§§ 2.1–2.3 : Axioms of probability

Intuitive axioms of probability

From Expl 3.3d:

alpha

Blood Test for disease α:

gives true pos. to 95% of sick people

gives false pos. to 1% of healthy people

0.5% of pop. has disease α.

Random person β takes the test, gets pos. result.

beta

What's the probability β has disease α?

We assign probabities to propositions (declarative sentences with a truth value). E.g.

keep on board

E = "Person β is a random member of the population."

F = "Person β takes the test."

G = "Person β gets a pos. result."

H = "Person β has disease α."

and

We are asked to compute $\underbrace{P(H \mid E \,\&\, F \,\&\, G)}_{\text{probability of } H \text{ given } E \,\&\, F \,\&\, G}$

We are given:

$P(G \mid F \,\&\, H) = 0.95$

$P(G \mid F \,\&\, \neg H) = 0.01$

$\uparrow$ not

$P(H \mid E) = 0.005$

To go from the givens to the desired quantity, we use axioms.

<u>Axiom I</u>: all probabilities are between 0 and 1

<u>Axiom II</u>: if $B \Rightarrow A$, then $P(A \mid B) = 1$.

$\uparrow$ logically implies

<u>Axiom III</u>: if $C \Rightarrow \neg(A \,\&\, B)$, then

$$P(A \text{ or } B \mid C) = P(A \mid C) + P(B \mid C).$$ ← addition rule

<u>Axiom IV</u>: if $P(A \mid C) > 0$, then

$$P(A \,\&\, B \mid C) = P(A \mid C)\, P(B \mid A \,\&\, C)$$ ← multiplication rule

<u>Axiom V</u>: if $A_1 \Rightarrow A_2 \Rightarrow A_3 \Rightarrow \cdots$, then

$$P(A_1 \text{ or } A_2 \text{ or } A_3 \text{ or } \cdots) = \lim_{n\to\infty} P(A_n)$$ ← continuity rule

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To solve this problem, we'll also need one thm.

<u>Thm VI</u> $P(\neg A \mid C) = 1 - P(A \mid C)$.

Pf: $A \& \neg A$ is always false, so $C \Rightarrow \neg(A \& \neg A)$.

$\therefore$ by add. rule (Axiom III),

$$P(A \text{ or } \neg A \mid C) = P(A \mid C) + P(\neg A \mid C).$$

But A or $\neg A$ is always true, so $C \Rightarrow A$ or $\neg A$. By Axiom II, $P(A \text{ or } \neg A \mid C) = 1$. $\therefore$

$$1 = P(A \mid C) + P(\neg A \mid C)$$

$$\Rightarrow P(\neg A \mid C) = 1 - P(A \mid C). \square$$

In all the computations we will consider, $\underbrace{E \& F}_{\text{background information}}$ will always be true.

Rename bg info: $\Omega = E \& F$.

Rewrite problem:

<u>Given</u>

$$P(G \mid \Omega \& H) = 0.95 = \frac{19}{20}$$

$$P(G \mid \Omega \& \neg H) = 0.01 = \frac{1}{100}$$

$$P(H \mid \Omega) = 0.005 = \frac{1}{200}$$

<u>Goal</u>

$$P(H \mid \Omega \& G) = ?$$

<u>Soln</u>

mult. rule

$$\underbrace{\cancel{P(H \mid \Omega)}}_{\frac{1}{200}} \underbrace{\cancel{P(G \mid \Omega \& H)}}_{\frac{19}{20}} = P(G \& H \mid \Omega) = P(G \mid \Omega) P(H \mid \Omega \& G)$$

$$P(H \mid \Omega \& G) = \frac{19}{4000} \cdot \frac{1}{P(G \mid \Omega)}$$

$$P(G|\Omega) = P\big(\overbrace{(G\&H)}^{\text{these sentences cannot both be true}} \text{ or } \overbrace{(G\&\neg H)}\,|\,\Omega\big)$$

these sentences are logically the same

$$\overset{\text{add. rule}}{=} P(G\&H|\Omega) + P(G\&\neg H|\Omega)$$

$$\overset{\text{mult. rule}}{=} \underbrace{\cancel{P(H|\Omega)}}_{\frac{1}{200}}\underbrace{\cancel{P(G|\Omega\&H)}}_{\frac{19}{20}} + P(\neg H|\Omega)\underbrace{\cancel{P(G|\Omega\&\neg H)}}_{\frac{1}{100}}$$

$$\overset{\text{Thm VI}}{=} \frac{19}{4000} + \frac{1}{100}\big(1 - \underbrace{\cancel{P(H|\Omega)}}_{\frac{1}{200}}\big)$$

$$= \frac{19}{4000} + \frac{1}{100}\cdot\frac{199}{200} = \frac{19}{4000} + \frac{199}{20000}$$

$$= \frac{95+199}{20000} = \frac{294}{20000} = \frac{147}{10000}$$

$$P(H|\Omega\&G) = \frac{19}{\cancel{4000}_{2}} \cdot \frac{\cancel{10000}^{5}}{147} = \boxed{\frac{95}{294}} \approx 0.3231$$

Ω is always given, so drop it from the notation:

$$P(A) = P(A|\Omega) \quad , \quad P(A|B) = P(A|\Omega\&B)$$

Problem looks simpler then:

