Variations of Stochastic Processes: Alternative Approaches

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Abstract

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The work in this dissertation consists of three main parts. The first part generalizes some of the results of Grannan and Swindle [14] regarding the scaled limit of transaction costs by investigating the scaled limit of the p-th variation of a Brownian martingale, then applying this to a portfolio process with $p = 1$.

In the second part, it is shown that the scaled median of n independent, standard, one-dimensional Brownian motions converges weakly as n goes to infinity to a continuous, centered Gaussian process. An explicit formula for the covariance of the limiting process is derived.

Finally, in the third part, the solution to a certain stochastic heat equation is considered. This solution is a random function of time and space. For a fixed point in space, the resulting random function of time (call it F_t) has the same local behavior as a fractional Brownian motion with Hurst parameter $H = 1/4$. The process F_t, therefore, has infinite quadratic variation and, hence, is not a semimartingale. It follows, then, that the classical Ito calculus does not apply to F_t. Heuristic ideas about a possible new calculus for this process lead, in a natural way, to the introduction and study of the "signed" quadratic variation of F_t. This signed variation, as a process of t, is shown to converge weakly to a Brownian motion.

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DEDICATION

To my wife, Sona, for reminding me, when I needed it, of what is truly important in life, and for sharing with me all of her patience, tolerance, advice, encouragement, and love.

Chapter 1

INTRODUCTION

The work in this dissertation consists of three main parts. Each of these can stand alone, yet they are all related in that they all stem from one original question. In this introduction, I would like to describe this question and explain how the three projects arose out of its consideration.

Consider a stock market that consists of N independent, identically distributed stocks

$$dX_t^{(i)} = \sigma X_t^{(i)} dW_t^{(i)}, \;\; X_0^{(i)} = 1$$

(where $1 \leq i \leq N$, $\sigma > 0$, and $W_t^{(i)}$ are independent, standard one-dimensional Brownian motions) and a unit bond

$$X_t^{(0)} \equiv 1.$$

Let M_t denote the median of $X_t^{(1)}, \dots, X_t^{(N)}$ and $m_t = N^{-1} \sum_1^N X_t^{(i)}$ the mean. Consider a European option with expiration time $T = 1$ and value at time T given by m_1. Let $a_t = (a_t^{(1)}, \dots, a_t^{(N)})$ be given by $a_t^{(i)} \equiv N^{-1}$. Then

$$1 + \int_0^1 a_t \, dX_t = m_1 \;\text{ a.s.}$$

and, thus, a_t is the portfolio that hedges this option. Since the portfolio is constant, there will be no transactions involved in implementing it. It is in this sense that *the mean is the "easiest" option to hedge.*

Now consider the same option, except this time its value at time $T = 1$ is given by M_1. Let a_t be the portfolio that hedges this option. To get an idea of what a_t looks like, we can consider the much simpler situation where we are trying to hedge the maximum of two stocks. Or, even simpler, we can consider hedging the maximum of one stock and a constant q, i.e. a European call option.

In this case, the "naive" portfolio is simply $\tilde{a}_t = 1_{\{X_t \geq q\}}$. This of course fails due to local time. The Black-Scholes equation gives that the correct portfolio is

$$a_t = \Phi(\rho_+(1-t, X_t)) + \frac{1}{\sigma\sqrt{1-t}}\Phi'(\rho_+(1-t, X_t)) - \frac{q}{\sigma\sqrt{1-t}X_t}\Phi'(\rho_-(1-t, X_t)),$$

where $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-u^2/2}\,du$ and $\rho_\pm(t,x) = (\sigma^2 t)^{-1/2}[\log(x/q) \pm t\sigma^2/2]$. Now, $X_1 > q$ implies that $\rho_\pm(1-t, X_t) \to \infty$ as $t \to 1$ and $X_1 < q$ implies that $\rho_\pm(1-t, X_t) \to -\infty$ as $t \to 1$. It's not hard to show, then, that $a_t \to 1_{\{X_1 \geq q\}}$ a.s. In some sense then, we can consider a_t to be a smoothed-out version of $\tilde{a}_t$ and use $\tilde{a}_t$ to get a heuristic idea about the behavior of a_t, at least for times t near 1.

Returning to our N stock model and the European option with terminal value M_1, the naive hedge is simply

$$\tilde{a}_t^{(i)} = 1_{\{X_t^{(i)} = M_t\}}.$$

If the true hedge is qualitatively similar to this, then we see that for N large, this portfolio will require a very large "number" of transactions. (Actually an infinite number, and this is one of the problems that make it so difficult to quantify this statement.) This stands in sharp contrast to the transaction-free method of hedging the mean value of the stocks. Moreover, when N is very large, we might expect the mean and median to be very close, so that these options, which generate very different hedging portfolios, will in fact have very similar terminal values. This leads to the following question:

Question 1 *Is there a sense in which the median is the "hardest" option to hedge.*

Measuring the difficulty of hedging an option in this sense amounts to measuring the amount of transactions involved. This is exactly the challenge faced when trying to incorporate transaction costs into a financial model. In this case, the transaction costs are related to the total variation of the portfolio process, a_t. However, unless a_t is constant, it will be of unbounded variation, making this impossible. The first chapter of this dissertation approaches the problem of transaction costs by using scaling.

A second question one can naturally ask in this situation is the following:

Question 2 *What is the limiting behavior of the median of N independent, standard one dimensional Brownian motions as $N \to \infty$?*

The work of Harris [15] suggests that the median should behave, in the limit, like a fractional Brownian motion F_t with exponent 1/4, i.e. $E|F_{t+\Delta t} - F_t|^2 \approx |\Delta t|^{1/2}$ for Δt small. His model looks at the "median" of infinitely many particles distributed on the real line according to a Poisson distribution.

Question 2 is answered in the second chapter of this dissertation. It is true that, in the limit, the median behaves locally like a fractional Brownian motion. Globally, however, it behaves like a Brownian motion. If we write M_t^N for the median of N independent, standard one-dimensional Brownian motions, then $\sqrt{N} M_t^N \to G_t$ weakly, where G_t is a Gaussian process with covariance

$$E[G_s G_t] = \sqrt{st}\ \sin^{-1}\left(\frac{s \wedge t}{\sqrt{st}}\right).$$

This process has the property that $E|G_{t+\Delta t} - G_t|^2 \approx |\Delta t|^{1/2}$ for Δt small. However, for $t - s$ large, $E|G_t - G_s|^2 \approx |t - s|$.

The work in the third chapter of this dissertation is motivated by the results obtained in studying Question 1. These results suggest a way to measure transaction costs based on the scaled p-th variation of a Brownian martingale. The analysis of the variations (particularly, the quadratic variation) of martingales is at the heart of Ito's stochastic calculus. This calculus is now a deep and rich field of mathematics that is used in a wide array of applications including economics, engineering, telecommunications, hydrology, biology, and many others. The power of this calculus is its ability to give meaning to and to analyze differential equations driven by random noise. An Ito stochastic differential equation is driven by what is often called a "white" noise, which is the derivative, in a certain sense, of Brownian motion. However, in many applications, Brownian motion may not be the most realistic process to use. Brownian motion has independent increments, whereas most real-world processes do not. An alternative to Brownian motion in these applications is fractional Brownian motion. In this case, though, Ito's stochastic calculus cannot be used, since fractional Brownian motion is not a semimartingale.

Many approaches have been taken to constructing an alternative stochastic calculus for fractional Brownian motion. Heuristic ideas about a possible new calculus for this process will be presented in the third chapter of this dissertation. These ideas lead us, in a natural way, to introduce and study a new kind of variation for the process: the “signed” quadratic variation. It will be proven that this signed variation converges, as a function of t, to a Brownian motion.

Chapter 2

SCALED VARIATIONS OF BROWNIAN MARTINGALES

2.1 Introduction

It is a fundamental fact of the Ito calculus that the quadratic variation of a stochastic integral, $M_t = \int_0^t \theta_s dB_s$, where B_t is a Brownian motion, is given by $\langle M \rangle_t = \int_0^t \theta_s^2 ds$, and that this is the limit, in probability, of the sum of the squares of its increments. More precisely, if $\Pi = \{t_0, \ldots, t_n\}$, $0 = t_0 \le \cdots \le t_n = t$ denotes a partition of the interval $[0, t]$, $\|\Pi\| = \max(t_j - t_{j-1})$ is its mesh size, and

$$V_t^{(p)}(\Pi) = \sum_{j=1}^{n} |M_{t_j} - M_{t_{j-1}}|^p,$$

then given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$P(|V_t^{(2)}(\Pi) - \langle M \rangle_t| \ge \varepsilon) < \varepsilon$$

whenever $\|\Pi\| < \delta$. Note that this does not depend on the particular choice of partition, only that its mesh size is sufficiently small. As a consequence, if $p > 2$, $V_t^{(p)}(\Pi)$ tends to zero, and if $p < 2$, $V_t^{(p)}(\Pi)$ tends to infinity. Nevertheless, for values of p other than 2, $V_t^{(p)}(\Pi)$ will tend to a finite, nontrivial limit, provided that it is properly scaled. This time, however, the limit is not independent of the choice of partition. Exactly what the appropriate scaling is, what the limit is, and how it depends on the partition are the subjects of Theorem 2.2.1.

The application to mathematical finance is as follows: Let X_t denote the price process of a stock or other risky asset and let a_t denote a portfolio process, i.e. a_t specifies the number of shares of the asset to be held at time t. Let us assume the presence of proportional transaction costs. If a_t is of bounded variation, then the cost of maintaining the portfolio is proportional to the total variation of the process $\zeta_t = \int_0^t X_s da_s$. Typically, however, a_t is a semimartingale with nontrivial martingale part, and therefore ζ_t is of unbounded first variation. According to Theorem 2.2.1, we may appropriately scale the discrete approximations to the transaction costs in order to reach a finite, nontrivial limit. Theorem 2.2.2 develops this idea in the case that $a_t = g(t, X_t)$, where g is a sufficiently smooth function.

