbrace{4!}_{\text{deal the aces}} \cdot \underbrace{\binom{48}{12,12,12,12}}_{\text{deal the remaining cards}}$$

$$|A| = \frac{4!\,48!}{(12!)^4}$$

$$P(A) = \frac{4!\,\cancel{48!}}{(\cancel{12!})^4} \cdot \frac{(13\cancel{!})^4}{\underset{52\cdot 51\cdot 50\cdot 49}{\cancel{52!}}}$$

$$= \frac{3!\,\cancel{4!}\,13^{\cancel{4}\,3}}{\cancel{52}\cdot 51\cdot 50\cdot 49}$$

$$= \frac{\cancel{3}\cdot \cancel{2}\cdot 13^3}{\underset{17}{\cancel{51}}\cdot \underset{25}{\cancel{50}}\cdot 49}$$

$$= \boxed{\frac{13^3}{17\cdot 25\cdot 49}} \approx 0.1055$$

Example 2.5k

A football team consists of 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determining roommates. If the pairing is done at random, what is the probability that there are no offensive–defensive roommate pairs? ~~What is the probability that there are 2i offensive–defensive roommate pairs, i = 1, 2, ..., 10?~~

Ω = {all ways of pairing of the 40 players}

P = equally likely

A = "There are no off-def pairs."

P(A) = ?

$$P(A) = \frac{|A|}{|\Omega|}, \qquad |\Omega| = ?$$

If the pairs were distinguishable, there would be $\binom{40}{\underbrace{2,2,\ldots 2}_{20 \text{ copies}}} = \frac{40!}{(2!)^{20}}$ ways.

But these 20 pairs are indistinguishable, so divide by the # of permutations of the pairs, $20!$

(See Examples 5b and 5c in Section 1.5)

$$\text{So } |\Omega| = \frac{\left(\frac{40!}{(2!)^{20}}\right)}{20!} = \frac{40!}{20!\,2^{20}}$$

$$|A| = \underbrace{\left(\frac{20!}{10!\,2^{10}}\right)}_{\substack{\text{break the 20 offensive players into}\\ \text{10 indistinguishable groups}}} \cdot \underbrace{\left(\frac{20!}{10!\,2^{10}}\right)}_{\substack{\text{do the same for}\\ \text{the defensive players}}}$$

$$= \left(\frac{20!}{10!\,2^{10}}\right)^2$$

$$P(A) = \frac{\left(\frac{20!}{10!\,2^{10}}\right)^2}{\left(\frac{40!}{20!\,2^{20}}\right)} = \frac{(20!)^2}{(10!)^2\,\cancel{2^{20}}} \cdot \frac{20!\,\cancel{2^{20}}}{40!}$$

$$= \boxed{\frac{(20!)^3}{(10!)^2\,40!}} \approx 1.3403 \times 10^{-6}$$

(The birthday "paradox")

Example 2.5i If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $\frac{1}{2}$?

$\Omega = \{$all ways to assign birthdays to the n people.$\}$

P = equally likely

A = "No birthdays are the same."

$P(A) = ?$

$$P(A) = \frac{|A|}{|\Omega|}, \qquad |\Omega| = 365^n$$

$$|A| = \underbrace{365 \cdot 364 \cdot 363 \cdot \cdots}_{n \text{ factors}}$$

$$= 365 \cdot 364 \cdot 363 \cdot \cdots \cdot (365 - (n-1))$$

$$= \frac{365!}{(365-n)!}$$

$$P(A) = \frac{365!}{(365-n)!\, 365^n}$$

Easier to simplify using this informal notation

$$P(A) = \frac{365 \cdot 364 \cdot 363 \cdot \cdots \cdot (365-(n-1))}{365^n}$$

n factors in numerator

$$= \frac{365 \cdot 364 \cdot 363 \cdot \cdots \cdot (365-(n-1))}{365 \cdot 365 \cdot 365 \cdot \cdots \cdot 365}$$

$$= \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \cdots \cdot \frac{365-(n-1)}{365}$$

$$= 1 \cdot \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \cdot \cdots \cdot \left(1 - \frac{n-1}{365}\right)$$

$$= \boxed{\prod_{j=1}^{n-1} \left(1 - \frac{j}{365}\right)}$$

This probability decreases as n increases.

Start calculating and you'll find:

for $n=22$, $P(A) \approx 0.5243$

for $n=23$, $P(A) \approx 0.4927$

With as few as 23 people, they're more likely than not to have a shared birthday.

Example 2.5j A deck of 52 playing cards is shuffled, and the cards are turned up one at a time until the first ace appears. Is the next card—that is, the card following the first ace—more likely to be the ace of spades or the two of clubs?

Ω = {all ways of shuffling the deck}

P = equally likely

A = "Ace of spades is immediately after the 1st ace."

T = "Two of clubs " "

Is $P(A) < P(T)$, $P(A) = P(T)$, or $P(A) > P(T)$?

$$P(A) = \frac{|A|}{|\Omega|}, \quad |\Omega| = 52!$$

$|A| = 51! \cdot 1$

arrange all but the ace of spades in some order

find the 1st ace in that arrangement and place the ace of spades immediately after it.

$|A| = 51!$, $P(A) = \frac{51!}{52!} = \frac{1}{52}$

$P(T) = \frac{|T|}{|S|}$

$|T| = 51! \cdot 1$

arrange all but the two of clubs in some order

find the 1st ace in that arrangement and place the two of clubs immediately after it.

$|T| = 51!$, $P(T) = \frac{51!}{52!} = \frac{1}{52}$

So $\boxed{P(A) = P(T)}$

HW: Ch. 2: 14, 20, 25, 33, 39, 45, 49, 56