<u>Given:</u>

$$P(G|H) = \frac{19}{20}$$

$$P(G|\neg H) = \frac{1}{100}$$

$$P(H) = \frac{1}{200}$$

<u>Goal:</u>

$$P(H|G) = ?$$

$$\underbrace{\cancel{P(H)}}_{\frac{1}{200}}\underbrace{\cancel{P(G|H)}}_{\frac{19}{20}} = P(G\&H) = P(G)P(H|G)$$

$$P(H|G) = \frac{19}{4000}\cdot\frac{1}{P(G)}$$

$$P(G) = P((G\&H) \text{ or } (G\&\neg H)) = P(G\&H) + P(G\&\neg H)$$

$$= \underbrace{\cancel{P(H)}}_{\frac{1}{200}}\underbrace{\cancel{P(G|H)}}_{\frac{19}{20}} + \underbrace{\cancel{P(\neg H)}}_{\frac{199}{200}}\underbrace{\cancel{P(G|\neg H)}}_{\frac{1}{100}} = \cdots = \frac{147}{10000}$$

$$P(H|G) = \frac{19}{4000}\cdot\frac{10000}{147} = \cdots = \boxed{\frac{95}{294}}$$

Two ingredients in a probability model:

Ω: background information
propositions that are always true in the model

P: probability function
takes two propositions, A and B,
and returns $P(A|\Omega \& B)$

or $P(A|B)$ in shorthand

To know P, it's enough to know $P(A)$ for all A.

mult. rule : $P(A \& B) = P(B)\,P(A|B)$

$$\Rightarrow P(A|B) = \frac{P(A \& B)}{P(B)}$$

Formal axioms of probability

All truth assignments to E, F, G, H that make $E \& F$ true

E	F	G	H		
T	T	T	T	$\leftarrow \omega_1$	outcomes
T	T	T	F	$\leftarrow \omega_2$	
T	T	F	T	$\leftarrow \omega_3$	
T	T	F	F	$\leftarrow \omega_4$	

ω_1: outcome where person has the disease & gets. pos. result

ω_2: " is healthy " " "

ω_3: " has the disease " neg. "

ω_4: " is healthy " " "

Propositions become sets:

$G = \{\omega_1, \omega_2\}$ $\leftarrow$ outcomes where G is true

$H = \{\omega_1, \omega_3\}$ $\leftarrow$ " H "

$\neg H = \{\omega_2, \omega_4\}$ $\leftarrow$ " $\neg H$ "

$G \text{ or } H = \{\omega_1, \omega_2, \omega_3\}$ $\leftarrow$ " G or H "

$G \,\&\, H = \{\omega_1\}$ $\leftarrow$ " $G \,\&\, H$ "

In general,

$\neg A = A^c$, $A \text{ or } B = A \cup B$, $A \,\&\, B = A \cap B$

Also, $A \Rightarrow B$ means $A \subseteq B$. ($\omega \in A$ means A is true under outcome ω)

Finally,

$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ $\leftarrow$ outcomes where $\underbrace{E \,\&\, F}_{\text{bg info.}}$ is true

the set of all outcomes

aka (<u>sample space</u>)

A set of outcomes is called an <u>event</u>

The "description" of an event is the proposition that corresponds to it.

<u>Axiom 1</u> $0 \leq P(\underset{\text{an event}}{A}) \leq 1$

<u>Axiom 2</u> $P(\Omega) = 1$

<u>Axiom 3</u> If $A_1, A_2, \ldots$ are <u>mutually exclusive</u> (means that $A_i \cap A_j = \emptyset$ whenever $i \neq j$), then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$ ← gives add. rule and cont. rule

<u>Definition</u> (of conditional probability) If $P(B) > 0$, then $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ ← gives mult. rule

(Axioms in §2.3, Defn in §3.2)

<u>Prop. 2.4.1</u> $P(A^c) = 1 - P(A)$.

<u>Pf:</u> Axiom 2

$$1 = P(\Omega) = P(\underbrace{A \cup A^c}_{=\Omega}) = P(A) + P(A^c)$$

mut. excl.   Axiom 3

$\Rightarrow P(A^c) = 1 - P(A)$. $\square$

Here's our example in the formal axioms:

<u>Given:</u>

$P(G|H) = \frac{19}{20}$

$P(G|H^c) = \frac{1}{100}$

$P(H) = \frac{1}{200}$

<u>Goal:</u>

$P(H|G) = ?$

$$\underset{\frac{1}{200}}{\cancel{P(H)}}\,\underset{\frac{19}{20}}{\cancel{P(G\mid H)}} \overset{\text{defn of CP}}{=} P(G\cap H) \overset{\text{defn of CP}}{=} P(G)\,P(H\mid G)$$

$$P(H\mid G) = \frac{19}{4000}\cdot\frac{1}{P(G)}$$

$$P(G) = P\big(\underbrace{(G\cap H)\cup(G\cap H^c)}_{\text{mut. excl.}}\big) \underset{\text{Axiom 3}}{=} P(G\cap H) + P(G\cap H^c)$$

$$\overset{\text{Defn. of CP}}{=} \underset{\frac{1}{200}}{\cancel{P(H)}}\,\underset{\frac{19}{20}}{\cancel{P(G\mid H)}} + P(H^c)\,\underset{\frac{1}{100}}{\cancel{P(G\mid H^c)}}$$

$$\overset{\text{Prop. 2.4.1}}{=} \frac{19}{4000} + \frac{1}{100}\Big(1 - \underset{\frac{1}{200}}{\cancel{P(H)}}\Big) = \cdots = \frac{147}{10000}$$

$$P(H\mid G) = \frac{19}{4000}\cdot\frac{10000}{147} = \cdots = \boxed{\frac{95}{294}}$$

<u>HW</u> Ch.2: 3, 6, 8 (highlights are focus problems)