Theorem 2.2.4 is primarily a special case of Theorem 2.2.2, where g is taken to be a partial derivative of a solution, h, to a Black-Scholes type PDE. The conditions on g in Theorem 2.2.2 are reformulated in Theorem 2.2.4 as conditions on the terminal values, $u(x)$, of the PDE. The results of Theorem 2.2.4, under more restrictive conditions (namely, the concavity of u and a strict boundedness condition on h), are given in [14].

2.2 Results

The following definitions will be used throughout the remainder of this chapter. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{B_t, \mathcal{F}_t\}$ a standard, one-dimensional Brownian motion, where $\mathcal{F}_t$ is the augmentation of the filtration, $\mathcal{F}_t^B$, generated by B_t. Let $\{\theta_t, \mathcal{F}_t\}$ be a progressively measurable, real-valued process and f a C^1, increasing diffeomorphism of $[0, T]$ for some fixed $T > 0$. For each $n \in \mathbb{N}$, define $t_j = f^{-1}(\frac{jT}{n})$ for $0 \leq j \leq n$.

Theorem 2.2.1 *Let $p \in [0, \infty)$ and suppose that $\int_0^T |\theta_s|^{p\vee 2} ds < \infty$, P-a.s. If $M_t = \int_0^t \theta_s dB_s$, then*

$$\left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} |M_{t_j} - M_{t_{j-1}}|^p \to C_p \int_0^T |\theta_s|^p (f'(s))^{1-\frac{p}{2}} ds$$

in probability as $n \to \infty$, where $C_p = E|\xi|^p$, $\xi \sim N(0,1)$. Moreover, if $E\int_0^T |\theta_s|^{2p\vee 2} ds < \infty$, then convergence is in L^2.

Definition 2.2.1 *A continuous function, $g : [0, T) \times (0, \infty)$ is of class $C^{n,m}([0, T) \times (0, \infty))$ if for every $0 \leq j \leq n$, $0 \leq k \leq m$, the partial derivatives $\partial_t^j g$ and $\partial_x^k g$ are of class $C((0, T) \times (0, \infty))$ and have continuous extensions to $[0, T) \times (0, \infty)$. Similar definitions hold for $C^{n,m}((0, T] \times (0, \infty))$ and $C^{n,m}([0, T] \times (0, \infty))$.*

Theorem 2.2.2 *Suppose $g(t, x) \in C^{1,2}([0, T) \times (0, \infty))$ and $dX_t = \mu X_t dt + \sigma X_t dB_t$, $\sigma > 0$, $X_0 = x$. For $t \in [0, T)$, let $\varphi_t = g_t(t, X_t) + \mu X_t g_x(t, X_t) + \frac{1}{2}\sigma^2 X_t^2 g_{xx}(t, X_t)$ and $\theta_t =$*

$\sigma X_t g_x(t, X_t)$ so that

$$g(t, X_t) = g(0, X_0) + \int_0^t \varphi_s ds + \int_0^t \theta_s dB_s, \quad t \in [0, T) \tag{2.2.1}$$

If either

(a) there exists $q > 2$ such that $E\int_0^T (|\varphi_s|^q + |\theta_s|^q)ds < \infty$, or

(b) $g \in C^{1,2}([0, T] \times (0, \infty))$,

then

$$\sqrt{\frac{T}{n}} \sum_{j=1}^n X_{t_j} |g(t_j, X_{t_j}) - g(t_{j-1}, X_{t_{j-1}})| \to \sigma \sqrt{\frac{2}{\pi}} \int_0^T X_s^2 \left| \frac{\partial g}{\partial x}(s, X_s) \right| \sqrt{f'(s)} ds$$

in probability as $n \to \infty$. Moreover, if (a) holds, then convergence is in L^2.

Definition 2.2.2 *A function, $u : (0, \infty) \to \mathbb{R}$, is of class S^k if $u \in C^k((0, \infty))$ and there exists $a, C > 0$ such that $|u^{(k)}(e^x)| \leq Ce^{a|x|}$ for all $x \in \mathbb{R}$.*

Let $\tilde{P}$ denote the risk-neutral probability measure on $(\Omega, \mathcal{F})$ so that $dX_t = \sigma X_t d\tilde{B}_t$, where $\tilde{B}_t$ is a Brownian motion under $\tilde{P}$.

Lemma 2.2.3 *If $u \in S^k$, $k \geq 0$ and $w \in C([0, T])$ with $w \geq 1$, then there is a unique $h(t, x) \in C([0, T] \times (0, \infty)) \cap C^{1,2}([0, T) \times (0, \infty))$ that satisfies $|h(t, e^x)| \leq Ce^{a|x|}$ for some $a, C \in \mathbb{R}$, for all $(t, x) \in [0, T] \times \mathbb{R}$ and solves*

$$\frac{\partial h}{\partial t} + \frac{1}{2}\sigma^2 w(t) x^2 \frac{\partial^2 h}{\partial x^2} = 0; \qquad (t, x) \in [0, T) \times (0, \infty) \tag{2.2.2}$$

$$h(T, x) = u(x); \qquad x \in (0, \infty) \tag{2.2.3}$$

Moreover, $h(t, x) = \tilde{E}_x[u(X_{s(t)})]$, where $s(t) = \int_t^T w(u)du$, and satisfies

(a) $h \in C^{0,k}([0, T] \times (0, \infty))$, and

(b) $\exists b, K \in \mathbb{R}$ such that for all $(t, x) \in [0, T] \times \mathbb{R}$ and all $0 \leq j \leq k$, $|\partial_x^j h(t, e^x)| \leq Ke^{b|x|}$.

Theorem 2.2.4 *If $u \in S^3$ and $w(t)$, $h(t, x)$ are as in Lemma 2.2.3, then*

$$\sqrt{\frac{T}{n}} \sum_{j=1}^n X_{t_j} |h_x(t_j, X_{t_j}) - h_x(t_{j-1}, X_{t_{j-1}})| \to \sigma \sqrt{\frac{2}{\pi}} \int_0^T X_s^2 \left| \frac{\partial^2 h}{\partial x^2}(s, X_s) \right| \sqrt{f'(s)} ds \tag{2.2.4}$$

in L^2 as $n \to \infty$.

2.3 Proof of Theorem 2.2.1

The idea of the proof in the case where $f(t) = t$ is simple, though the details are somewhat tedious. Informally, we have

$$\begin{aligned} \left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} |M_{t_j} - M_{t_{j-1}}|^p &= \left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} \left| \int_{t_{j-1}}^{t_j} \theta_s dB_s \right|^p \\ &\approx \left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} |\theta_{t_{j-1}}|^p |B_{t_j} - B_{t_{j-1}}|^p \\ &\approx \left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} |\theta_{t_{j-1}}|^p E|B_{t_j} - B_{t_{j-1}}|^p \\ &= \left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} |\theta_{t_{j-1}}|^p |t_j - t_{j-1}|^{p/2} C_p \\ &= C_p \left(\frac{T}{n}\right) \sum_{j=1}^{n} |\theta_{t_{j-1}}|^p \\ &\to C_p \int_0^T |\theta_s|^p ds \end{aligned}$$

The general case is treated by a time change.

The formal proof requires three lemmas, the first of which uses the following notation. Let $\Pi = \{t_0, \ldots, t_n\}$, $0 = t_0 \leq \cdots \leq t_n = T$ denote a partition of the interval $[0, T]$ and $\|\Pi\| = \max_{1 \leq j \leq n}(t_j - t_{j-1})$. We will say a simple process, $\zeta(t) = \zeta_0 1_{\{0\}}(t) + \sum_{j=1}^{n} \zeta_{j-1} 1_{(t_{j-1}, t_j]}(t)$, where $\zeta_j \in \mathcal{F}_{t_j}$, is supported on Π.

Lemma 2.3.1 *Let $\alpha \in (0, \infty)$. If there exists $K < \infty$ such that*

$$P\left(\sup_{t \in [0,T]} |\theta_t| \leq K\right) = 1,$$

then given any sequence of partitions, $\{\Pi_n\}$, with $\|\Pi_n\| \to 0$, there exists a subsequence $\{\Pi_{n_j}\}$ and a sequence, $\{\zeta^{(j)}\}$, of simple processes that satisfy

(a) $\zeta^{(j)}$ is supported on Π_{n_j},

(b) $P(\sup_{t \in [0,T]} |\zeta^{(j)}(t)| \leq K) = 1$, and

(c) $E\int_0^T |\theta_t - \zeta^{(j)}(t)|^\alpha dt \to 0$ *as* $j \to \infty$.

Proof. Let $F_t = \int_0^t \theta_s ds$ and for each $m \in \mathbb{N}$, $\theta_t^{(m)} = m(F_t - F_{(t-1/m)\vee 0})$.

Now fix $m \in \mathbb{N}$. Given $n \in \mathbb{N}$, write $\Pi_n = \{t_0, \ldots, t_k\}$ and define $\theta_t^{(m,n)} = \theta_0^{(m)} 1_{\{0\}}(t) + \sum_{j=1}^k \theta_{t_{j-1}}^{(m)} 1_{(t_{j-1},t_j]}(t)$. Since $\theta_t^{(m)}$ is continuous, $\int_0^T |\theta_t^{(m)} - \theta_t^{(m,n)}|^\alpha dt \to 0$ as $n \to \infty$, P-a.s. Thus, by bounded convergence, $E\int_0^T |\theta_t^{(m)} - \theta_t^{(m,n)}|^\alpha dt \to 0$ as $n \to \infty$.

