Ch.1 Combinatorics

§§1.1–1.3 Permutations

Expl 1.2b

committee 3F, 4S, 5J, 2S

subcomm. 1 of each

How many possible subcommittees

$$\underbrace{\underset{\substack{\text{choose} \\ \text{a F}}}{3} \cdot \underset{\substack{\text{choose} \\ \text{a S}}}{4} \cdot \underset{\substack{\text{choose} \\ \text{a J}}}{5} \cdot \underset{\substack{\text{choose} \\ \text{a S}}}{2}}_{\text{basic principle of counting}} = 120$$

Expl 1.3a

Baseball team of 9 players.
How many batting orders?

$$\underset{\substack{\text{choose} \\ \text{1st batter}}}{9} \cdot \underset{\substack{\text{choose} \\ \text{2nd}}}{8} \cdot \underset{\text{etc.}}{7} \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot \underline{1}$$

$= 362880 \leftarrow$ # of ways to "permute" the players

i.e. # of "permutations"

Defn $0! = 1$,

$n! = n \cdot (n-1)!$ for $n \geq 1$

↑ *factorial*

$$n! = \underbrace{n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}_{\#\text{ of permutations of } n \text{ objects}}$$

Expl 1.3d

How many arrangements of the letters in PEPPER are possible?

Permutations require distinct objects

$P_1 E_1 P_2 P_3 E_2 R \longleftarrow$ 6! permutations of these. But that's too many

e.g. $\left.\begin{array}{l} P_1 P_2 E_1 P_3 E_2 R \\ P_3 P_1 E_2 P_2 E_1 R \end{array}\right\}$ both are just PPEPER

How many of the 6! correspond to PPEPER?

Answer: $\underbrace{3!}_{\text{\# of ways to order the P's}}\ \underbrace{2!}_{\text{\# of ways to order the E's}}$

Same is true for ERPEPP or any other arrangement.

So final answer is

$$\frac{6!}{3!2!} = \frac{\cancel{6}\cdot 5\cdot 4\cdot 3\cdot \cancel{2}\cdot \cancel{1}}{\cancel{3\cdot 2}\cdot \cancel{1}\cdot \cancel{2}\cdot \cancel{1}} = 5\cdot 4\cdot 3 = \boxed{60}$$

__Expl 1.3f__

$\boxed{4W,\ 3R,\ 2B}$ $\longrightarrow$ put all 9 in a line

How many arrangements?

Same reasoning.

$$\frac{9!}{4!3!2!} = \frac{9\cdot \overset{4}{\cancel{8}}\cdot 7\cdot \cancel{6}\cdot 5\cdot \cancel{4\cdot 3\cdot 2}}{\cancel{4\cdot 3\cdot 2}\cdot \cancel{3\cdot 2}\cdot \cancel{2}}$$

$$= 9\cdot 7\cdot 5\cdot 4 = 63\cdot 20 = \boxed{1260}$$

__Expl__

How many 4 digit #'s are there with distinct digits and no 0's?

$$9 \cdot 8 \cdot 7 \cdot 6 = \boxed{3024}$$

choose 1st digit, choose 2nd, choose 3rd, choose 4th

$$\frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)\cancel{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}}{\cancel{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}}$$

$$= \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{\#\text{ of factors }:\ n-(n-r)=r}$$

# of ways to choose r distinct objects from n possible objects, if order matters.

1.4 Combinations

Expl 1.4a

20 people $\longrightarrow$ committee of 3

How many committees?

$$20 \cdot 19 \cdot 18 \longleftarrow \text{too many.}$$

order chosen doesn't matter

choose 1st member, choose 2nd, choose 3rd e.g. ABC, ACB, BAC, BCA, CAB, CBA

are all the same committee

Need to divide by 6, which is 3!

\# of ways to permute the choices

$$\text{Final answer: } \frac{\overset{10}{\cancel{20}} \cdot 19 \cdot \overset{6}{\cancel{18}}}{\cancel{3} \cdot \cancel{2} \cdot 1} = 19 \cdot 60 = 20 \cdot 60 - 60$$

$$= 1200 - 60 = \boxed{1140}$$

Or...

$$\frac{\left(\frac{20!}{(20-3)!}\right)}{3!}$$

$\left(\frac{20!}{(20-3)!}\right)$ ← # of ways to choose 3 where order matters

$3!$ ← # of ways to order the 3

$$= \frac{20!}{(20-3)!\,3!}$$

← # of ways to choose 3 where order doesn't matter

$$\underbrace{\binom{n}{r}}_{\text{"n choose r"}} = \underbrace{\frac{n!}{(n-r)!\,r!}}_{}$$