As in part (b) of the proof of Lemma 3.2.4 in [17], $E\int_0^T |\theta_t^{(m)} - \theta_t|^\alpha dt \to 0$. Thus, we may extract a subsequence, $\zeta^{(j)}(t) = \theta_t^{(m_j,n_j)}$, with $n_j > n_{j-1}$, that satisfies the conditions of the lemma. ■

Lemma 2.3.2 *If* $p \in [1, \infty)$*, then*

(a) $||a|^p - |b|^p| \le p|a-b|(|a|^{p-1} + |b|^{p-1})$, $\forall a, b \in \mathbb{R}$

(b) $\left|\sum_{j=1}^n a_n\right|^p \le n^{p-1} \sum_{j=1}^n |a_n|^p$, $\forall a_n \in \mathbb{R}$

(c) $\left|\int_a^b f(t)dt\right|^p \le (b-a)^{p-1} \int_a^b |f(t)|^p dt$, $\forall f \in L^p((a,b))$

If $p \in (0, 1]$*, then*

(d) $||a|^p - |b|^p| \le |a-b|^p$, $\forall a, b \in \mathbb{R}$

(e) $\sum_{j=1}^n |a_n|^p \le n^{1-p} \left|\sum_{j=1}^n |a_n|\right|^p$, $\forall a_n \in \mathbb{R}$

(f) $\int_a^b |f(t)|^p dt \le (b-a)^{1-p} \left|\int_a^b |f(t)|dt\right|^p$, $\forall f \in L^1((a,b))$

Proof. For (a) and (d), we may assume, by symmetry, that $|a| > |b| > 0$. Set $t = |b|/|a| \in (0,1)$. Since, for every $p \in (0,\infty)$, $t \mapsto (1-t^p)/(1-t)$ is monotone on $t \in [0,1]$, and $(1-t^p)/(1-t) \to p$ as $t \to 1$, we have $(1-t^p) \le (p \vee 1)(1-t)$. Thus, $||a|^p - |b|^p| = |a|^p(1-t^p) \le (p \vee 1)|a|^p(1-t)$.

Thus, for any $p \in (0,\infty)$, $||a|^p - |b|^p| \le (p\vee 1)|a|^{p-1}(|a|-|b|) \le (p\vee 1)|a-b|(|a|^{p-1}+|b|^{p-1})$, which gives (a). If $p \in (0,1]$, then $||a|^p - |b|^p| \le (|a|(1-t)^{1/p})^p \le (|a|(1-t))^p \le |a-b|^p$, which gives (d).

Parts (b) and (c) are just Jensen's inequality; (e) and (f) are derived by applying (b) and (c) with $1/p$. ■

Lemma 2.3.3 *Let $p > 0$, $1 \le r \le 2$. Let μ_t, ν_t be progressively measurable processes and set $\Delta M_j = \int_{t_{j-1}}^{t_j} \mu_s dB_s$, $\Delta N_j = \int_{t_{j-1}}^{t_j} \nu_s dB_s$. Let $X_n = \left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^n ||\Delta M_j|^p - |\Delta N_j|^p|$. Suppose α_t is progressively measurable and $|\mu_t| \le \alpha_t$, $|\nu_t| \le \alpha_t$. Then there exists $C < \infty$, depending only on p, r, and T, such that*

(a) $E|X_n|^r \le C\left(E\int_0^T |\mu_s - \nu_t|^2 ds\right)^{pr/2}$, *if* $p \le 1$;

(b) $E|X_n|^r \le C\left(E\int_0^T |\alpha_s|^2 ds\right)^{r(p-1)/2}\left(E\int_0^T |\mu_s - \nu_t|^2 ds\right)^{r/2}$, *if* $1 \le p \le 2/r$;

(c) $E|X_n|^r \le C\left(E\int_0^T |\alpha_s|^{pr} ds\right)^{(p-1)/p}\left(E\int_0^T |\mu_s - \nu_t|^{pr} ds\right)^{1/p}$, *if* $p \ge 2/r$;

Proof. First, by Lemma 2.3.2(b),

$$\begin{aligned} E|X_n|^r &\le \left(\frac{T}{n}\right)^{r-pr/2} n^{r-1} \sum_{j=1}^n E||\Delta M_j|^p - |\Delta N_j|^p|^r \\ &= \kappa_1 n^{pr/2-1} \sum_{j=1}^n E||\Delta M_j|^p - |\Delta N_j|^p|^r \end{aligned}$$

where $\kappa_1 = T^{r-pr/2}$.

Now suppose $p \le 1$. In this case, Lemma 2.3.2(d) gives

$$\begin{aligned} E|X_n|^r &\le \kappa_1 n^{pr/2-1} \sum_{j=1}^n E|\Delta M_j - \Delta N_j|^{pr} \\ &= \kappa_1 n^{pr/2-1} \sum_{j=1}^n E\left|\int_{t_{j-1}}^{t_j} (\mu_s - \nu_s) dB_s\right|^{pr} \\ &\le \kappa_1 n^{pr/2-1} \sum_{j=1}^n KE\left|\int_{t_{j-1}}^{t_j} |\mu_s - \nu_s|^2 ds\right|^{pr/2} \end{aligned}$$

by the Burkholder-Davis-Gundy inequalities, where K depends only on the product pr. By the assumptions on p and r, $pr/2 \le 1$, so Jensen's inequality followed by Lemma 2.3.2(e) gives

$$\begin{aligned} E|X_n|^r &\leq \kappa_1 K n^{pr/2-1} \sum_{j=1}^{n} \left| E\int_{t_{j-1}}^{t_j} |\mu_s - \nu_s|^2 ds \right|^{pr/2} \\ &\leq \kappa_1 K \left| \sum_{j=1}^{n} E\int_{t_{j-1}}^{t_j} |\mu_s - \nu_s|^2 ds \right|^{pr/2} \\ &= \kappa_1 K \left(E\int_0^T |\mu_s - \nu_s|^2 ds \right)^{pr/2}. \end{aligned}$$

Now suppose $p > 1$. In this case, Lemma 2.3.2(a) gives

$$E|X_n|^r \leq \kappa_1 n^{pr/2-1} \sum_{j=1}^{n} E(p|\Delta M_j - \Delta N_j| \cdot ||\Delta M_j|^{p-1} + |\Delta N_j|^{p-1}|)^r.$$

Using Hölder's inequality with the conjugate exponents p and $p/(p-1)$ gives

$$\begin{aligned} E|X_n|^r &\leq \kappa_1 p^r n^{pr/2-1} \left(\sum_{j=1}^{n} E|\Delta M_j - \Delta N_j|^{pr} \right)^{\frac{1}{p}} \\ &\quad \cdot \left(\sum_{j=1}^{n} E||\Delta M_j|^{p-1} + |\Delta N_j|^{p-1}|^{pr/(p-1)} \right)^{\frac{p-1}{p}}. \end{aligned}$$

Now note that

$$\begin{aligned} E|\Delta M_j|^{pr} &= E\left| \int_{t_{j-1}}^{t_j} \mu_s dB_s \right|^{pr} \\ &\leq KE\left| \int_{t_{j-1}}^{t_j} |\mu_s|^2 ds \right|^{pr/2} \\ &\leq KE\left| \int_{t_{j-1}}^{t_j} |\alpha_s|^2 ds \right|^{pr/2} \end{aligned}$$

and similarly,

$$E|\Delta N_j|^{pr} \leq KE\left| \int_{t_{j-1}}^{t_j} |\alpha_s|^2 ds \right|^{pr/2}.$$

Thus, using $||a| + |b||^q \leq 2^q(|a|^q + |b|^q)$, we have

$$\begin{aligned} E|X_n|^r &\leq \kappa_1 p^r n^{pr/2-1} \left(\sum_{j=1}^{n} KE\left| \int_{t_{j-1}}^{t_j} |\mu_s - \nu_s|^2 ds \right|^{pr/2} \right)^{\frac{1}{p}} \\ &\quad \cdot \left(\sum_{j=1}^{n} 2^{pr/(p-1)+1} KE\left| \int_{t_{j-1}}^{t_j} |\alpha_s|^2 ds \right|^{pr/2} \right)^{\frac{p-1}{p}}, \end{aligned}$$

i.e.

$$E|X_n|^r \le \kappa_2 n^{pr/2-1}\left(\sum_{j=1}^n E\left|\int_{t_{j-1}}^{t_j}|\mu_s-\nu_s|^2ds\right|^{pr/2}\right)^{\frac{1}{p}} \cdot\left(\sum_{j=1}^n E\left|\int_{t_{j-1}}^{t_j}|\alpha_s|^2ds\right|^{pr/2}\right)^{\frac{p-1}{p}} \qquad (2.3.1)$$

where $\kappa_2 = \kappa_1 p^r K 2^{r+(p-1)/p}$.

If $p \le 2/r$, so that $pr/2 \le 1$, then Jensen's inequality followed by Lemma 2.3.2(e) gives

$$\begin{aligned} E|X_n|^r &\le \kappa_2 n^{pr/2-1}\left(n^{1-pr/2}\left|E\int_0^T|\mu_s-\nu_s|^2ds\right|^{pr/2}\right)^{\frac{1}{p}} \\ &\quad\cdot\left(n^{1-pr/2}\left|E\int_0^T|\alpha_s|^2ds\right|^{pr/2}\right)^{\frac{p-1}{p}} \\ &= \kappa_2\left(E\int_0^T|\alpha_s|^2ds\right)^{\frac{r(p-1)}{2}}\left(E\int_0^T|\mu_s-\nu_s|^2ds\right)^{\frac{r}{2}}. \end{aligned}$$

If $p \ge 2/r$, so that $pr/2 \ge 1$, then Lemma 2.3.2(c) applied to (2.3.1) gives

$$\begin{aligned} E|X_n|^r &\le \kappa_2 n^{pr/2-1}\left(\sum_{j=1}^n\left(\frac{T}{n}\right)^{pr/2-1}E\int_{t_{j-1}}^{t_j}|\mu_s-\nu_s|^{pr}ds\right)^{\frac{1}{p}} \\ &\quad\cdot\left(\sum_{j=1}^n\left(\frac{T}{n}\right)^{pr/2-1}E\int_{t_{j-1}}^{t_j}|\alpha_s|^{pr}ds\right)^{\frac{p-1}{p}} \\ &= \kappa_3\left(E\int_0^T|\alpha_s|^{pr}ds\right)^{\frac{p-1}{p}}\left(E\int_0^T|\mu_s-\nu_s|^{pr}ds\right)^{\frac{1}{p}} \end{aligned}$$

where $\kappa_3 = \kappa_2 T^{pr/2-1}$. ■

Proof of Theorem 2.2.1. In this proof, several different cases are considered.

Case 1: M and θ are bounded, $f(t) = t$.

Suppose there exists $K < \infty$ such that, with probability one, $|M_t| \le K$ and $|\theta_t| \le K$, $\forall t \in [0,T]$. It will be shown that

$$\left(\frac{T}{n}\right)^{1-\frac{p}{2}}\sum_{j=1}^n|M_{t_j}-M_{t_{j-1}}|^p \to C_p\int_0^T|\theta_s|^pds$$

in L^2 as $n \to \infty$.

By Lemma 2.3.1, it may be assumed without loss of generality that for each $n \in \mathbb{N}$, $0 \le j < n$, $\exists \xi_{t_j}^{(n)} \in \mathcal{F}_{t_j}$ such that $|\xi_{t_j}^{(n)}| \le K$ a.s. and $\theta_t^{(n)} = \xi_0^{(n)} 1_{\{0\}}(t) + \sum_{j=1}^n \xi_{t_{j-1}}^{(n)} 1_{(t_{j-1}, t_j]}(t)$ satisfies $E\int_0^T |\theta_t - \theta_t^{(n)}|^{2p\vee 2} dt \to 0$ as $n \to \infty$.

Now write

$$\left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^n |M_{t_j} - M_{t_{j-1}}|^p - C_p \int_0^T |\theta_s|^p ds = X_1^{(n)} + X_2^{(n)} + X_3^{(n)}$$

where

$$\begin{aligned}
X_1^{(n)} &= \left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^n |M_{t_j} - M_{t_{j-1}}|^p - \left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^n |\xi_{t_{j-1}}^{(n)}|^p |B_{t_j} - B_{t_{j-1}}|^p, \\
X_2^{(n)} &= \left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^n |\xi_{t_{j-1}}^{(n)}|^p |B_{t_j} - B_{t_{j-1}}|^p - C_p \left(\frac{T}{n}\right) \sum_{j=1}^n |\xi_{t_{j-1}}^{(n)}|^p, \\
X_3^{(n)} &= C_p \left(\frac{T}{n}\right) \sum_{j=1}^n |\xi_{t_{j-1}}^{(n)}|^p - C_p \int_0^T |\theta_s|^p ds
\end{aligned}$$

It will be shown that each $X_j^{(n)} \to 0$ in L^2 as $n \to \infty$.

First, $X_3^{(n)} = C_p \int_0^T (|\theta_s^{(n)}|^p - |\theta_s|^p) ds$. If $p \le 1$, then Lemma 2.3.2(d) gives $|X_3^{(n)}| \le C_p \int_0^T |\theta_s^{(n)} - \theta_s|^p ds$. Hence, by Lemma 2.3.2(c),

$$E|X_3^{(n)}|^2 \le C_p^2 T E \int_0^T |\theta_s^{(n)} - \theta_s|^{2p} ds$$

which tends to zero as $n \to \infty$.

If $p \ge 1$, then Lemma 2.3.2(a) gives

$$\begin{aligned}
|X_3^{(n)}| &\le pC_p \int_0^T |\theta_s^{(n)} - \theta_s| (|\theta_s^{(n)}|^{p-1} + |\theta_s|^{p-1}) ds \\
&\le C \int_0^T |\theta_s^{(n)} - \theta_s| ds
\end{aligned}$$

where $C = 2pC_p K^{p-1}$. Thus,

$$E|X_3^{(n)}|^2 \le C^2 T E \int_0^T |\theta_s^{(n)} - \theta_s|^2 ds$$

which also tends to zero as $n \to \infty$.

Next, $X_2^{(n)} = \frac{T}{n}\sum_{j=1}^n |\xi_{t_{j-1}}^{(n)}|^p Y_j^{(n)}$, where $Y_j^{(n)} = \left(\frac{T}{n}\right)^{-p/2}|B_{t_j} - B_{t_{j-1}}|^p - C_p$. If $k > j$, then $Y_k^{(n)}$ is independent of $|\xi_{t_{j-1}}^{(n)}\xi_{t_{k-1}}^{(n)}|^p Y_j^{(n)}$; and $\left(\frac{T}{n}\right)^{-1/2}(B_{t_j} - B_{t_{j-1}})$ is normal with mean zero and variance one, so that $EY_k^{(n)} = 0$. Therefore

$$E|X_2^{(n)}|^2 = \frac{T^2}{n^2}\sum_{j=1}^n E\left(|\xi_{t_{j-1}}^{(n)}|^{2p}|Y_j^{(n)}|^2\right) \le \frac{T^2K^{2p}}{n^2}\sum_{j=1}^n E|Y_j^{(n)}|^2.$$

But $E|Y_j^{(n)}|^2$ is just a constant which does not depend on j or n. Thus, $E|X_2^{(n)}|^2 \le \frac{C}{n}$ and $X_2^{(n)} \to 0$ in L^2 as $n \to \infty$.

Finally, for $X_1^{(n)}$, write $M_t^{(n)} = \int_0^t \theta_s^{(n)} dB_s$ and adopt the notation $\Delta M_j = M_{t_j} - M_{t_{j-1}}$ and similarly for $\Delta M_j^{(n)}$, so that

$$X_1^{(n)} = \left(\frac{T}{n}\right)^{1-\frac{p}{2}}\sum_{j=1}^n (|\Delta M_j|^p - |\Delta M_j^{(n)}|^p).$$

Applying Lemma 2.3.3 with $r = 2$ gives that if $p \le 1$, then

$$E|X_1^{(n)}|^2 \le C\left(E\int_0^T |\theta_s - \theta_s^{(n)}|^2 ds\right)^p$$

and if $p \ge 1$, then

$$\begin{aligned} E|X_1^{(n)}|^2 &\le C\left(E\int_0^T K^{2p} ds\right)^{\frac{p-1}{p}}\left(E\int_0^T |\theta_s - \theta_s^{(n)}|^{2p} ds\right)^{\frac{1}{p}} \\ &= \tilde{C}\left(E\int_0^T |\theta_s - \theta_s^{(n)}|^{2p} ds\right)^{\frac{1}{p}}. \end{aligned}$$

In either case, $X_1^{(n)} \to 0$ in L^2 as $n \to \infty$.

Case 2: $E\int_0^T |\theta_s|^{2p\vee 2} ds < \infty$, $f(t) = t$.

For $k \in \mathbb{N}$, let $\theta_t^{(k)} = -k \vee (\theta_t \wedge k)$ and $\tau_k = T \wedge \inf\{t \in [0,T] : \left|\int_0^t \theta_s^{(k)} dB_s\right| \ge k\}$. Define $M_t^{(k)} = \int_0^{t\wedge\tau_k} \theta_s^{(k)} dB_s = \int_0^t \tilde{\theta}_s^{(k)} dB_s$, where $\tilde{\theta}_t^{(k)} = \theta_t^{(k)} 1_{[0,\tau_k]}(t)$. Note that, with probability one, $\tilde{\theta}_t^{(k)} \to \theta_t$ pointwise on $[0,T]$.

Now write

$$\left(\frac{T}{n}\right)^{1-\frac{p}{2}}\sum_{j=1}^n |M_{t_j} - M_{t_{j-1}}|^p - C_p\int_0^T |\theta_s|^p ds = X_1^{(n,k)} + X_2^{(n,k)} + X_3^{(k)}$$

where

$$
\begin{aligned}
X_1^{(n,k)} &= \left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} (|\Delta M_j|^p - |\Delta M_j^{(k)}|^p), \\
X_2^{(n,k)} &= \left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} |\Delta M_j^{(k)}|^p - C_p \int_0^T |\tilde{\theta}_s^{(k)}|^p ds, \\
X_3^{(k)} &= C_p \int_0^T (|\tilde{\theta}_s^{(k)}|^p - |\theta_s|^p) ds.
\end{aligned}
$$

By case 1, $X_2^{(n,k)} \to 0$ in L^2 as $n \to \infty$ for each fixed k. Also, $E|X_3^{(k)}|^2 \to 0$ by dominated convergence.

It will be shown that there exists $h(k)$ such that

(i) $h(k) \to 0$ as $k \to \infty$, and

(ii) $E|X_1^{(n,k)}|^2 \leq h(k)$ for all $n \in \mathbb{N}$,

from which it follows that

$$
\left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} |M_{t_j} - M_{t_{j-1}}|^p \to C_p \int_0^T |\theta_s|^p ds
$$

in L^2 as $n \to \infty$.

As in case 1, Lemma 2.3.3 shows that if $p \leq 1$, then

$$
E|X_1^{(n,k)}|^2 \leq C \left(E \int_0^T |\theta_s - \tilde{\theta}_s^{(k)}|^2 ds \right)^p
$$

and if $p \geq 1$, then

$$
E|X_1^{(n,k)}|^2 \leq C \left(E \int_0^T |\theta_s|^{2p} ds \right)^{\frac{p-1}{p}} \left(E \int_0^T |\theta_s - \tilde{\theta}_s^{(k)}|^{2p} ds \right)^{\frac{1}{p}}.
$$

Since $E \int_0^T |\theta_s|^{2p} ds < \infty$, this completes the proof of case 2.

Case 3: $f(t) = t$.

Let $\tau_k = T \wedge \inf\{t \in [0,T] : \int_0^t |\theta_s|^{p \vee 2} ds \geq k\}$. Let $\theta_t^{(k)} = \theta_t 1_{[0,\tau_k]}(t)$ and $M_t^{(k)} = \int_0^{t \wedge \tau_k} \theta_s dB_s = \int_0^t \theta_s^{(k)} dB_s$. Note that $P(\tau_k = T) \nearrow 1$ as $k \to \infty$ and $M_t = M_t^{(k)}$, $\theta_t = \theta_t^{(k)}$

on $\{\tau_k = T\}$. Hence, since

$$\begin{aligned} & P\left(\left|\left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} |\Delta M_j|^p - C_p \int_0^T |\theta_s|^p ds\right| \geq \varepsilon\right) \\ \leq \quad & P(\tau_k < T) + P\left(\left|\left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} |\Delta M_j^{(k)}|^p - C_p \int_0^T |\theta_s^{(k)}|^p ds\right| \geq \varepsilon\right) \end{aligned}$$

it will suffice to show that for each fixed $k \in \mathbb{N}$,

$$\left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} |\Delta M_j^{(k)}|^p \to C_p \int_0^T |\theta_s^{(k)}|^p ds$$

in L^1, and therefore in probability, as $n \to \infty$.