← n, r integers, $0 \le r \le n$

\# of ways to choose r objects from n objects, where order doesn't matter

If n, r are integers and $r < 0$ or $r > n$, then $\binom{n}{r} = 0$. } convention

- $\binom{n}{0} = \frac{n!}{(n-0)!0!} = \frac{n!}{n! \cdot 1} = 1$

- $\binom{n}{1} = \frac{n!}{(n-1)!1!} = \frac{n \cdot \cancel{(n-1)!}}{\cancel{(n-1)!}} = n$

- $\binom{n}{n-r} = \frac{n!}{(\underbrace{n-\cancel{(n-r)}}_{r})!(n-r)!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$

e.g. $\binom{9}{2} = \binom{9}{7}$, also $\binom{n}{n} = 1$, $\binom{n}{n-1} = n$

Binomial Theorem

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

$\binom{n}{r}$ also called a "binomial coefficient"

Expl 1.4d

$$(x+y)^3 = \sum_{r=0}^{3} \binom{3}{r} x^r y^{3-r}$$

$$= \underbrace{\binom{3}{0}}_{1} x^0 y^3 + \underbrace{\binom{3}{1}}_{3} x^1 y^2 + \underbrace{\binom{3}{2}}_{3} x^2 y^1 + \underbrace{\binom{3}{3}}_{1} x^3 y^0$$

$$= y^3 + 3xy^2 + 3x^2y + x^3$$

1.5 Multinomial coefficients

Expl 1.5a

10 officers → 5 streets, 2 station, 3 reserve

How many assignments?

$$\binom{10}{5}\binom{5}{2}\binom{3}{3} = \frac{10!}{\cancel{5!}\,5!}\cdot\frac{\cancel{5!}}{\cancel{3!}\,2!}\cdot\frac{\cancel{3!}}{1!\,3!}$$

choose streets — from the remaining, choose station — rest go on reserve

$$= \frac{10!}{5!\,2!\,3!}$$

OR

$$\binom{10}{2}\binom{8}{3}\binom{5}{5} = \frac{10!}{\cancel{8!}\,2!}\cdot\frac{\cancel{8!}}{5!\,3!}\cdot\frac{\cancel{5!}}{1!\,5!} = \frac{10!}{2!\,3!\,5!}$$

same

Notation: $\underbrace{\binom{10}{5,3,2}}_{\substack{\#\text{ of ways of breaking 10 objects}\\ \text{into 3 distinct groups (group of 5, 3, and 2)}}} = \dfrac{10!}{5!\,3!\,2!}$

more on this later

$$\frac{10!}{5!\,3!\,2!} = \frac{10\cdot 9\cdot \overset{4}{\cancel{8}}\cdot 7\cdot \cancel{6}}{\cancel{3\cdot 2}\cdot \cancel{2}} = 10\cdot 63\cdot 4$$

$$= 10\cdot 252 = \boxed{2520}$$

multinomial coefficient

In general,

$$\binom{n}{\underbrace{n_1, n_2, \ldots, n_r}_{\text{must add up to } n}} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

Also,

$$\binom{n}{n_1, n_2} = \binom{n}{n_1} = \binom{n}{n_2}$$

because $n_1 + n_2 = n \Rightarrow n_1 = n - n_2$ and $n_2 = n - n_1$

Expl 1.5b

$\boxed{\text{10 kids}}$ → Team A (5 kids)

→ Team B (5 kids)

How many assignments

Teams A and B are meant to be distinct.

e.g. A: red jerseys
B: blue jerseys

e.g. oldest in red, youngest in blue $\neq$ oldest in blue, youngest in red

Because distinct, can use mult. coeff.

$$\binom{10}{5,5} = \frac{10!}{5!\,5!} = \frac{\cancel{10}\cdot \overset{3}{\cancel{9}}\cdot \overset{2}{\cancel{8}}\cdot 7\cdot 6}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}$$

$$= 42\cdot 6 = \boxed{252}$$

Expl 1.5c

$\boxed{\text{10 kids}}$ → break into 2 teams of 5

How many way?

Teams not distinct. Only one assignment w/ oldest on a team

$$252 = \boxed{?}\cdot 2!$$

252: ways w/ colored jerseys

?: ways w/o

2!: # of ways to assign jerseys

Answer: $\frac{252}{2!} = \boxed{126}$

HW: Ch. 1: 5, 8, 10, 15, 17, 22, 23, 27, 29, 30, 33

13, 15, 20, 21, 24, 26, 27, 30

9th ed. numbering