Fix $k \in \mathbb{N}$ and for each $l \in \mathbb{N}$, let $\theta_t^{(k,l)} = -l \vee (\theta_t^{(k)} \wedge l)$ and $M_t^{(k,l)} = \int_0^t \theta_s^{(k,l)} dB_s$. Write

$$\left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} |\Delta M_j^{(k)}|^p - C_p \int_0^T |\theta_s^{(k)}|^p ds = X_1^{(n,l)} + X_2^{(n,l)} + X_3^{(l)}$$

where

$$\begin{aligned} X_1^{(n,l)} &= \left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} (|\Delta M_j^{(k)}|^p - |\Delta M_j^{(k,l)}|^p), \\ X_2^{(n,l)} &= \left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} |\Delta M_j^{(k,l)}|^p - C_p \int_0^T |\theta_s^{(k,l)}|^p ds, \\ X_3^{(l)} &= C_p \int_0^T (|\theta_s^{(k,l)}|^p - |\theta_s^{(k)}|^p) ds. \end{aligned}$$

By case 2, for each fixed l, $X_2^{(n,l)} \to 0$ in L^2, and therefore in L^1, as $n \to \infty$. By dominated convergence, $X_3^{(l)} \to 0$ in L^1 as $n \to \infty$.

It will be shown that there exists $h(l)$ such that

(i) $h(l) \to 0$ as $l \to \infty$, and

(ii) $E|X_1^{(n,l)}| \leq h(l)$ for all $n \in \mathbb{N}$,

which will complete the proof in this case.

Applying Lemma 2.3.3 with $r = 1$ shows that if $p \leq 1$, then

$$E|X_1^{(n,l)}| \leq C\left(E\int_0^T |\theta_s^{(k)} - \theta_s^{(k,l)}|^2 ds\right)^{\frac{p}{2}};$$

if $1 \leq p \leq 2$, then

$$E|X_1^{(n,l)}| \leq C\left(E\int_0^T |\theta_s^{(k)}|^2 ds\right)^{\frac{p-1}{2}} \left(E\int_0^T |\theta_s^{(k)} - \theta_s^{(k,l)}|^2 ds\right)^{\frac{1}{2}};$$

and if $p \geq 2$, then

$$E|X_1^{(n,l)}| \leq C\left(E\int_0^T |\theta_s^{(k)}|^p ds\right)^{\frac{p-1}{p}} \left(E\int_0^T |\theta_s^{(k)} - \theta_s^{(k,l)}|^p ds\right)^{\frac{1}{p}}.$$

As before, since $E\int_0^T |\theta_s^{(k)}|^{p\vee 2} ds \leq k$, this completes the proof of case 3.

Case 4: General case.

Let $X_t = \int_0^t \sqrt{f'(s)}dB_s$, so that $\langle X\rangle_t = f(t)$ and, hence, $\tilde{B}_t = X_{g(t)}$, where $g = f^{-1}$, is a Brownian motion with respect to the filtration, $\mathcal{G}_t = \mathcal{F}_{g(t)}$. Define $N_t = \int_0^t \varphi_s d\tilde{B}_s$, where $\varphi_t = \theta_{g(t)}\sqrt{g'(t)}$.

Claim: $N_t = M_{g(t)}$.

Proof of claim: Let $\{Y_t, \mathcal{G}_t\}$ be a continuous, square integrable martingale, so that

$$\begin{aligned} \langle N, Y\rangle_t &= \int_0^t \varphi_s d\langle \tilde{B}, Y\rangle_s \\ &= \int_0^t \theta_{g(s)}\sqrt{g'(s)}d\left(\langle X, Y_{f(\cdot)}\rangle_{g(s)}\right) \\ &= \int_0^{g(t)} \theta_s\sqrt{g'(f(s))}d\langle X, Y_{f(\cdot)}\rangle_s \\ &= \langle \Phi, Y_{f(\cdot)}\rangle_{g(t)} \end{aligned}$$

where

$$\Phi_t = \int_0^t \theta_s \frac{1}{\sqrt{f'(s)}} dX_s = \int_0^t \theta_s dB_s = M_t.$$

Thus, $\langle N, Y\rangle_t = \langle M, Y_{f(\cdot)}\rangle_{g(t)} = \langle M_{g(\cdot)}, Y\rangle_t$. Since Y was arbitrary, this proves the claim./‌//

Now note that for any $p \in (0, \infty)$, $\int_0^T |\varphi_s|^p ds = \int_0^T |\theta_s|^p (f'(s))^{1-\frac{p}{2}} ds$. Since f is a C^1, increasing diffeomorphism, f' is both bounded above and bounded away from zero. Hence, $\int_0^T |\varphi_s|^{p\vee 2} ds < \infty$, P-a.s. By the previous cases, then,

$$\left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^n |N_{f(t_j)} - N_{f(t_{j-1})}|^p \to C_p \int_0^T |\varphi_s|^p ds,$$

i.e.

$$\left(\frac{T}{n}\right)^{1-\frac{p}{2}} \sum_{j=1}^{n} |M_{t_j} - M_{t_{j-1}}|^p \to C_p \int_0^T |\theta_s|^p (f'(s))^{1-\frac{p}{2}} ds$$

in probability as $n \to \infty$. If $E\int_0^T |\theta_s|^{2p\vee 2} ds < \infty$, then $E\int_0^T |\varphi_s|^{2p\vee 2} ds < \infty$ and convergence is in L^2. ■

2.4 Proof of Theorem 2.2.2

In this section, let $g_j = g(t_j, X_{t_j})$, $\Delta g_j = g_j - g_{j-1}$, and $T_n = \sqrt{\frac{T}{n}} \sum_{j=1}^n X_{t_j} |\Delta g_j|$. As before, the idea of the proof is straightforward: ignore the bounded variation part of the semimartingale, $g(t, X_t)$, and apply Theorem 2.2.1. Informally,

$$\begin{aligned}
\sqrt{\frac{T}{n}} \sum_{j=1}^{n} X_{t_j} |\Delta g_j| &\approx \sqrt{\frac{T}{n}} \sum_{j=1}^{n} X_{t_{j-1}} |\Delta g_j| \\
&\approx \sqrt{\frac{T}{n}} \sum_{j=1}^{n} X_{t_{j-1}} \left| \int_{t_{j-1}}^{t_j} \theta_s dB_s \right| \\
&= \sqrt{\frac{T}{n}} \sum_{j=1}^{n} \left| \int_{t_{j-1}}^{t_j} X_{t_{j-1}} \theta_s dB_s \right| \\
&\approx \sqrt{\frac{T}{n}} \sum_{j=1}^{n} \left| \int_{t_{j-1}}^{t_j} X_s \theta_s dB_s \right| \\
&\to C_1 \int_0^T |X_s \theta_s| \sqrt{f'(s)} ds \\
&= \sigma \sqrt{\frac{2}{\pi}} \int_0^T X_s^2 \left| \frac{\partial g}{\partial x}(s, X_s) \right| \sqrt{f'(s)} ds
\end{aligned}$$

It should be noted that condition (a) in the statement of Theorem 2.2.2 is sharp in the sense illustrated by the following example: Let $T = 1/4$, $\mu = 1/2$, and $\sigma = 1$. Let $h \in C^\infty(\mathbb{R})$ satisfy $|h| \leq 1$, $h \equiv 0$ on $(-\infty, 0]$, and $h \equiv 1$ on $[1, \infty)$. Let $f(x) = h(x) e^{x^2 - x}$ and define $g(t, x) = f(\log x)$. With these choices, $g(t, X_t) = f(B_t)$, so that $\theta_t = f'(B_t)$ and $\varphi_t = \frac{1}{2} f''(B_t)$. It can be shown (see, for example, section 4.3 of [17]) that $t \mapsto E|\theta_t|^2$ and $t \mapsto E|\varphi_t|^2$ are continuous for $t \in [0, 1/4]$. Thus, in this example, $E\int_0^T (|\theta_s|^2 + |\varphi_s|^2) ds < \infty$.

On the other hand, there is no $n \in \mathbb{N}$ for which $T_n \in L^2(\Omega)$. To see this, assume that $T_n \in L^2(\Omega)$ and observe that for $0 < s \le t \le T$,

$$\begin{aligned} E|X_t g(s, X_s)|^2 &= E\left|\frac{X_t}{X_s}\right|^2 E|X_s f(B_s)|^2 \\ &= E\left|\frac{X_t}{X_s}\right|^2 \left(\frac{1}{\sqrt{2\pi s}}\int_{-\infty}^{\infty} e^{2x}|f(x)|^2 e^{-x^2/2s}dx\right) \\ &= \frac{1}{\sqrt{2\pi s}}E\left|\frac{X_t}{X_s}\right|^2 \left(\int_0^1 |f(x)|^2 e^{2x-x^2/2s}dx + \int_1^{\infty} e^{2x^2-x^2/2s}dx\right) \end{aligned}$$

is finite if and only if $s < T = 1/4$. Thus, for any $j < n$, $X_{t_j}|\Delta g_j| \in L^2(\Omega)$. Since $T_n \in L^2(\Omega)$ by assumption, it follows that $X_{t_n}|\Delta g_n| \in L^2(\Omega)$. Therefore,

$$\begin{aligned} |X_T g(T, X_T)| &= |X_{t_n}(g(t_n, X_{t_n}) - g(t_{n-1}, X_{t_{n-1}}) + g(t_{n-1}, X_{t_{n-1}}))| \\ &\le X_{t_n}|\Delta g_n| + X_{t_n}|g(t_{n-1}, X_{t_{n-1}})| \end{aligned}$$

implies that $X_T g(T, X_T) \in L^2(\Omega)$, which is a contradiction.

Lemma 2.4.1 *If $Z_t = \int_0^t \alpha_s ds + \int_0^t \beta_s dB_s$ with $E\int_0^T (|\alpha_s|^\gamma + |\beta_s|^\gamma)ds < \infty$, $\gamma > 2$, then there exists $C < \infty$, depending only on γ, such that for any $\varepsilon < 1$,*

$$E|Z_{t+\varepsilon} - Z_t|^\gamma \le CE\int_t^{t+\varepsilon} (|\alpha_s|^\gamma + |\beta_s|^\gamma)ds.$$

Moreover, if $\sup_{0\le t\le T}(E|\alpha_t|^\gamma + E|\beta_t|^\gamma) = M < \infty$, then

$$E|Z_{t+\varepsilon} - Z_t|^\gamma \le CM\varepsilon^{\gamma/2}.$$

Proof. We have

$$\begin{aligned} E|Z_{t+\varepsilon} - Z_t|^\gamma &\le 2^\gamma \left(E\left|\int_t^{t+\varepsilon} \alpha_s ds\right|^\gamma + E\left|\int_t^{t+\varepsilon} \beta_s dB_s\right|^\gamma\right) \\ &\le 2^\gamma \left(\varepsilon^{\gamma-1}E\int_t^{t+\varepsilon} |\alpha_s|^\gamma ds + K_\gamma E\left|\int_t^{t+\varepsilon} |\beta_s|^2 ds\right|^{\gamma/2}\right) \\ &\le C\left(\varepsilon^{\gamma-1}E\int_t^{t+\varepsilon} |\alpha_s|^\gamma ds + \varepsilon^{\gamma/2-1}E\int_t^{t+\varepsilon} |\beta_s|^\gamma ds\right) \end{aligned}$$

where $C = 2^\gamma \max\{1, K_\gamma\}$ and the lemma now follows. ■

Proof of Theorem 2.2.2. First observe that if either condition (a) or (b) holds, then (2.2.1) is valid on the entire closed interval, $[0, T]$.

Now, assume (a) holds. Write $T_n = \sum_{k=1}^4 T_n^{(k)}$, where

$$\begin{aligned}
T_n^{(1)} &= \sqrt{\frac{T}{n}} \sum_{j=1}^n (X_{t_j} - X_{t_{j-1}})|\Delta g_j|, \\
T_n^{(2)} &= \sqrt{\frac{T}{n}} \sum_{j=1}^n \left(|X_{t_{j-1}} \Delta g_j| - \left| X_{t_{j-1}} \int_{t_{j-1}}^{t_j} \theta_s dB_s \right| \right), \\
T_n^{(3)} &= \sqrt{\frac{T}{n}} \sum_{j=1}^n \left(\left| X_{t_{j-1}} \int_{t_{j-1}}^{t_j} \theta_s dB_s \right| - \left| \int_{t_{j-1}}^{t_j} X_s \theta_s dB_s \right| \right), \\
T_n^{(4)} &= \sqrt{\frac{T}{n}} \sum_{j=1}^n \left| \int_{t_{j-1}}^{t_j} X_s \theta_s dB_s \right|.
\end{aligned}$$

Since

$$E \int_0^T |X_s \theta_s|^2 ds \le \left(E \int_0^T |\theta_s|^q ds \right)^{2/q} \left(E \int_0^T |X_s|^{2r} ds \right)^{1/r} < \infty$$

where r is such that $\frac{1}{r} + \frac{2}{q} = 1$, we may apply Theorem 2.2.1 with $p = 1$ to conclude that

$$T_n^{(4)} \to C_1 \int_0^T |X_s \theta_s| \sqrt{f'(s)} ds = \sigma \sqrt{\frac{2}{\pi}} \int_0^T X_s^2 \left| \frac{\partial g}{\partial x}(s, X_s) \right| \sqrt{f'(s)} ds$$

in L^2 as $n \to \infty$. It will thus suffice to show that for $k = 1, 2, 3$, $T_n^{(k)} \to 0$ in L^2 as $n \to \infty$.

For $T_n^{(1)}$, Lemma 2.3.2(b) gives

$$\begin{aligned}
E|T_n^{(1)}|^2 &\le T \sum_{j=1}^n E[|X_{t_j} - X_{t_{j-1}}|^2 |\Delta g_j|^2] \\
&\le T \left(\sum_{j=1}^n E|X_{t_j} - X_{t_{j-1}}|^{2r} \right)^{1/r} \left(\sum_{j=1}^n E|\Delta g_j|^q \right)^{2/q}.
\end{aligned}$$

By Lemma 2.4.1, there is a constant, C, such that

$$E|T_n^{(1)}|^2 \le CT \left(\sum_{j=1}^n |t_j - t_{j-1}|^r \right)^{1/r} \left(\sum_{j=1}^n E \int_{t_{j-1}}^{t_j} (|\theta_s|^q + |\varphi_s|^q) ds \right)^{2/q}$$

which tends to zero as $n \to \infty$ since $r > 1$.

For $T_n^{(2)}$, we have

$$E|T_n^{(2)}|^2 \le T \sum_{j=1}^n E \left| X_{t_{j-1}} \int_{t_{j-1}}^{t_j} \varphi_s ds \right|^2.$$

Let $X^* = \sup_{0\le t\le T}|X_t|$, so that

$$\begin{aligned} E|T_n^{(2)}|^2 &\le T\sum_{j=1}^n \left(E|X^*|^{2r}\right)^{1/r}\left(E\left|\int_{t_{j-1}}^{t_j}\varphi_s ds\right|^q\right)^{2/q} \\ &\le Cn^{1-2/q}\left|\sum_{j=1}^n |t_j - t_{j-1}|^{q-1} E\int_{t_{j-1}}^{t_j}|\varphi_s|^q ds\right|^{2/q}. \end{aligned}$$

Since $f^{-1}\in C^1[0,T]$, we may write

$$\begin{aligned} E|T_n^{(2)}|^2 &\le \tilde{C}n^{1-2/q}\left|\sum_{j=1}^n n^{1-q}E\int_{t_{j-1}}^{t_j}|\varphi_s|^q ds\right|^{2/q} \\ &= \frac{\tilde{C}}{n}\left(E\int_0^T |\varphi_s|^q ds\right)^{2/q} \end{aligned}$$

which tends to zero as $n\to\infty$.

Finally, for $T_n^{(3)}$, we have

$$\begin{aligned} E|T_n^{(3)}|^2 &\le T\sum_{j=1}^n E\left|\int_{t_{j-1}}^{t_j}(X_s - X_{t_{j-1}})\theta_s dB_s\right|^2 \\ &= T\sum_{j=1}^n E\int_{t_{j-1}}^{t_j}|X_s - X_{t_{j-1}}|^2|\theta_s|^2 ds \\ &\le T\sum_{j=1}^n \int_{t_{j-1}}^{t_j}\left(E|X_s - X_{t_{j-1}}|^{2r}\right)^{1/r}\left(E|\theta_s|^q\right)^{2/q} ds. \end{aligned}$$

By Lemma 2.4.1 and the fact that $f^{-1}\in C^1[0,T]$, we can write

$$\begin{aligned} E|T_n^{(3)}|^2 &\le \frac{C}{n}\int_0^T (E|\theta_s|^q)^{2/q} ds \\ &\le \frac{C}{n}T^{1-2/q}\left(E\int_0^T|\theta_s|^q ds\right)^{2/q} \end{aligned}$$

which tends to zero as $n\to\infty$, and this completes the proof of (a).

Now suppose (b) holds. We may assume without loss of generality that $X_0 = 1$. Let $\tau_k = \inf\{t\ge 0 : X_t \notin [1/k, k]\}$. Choose compactly supported $g^{(k)} \in C^{1,2}([0,T]\times(0,\infty))$ such that $g^{(k)} \equiv g$ on $[0,T]\times[1/k,k]$ and define $\theta_t^{(k)}$, $\varphi_t^{(k)}$ as in (2.2.1).

First note that $\varphi_t^{(k)} = g_t^{(k)}(t,X_t) + \frac{1}{2}\mu X_t g_{xx}^{(k)}(t,X_t)$ and $\theta_t^{(k)} = \sigma X_t g_x^{(k)}(t,X_t)$. Thus, since $g^{(k)}$ has compact support, part (a) implies

$$T_n^{(k)} = \sqrt{\frac{T}{n}}\sum_{j=1}^n X_{t_j}|g^{(k)}(t_j, X_{t_j}) - g^{(k)}(t_{j-1}, X_{t_{j-1}})| \to \sqrt{\frac{2}{\pi}}\int_0^T |X_t\theta_t^{(k)}|\sqrt{f'(t)}dt$$

in L^2 as $n \to \infty$. Next, note that $\tau_k \nearrow \infty$ a.s. and on $\{t \leq \tau_k\}$, $g(t, X_t) = g^{(k)}(t, X_t)$, $\theta_t = \theta_t^{(k)}$, and $\varphi_t = \varphi_t^{(k)}$.

Now let $\varepsilon > 0$ be given and choose k sufficiently large so that $P(T > \tau_k) < \varepsilon$. For this particular choice of k, choose n_0 such that for all $n \geq n_0$,

$$P\left(\left|T_n^{(k)} - \sqrt{\frac{2}{\pi}}\int_0^T |X_t\theta_t^{(k)}|\sqrt{f'(t)}dt\right| \geq \varepsilon\right) < \varepsilon.$$

It now follows that for any $n \geq n_0$,

$$\begin{aligned} P\left(\left|T_n - \sqrt{\frac{2}{\pi}}\int_0^T |X_t\theta_t|\sqrt{f'(t)}dt\right| \geq \varepsilon\right) &\leq P(T > \tau_k) + \\ & \quad P\left(\left|T_n^{(k)} - \sqrt{\frac{2}{\pi}}\int_0^T |X_t\theta_t^{(k)}|\sqrt{f'(t)}dt\right| \geq \varepsilon\right) \\ &< 2\varepsilon \end{aligned}$$

which proves (b). ∎

2.5 Proofs of Lemma 2.2.3 and Theorem 2.2.4

Proof of Lemma 2.2.3. First note that for $u \in S^k$, $k \geq 1$,

$$\begin{aligned} \left|u^{(k-1)}(e^x)\right| &= \left|\int_0^x e^t u^{(k)}(e^t)dt + u^{(k-1)}(1)\right| \\ &\leq C\left|\int_0^x e^t e^{a|t|}dt\right| + K \\ &\leq C\left|\int_0^x e^{b|t|}dt\right| + K \\ &= C\int_0^{|x|} e^{bt}dt + K \\ &= \frac{C}{b}(e^{b|x|} - 1) + K \\ &\leq \left(\frac{C}{b} + \left|K - \frac{C}{b}\right|\right)e^{b|x|} \end{aligned}$$

where $K = u^{(k-1)}(1)$ and $b = 1 + a$. Thus, $S^{k-1} \subset S^k$, and by induction, $S^j \subset S^k$ for all $0 \leq j \leq k$.

Now, define $h(t, x) = \tilde{E}_x[u(X_{s(t)})]$ and note that

$$h(t, x) = g\left(s(t), \frac{\log x}{\sigma} - \frac{\sigma s(t)}{2}\right),$$

where $g(t,x) = E_x[f(B_t)]$, $f(x) = u(e^{\sigma x})$. Since

$$\int_{-\infty}^{\infty} e^{-bx^2}|f(x)|dx \le C\int_{-\infty}^{\infty} e^{-bx^2}e^{a\sigma|x|}dx < \infty$$

for all $b > 0$, it follows that $g(t,x)$ is well-defined, has derivatives of all orders, and satisfies $g_t = \frac{1}{2}g_{xx}$ on $(0,\infty)\times\mathbb{R}$. Moreover, since f is continuous, $g \in C([0,\infty)\times\mathbb{R})$ with $g(0,x) = f(x)$. (See section 4.3 of [17].) Consequently, it can be verified that $h \in C([0,T]\times(0,\infty)) \cap C^{1,2}([0,T)\times(0,\infty))$ and satisfies (2.2.2) and (2.2.3).

Moreover,

$$\begin{aligned} |g(t,x)| &\le \frac{C}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{a\sigma|y|}e^{-(x-y)^2/2t}dy \\ &\le \frac{C}{\sqrt{2\pi t}}e^{a\sigma|x|}\int_{-\infty}^{\infty} e^{a\sigma|v|}e^{-v^2/2t}dv. \end{aligned}$$

Since $t \mapsto \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{a\sigma|v|}e^{-v^2/2t}dv$ is continuous on $[0,\infty)$, there exists $M < \infty$ such that $|g(t,x)| \le Me^{a\sigma|x|}$ for all $(t,x) \in [0,s(0)]\times\mathbb{R}$. Thus, for all $(t,x) \in [0,T]\times\mathbb{R}$,

$$\begin{aligned} |h(t,e^x)| &= \left|g\left(s(t), \frac{x}{\sigma} - \frac{\sigma s(t)}{2}\right)\right| \\ &\le M\exp\left(a\sigma\left|\frac{x}{\sigma} - \frac{\sigma s(t)}{2}\right|\right) \\ &\le \tilde{M}e^{a|x|}, \end{aligned}$$

where $\tilde{M} = M\exp(a\sigma^2 s(0)/2)$.

Furthermore, if $\tilde{h}$ is another such function, we may set $\tilde{g}(t,x) = h(z(t), \exp(\sigma x + \sigma^2 t/2))$, where $z = s^{-1}$, in which case Tychonoff's Uniqueness Theorem gives that $\tilde{g} \equiv g$ and, hence, $\tilde{h} \equiv h$.

Finally, fix $j \in \{0,\cdots,k\}$. By induction, there exists $c_i \in \mathbb{R}$ such that

$$\partial_x^j h(t,x) = \sum_{i=1}^{j} c_i x^{-j}\partial_x^i g\left(s(t), \frac{\log x}{\sigma} - \frac{\sigma s(t)}{2}\right).$$

Now, applying integration by parts to the results of problem 4.3.1 in [17] gives

$$\begin{aligned} \partial_x^i g(t,x) &= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} f^{(i)}(y)e^{-(x-y)^2/2t}dy \\ &= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}\left[\sum_{n=1}^{i} p_n(e^{\sigma y})u^{(n)}(e^{\sigma y})\right]e^{-(x-y)^2/2t}dy \end{aligned}$$

for some polynomials, p_n. Since $u \in S^n$ for all $n \leq k$, each $p_n(x)u^{(n)}(x) \in S^0$. Therefore, as above, each $\partial_x^i g \in C([0,\infty) \times \mathbb{R})$, so that $\partial_x^j h \in C([0,T] \times (0,\infty))$, which proves (a). Also, as above, there exists M_n, a_n such that

$$\left| \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} p_n(e^{\sigma y}) u^{(n)}(e^{\sigma y}) e^{-(x-y)^2/2t} dy \right| \leq M_n e^{a_n |x|}$$

for all $(t,x) \in [0, s(0)] \times \mathbb{R}$. Hence, $|\partial_x^i g(t,x)| \leq N_i e^{b_i |x|}$, for all $(t,x) \in [0,s(0)] \times \mathbb{R}$, where $b_i = \max\{a_1, \cdots, a_i\}$ and $N_i = M_1 + \cdots + M_i$. Finally, then, for all $(t,x) \in [0,T] \times \mathbb{R}$,

$$\begin{aligned} |\partial_x^j h(t, e^x)| &\leq e^{-jx} \sum_{i=1}^{j} c_i N_i \exp\left(b_i \left| \frac{x}{\sigma} - \frac{\sigma s(t)}{2} \right| \right) \\ &\leq e^{j|x|} \sum_{i=1}^{j} \tilde{N}_i e^{\tilde{b}_i |x|} \\ &\leq N e^{b|x|} \end{aligned}$$

where $b = j + \max\{\tilde{b}_1, \cdots, \tilde{b}_j\}$ and $N = \tilde{N}_1 + \cdots + \tilde{N}_j$, which proves (b). ∎

Proof of Theorem 2.2.4. By Lemma 2.2.3(a) and Theorem 2.2.2, it will suffice to show that

$$E \int_0^T (|\varphi_s|^q + |\theta_s|^q) ds < \infty$$

for some $q > 2$, where $\theta_t = \sigma X_t h_{xx}(t, X_t)$ and

$$\varphi_t = h_{xt}(t, X_t) + \mu X_t h_{xx}(t, X_t) + \frac{1}{2}\sigma^2 X_t^2 h_{xxx}(t, X_t).$$

Note that

$$\begin{aligned} h_{xt} &= (h_t)_x \\ &= \left(-\frac{1}{2}\sigma^2 w(t) x^2 h_{xx} \right)_x \\ &= -\sigma^2 w(t) x h_{xx} - \frac{1}{2}\sigma^2 w(t) x^2 h_{xxx} \end{aligned}$$

so that

$$\varphi_t = (\mu - \sigma^2 w(t)) X_t h_{xx}(t, X_t) + \frac{1}{2}\sigma^2 (1 - w(t)) X_t^2 h_{xxx}.$$

It will thus suffice to show that for any $p_1, p_2 > 0$ and $j \in \{2, 3\}$, $E\int_0^T X_s^{p_1}|\partial_x^j h(s, X_s)|^{p_2} ds < \infty$. To see this, observe that by Lemma 2.2.3(b),

$$\begin{aligned}
|\partial_x^j h(t,x)| &\leq Ke^{b|\log x|} \\
&\leq K(e^{b\log x} + e^{-b\log x}) \\
&= K(x^b + x^{-b}).
\end{aligned}$$

Thus,

$$E\int_0^T X_s^{p_1}|\partial_x^j h(s, X_s)|^{p_2} ds \leq K^{p_2} E\int_0^T (X_s^{p_1+bp_2} + X_s^{p_1-bp_2})ds,$$

which is finite. ■

2.6 Conclusions

What follows are three topics I find interesting and worthy of further investigation. They represent possible directions for my continued research in this area.

To apply the results given above to the field of mathematical finance, we could utilize Theorem 2.2.4, taking for h the solution to the Black-Scholes equation, which is none other than (2.2.2) and (2.2.3) with $w \equiv 1$. We could then estimate the transaction costs incurred as we try to hedge the option $u(X_T)$. However, in the presence of transaction costs, h no longer succeeds in hedging the option, since our wealth at time T will be $u(X_T)$ minus the overall cumulative transaction costs. In the case that u has a strictly positive second derivative, the absolute value bars in the integrand of (2.2.4) disappear, and Grannan and Swindle [14] show that in this case, the appropriate h to consider is the solution to (2.2.2), (2.2.3) with

$$w(t) = 1 + \frac{2}{\sigma}\sqrt{\frac{2}{\pi}}\sqrt{f'(t)},$$

and that the terminal wealth, V_n, obtained by a discrete hedge (and subtracting the scaled transaction costs) converges in L^2 to $u(X_T)$. (It should be noted that the case considered in [14] is actually not quite this general. There, it is further assumed that u is such that the function, h, which solves their equation satisfies

$$\|h\|_{m,n,p} \equiv \sup_{\substack{0\leq x \\ 0\leq t\leq T}} \left[x^m \frac{\partial^{n+p}h(t,x)}{\partial x^n \partial t^p}\right] < \infty$$

for all nonnegative integers m, n, and p.)

When u is not convex, the appropriate h to consider seems to be one that solves a nonlinear PDE involving $|\partial_x^2 h|$. Questions that come to mind are whether or not this PDE has a unique solution and whether or not the discrete hedges converge to the desired limit.

Another important question is what can be said about the rate of convergence in all of these various situations and how does it depend on f. Not only is this important because actual transaction costs cannot be made arbitrarily small, but also because the freedom to choose f implies the freedom to make the limit in (2.2.4) virtually anything we want. Again, in [14], results are given on the rate of convergence of the V_n under the special restrictions on u.

Finally, an interesting question not considered in [14] is what happens when we let f be a random function. Certainly, in applications, we would have the ability to choose our partition as we go, so we should require nothing more of f than that it be adapted. In fact, it might be even more interesting to eliminate f altogether and consider partitions that consist of increasing sequences of stopping times. In either case, the mesh size would be a random variable, which could converge to zero in different ways, making it unclear as to what the "right" way is to even formulate possible new theorems.

Chapter 3

THE MEDIAN OF INDEPENDENT BROWNIAN MOTIONS

3.1 Introduction

Define the median function, $\mathcal{M} : \mathbb{R}^n \to \mathbb{R}$, as follows: given $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a bijection such that $y_j = x_{\tau(j)}$ satisfy $y_1 \le \cdots \le y_n$, and define $\mathcal{M}(\mathbf{x}) = y_k$, where $k = \lfloor (n+1)/2 \rfloor$. (Here, $\lfloor \cdot \rfloor$ denotes the greatest integer, or floor, function.)

Let $\{B_t^{(j)}\}_{j=1}^\infty$ be a sequence of independent, standard, one-dimensional Brownian motions and for each $n \in \mathbb{N}$, let $M_t^{(n)} = \mathcal{M}(B_t^{(1)}, \ldots, B_t^{(n)})$ and $X_t^{(n)} = \sqrt{n} M_t^{(n)}$.

The space of continuous, real-valued function on $[0, \infty)$ is denoted by $C[0, \infty)$, and is endowed with the metric of uniform convergence on compact intervals. Namely, for ω_1, $\omega_2 \in C[0, \infty)$, the metric is given by

$$d(\omega_1, \omega_2) = \sum_{n=1}^\infty \frac{1}{2^n} \max_{0 \le t \le n} (|\omega_1(t) - \omega_2(t)| \wedge 1).$$

The σ-algebra generated by the open sets in this metric space will be denoted by $\mathcal{B}(C[0, \infty))$. The random variables, $X^{(n)} = \{X_t^{(n)} : 0 \le t < \infty\}$, take values in this space, and it will be shown that they converge in distribution to the process described in the following theorem.

Theorem 3.1.1 *There exists a continuous, centered Gaussian process, $X = \{X_t : 0 \le t < \infty\}$, with covariance*

$$E[X_s X_t] = \rho(s, t) = \sqrt{st}\ \sin^{-1}\left(\frac{s \wedge t}{\sqrt{st}}\right),$$

where $\sin^{-1}$ takes values in $[-\pi/2, \pi/2]$, and which is locally Hölder-continuous with exponent γ for every $\gamma \in (0, 1/4)$.

The proof of this theorem will be postponed until after we have investigated the convergence of the finite-dimensional distributions of the scaled median processes.

The main result of this article is the following.

Theorem 3.1.2 *Let X be as in Theorem 3.1.1. Let $\{X^{(n)}\}_{n=1}^{\infty}$ be the scaled median processes as in the discussion preceding Theorem 3.1.1. Then $\{X_t^{(2n+1)} : 0 \le t < \infty\} \xrightarrow{d} \{X_t : 0 \le t < \infty\}$ as $n \to \infty$.*

The restriction to odd integers in Theorem 3.1.2 is for convenience only. The probability estimates we shall derive are "one-sided", i.e. we shall derive an upper bound for $P(X_t^{(n)} - X_s^{(n)} > \varepsilon)$, but not for $P(X_t^{(n)} - X_s^{(n)} < -\varepsilon)$. The latter estimate follows from the former in the case that $X^{(n)}_\cdot \overset{d}{=} -X^{(n)}_\cdot$, which holds if n is odd. Clearly, analogous methods can be used to separately derive the latter estimate in the case that n is even and remove the restriction from Theorem 3.1.2.

The chief difficulty in proving the main result will be in establishing tightness. Let us first give the definition of tightness.

Definition 3.1.1 *Let (S, ρ) be a metric space and let Π be a family of probability measures on $(S, \mathcal{B}(S))$. We say that Π is tight if for every $\varepsilon > 0$, there exists a compact set $K \subset S$ such that $P(K) \ge 1 - \varepsilon$, for every $P \in \Pi$. If $\{X_\alpha\}_{\alpha \in A}$ is a family of random variables, each one defined on a probability space $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$ and taking values in S, we say that this family is tight if the family of induced measures $\{P_\alpha X_\alpha^{-1}\}_{\alpha \in A}$ is tight.*

The proof of Theorem 3.1.2 will rely on the following result, which is Theorem 2.4.15 in [17].

Theorem 3.1.3 *Let $\{X^{(n)}\}_{n=1}^{\infty}$ be a tight sequence of continuous processes with the property that, whenever $0 \le t_1 < \cdots < t_d < \infty$, the sequence of random vectors $\{(X_{t_1}^{(n)}, \ldots, X_{t_d}^{(n)})\}_{n=1}^{\infty}$ converges in distribution. Let P_n be the measure induced on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ by $X^{(n)}$. Then $\{P_n\}_{n=1}^{\infty}$ converges weakly to a measure P, under which the coordinate mapping process $W_t(\omega) = \omega(t)$ on $C[0,\infty)$ satisfies*

$$(X_{t_1}^{(n)}, \ldots, X_{t_d}^{(n)}) \xrightarrow{d} (W_{t_1}, \ldots, W_{t_d}), \;\; 0 \le t_1 < \cdots < t_d < \infty,\, d \ge 1.$$

By Theorem 3.1.3, the main result will follow from the following two theorems.

Theorem 3.1.4 *For every $0 \le t_1 < \cdots < t_d < \infty$, $d \ge 1$,*

$$(X_{t_1}^{(n)}, \ldots, X_{t_d}^{(n)}) \xrightarrow{d} (X_{t_1}, \ldots, X_{t_d}).$$

Theorem 3.1.5 *The sequence* $\{X_t^{(2n+1)}\}_{n=1}^{\infty}$ *is tight.*

The finite dimensional convergence, Theorem 3.1.4, is easy to establish. However, the tightness for $\{X_t^{(2n+1)} : 0 \leq t < \infty\}_{n=1}^{\infty}$ is much harder to prove. The major portion of this chapter is devoted to establishing Theorem 3.1.5.

3.2 Convergence of Finite Dimensional Distributions

The starting point for the proof of Theorem 3.1.4 is the following result, which is a special case of Theorems 7.1.1 and 7.1.2 in [22].

Theorem 3.2.1 (Multi-Dimensional Median Central Limit Theorem) *Let* $\xi^{(n)} \in \mathbb{R}^d$, $n \in \mathbb{N}$, *be iid random vectors. Let* $F_j(x) = P(\xi_j^{(1)} \leq x)$ *and* $G_{ij}(x,y) = P(\xi_i^{(1)} \leq x, \xi_j^{(1)} \leq y)$. *Let* $M^{(n)} \in \mathbb{R}^d$ *have components* $M_j^{(n)} = \mathcal{M}(\xi_j^{(1)}, \ldots, \xi_j^{(n)})$ *and* $\rho_{ij} = G_{ij}(0,0) - 1/4$. *If*

(i) $F_j(0) = 1/2$ *and* $F_j'(0) > 0$ *for all* j*; and*

(ii) G_{ij} *is continuous at* $(0,0)$ *for all* i, j,

then $\sqrt{n}M^{(n)} \xrightarrow{d} N$, *where* N *is multi-normal with mean zero and covariance* σ, *where*

$$\sigma_{ij} = EN_iN_j = \frac{\rho_{ij}}{F_i'(0)F_j'(0)}.$$

Corollary 3.2.2 *If* $\xi^{(1)}, \xi^{(2)}, \ldots$ *are iid, mean zero, multinormal,* $\mathbb{R}^d$*-valued random vectors with covariance* σ *and* $M^{(n)} \in \mathbb{R}^d$ *has components* $M_j^{(n)} = \mathcal{M}(\xi_j^{(1)}, \ldots, \xi_j^{(n)})$, *then* $\sqrt{n}M^{(n)} \xrightarrow{d} Z$, *where* Z *is multinormal with mean zero and covariance*

$$\tau_{ij} = EZ_iZ_j = \sqrt{\sigma_{ii}\sigma_{jj}}\ \sin^{-1}\left(\frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}\right).$$

(Here, $\sin^{-1}$ *takes values in* $[-\pi/2, \pi/2]$*.)*

Proof. By Theorem 3.2.1, $\sqrt{n}M^{(n)} \xrightarrow{d} Z$, where Z is multinormal with mean zero and covariance

$$\tau_{ij} = \frac{\rho_{ij}}{F_i'(0)F_j'(0)},$$

where $\rho_{ij} = P(\xi_i^{(1)} \leq 0, \xi_j^{(1)} \leq 0) - 1/4$ and

$$F_j(x) = P(\xi_j^{(1)} \leq x) = \frac{1}{\sqrt{2\pi\sigma_{jj}}} \int_{-\infty}^{x} e^{-t^2/2\sigma_{jj}}\, dt.$$

Since $F_j'(0) = (2\pi\sigma_{jj})^{-1/2}$, it remains only to show that

$$\rho_{ij} = \frac{1}{2\pi} \sin^{-1}\left(\frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}\right).$$

For notational simplicity, let $X = \xi_i^{(1)}$, $Y = \xi_j^{(1)}$ and define

$$\begin{aligned} a^{\pm} &= 1 \pm \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} \\ \tilde{X}^{\pm} &= \frac{1}{\sqrt{2a^{\pm}}}\left(\frac{1}{\sqrt{\sigma_{ii}}} X \pm \frac{1}{\sqrt{\sigma_{jj}}} Y\right), \end{aligned}$$

so that $\tilde{X}^+$, $\tilde{X}^-$ are independent standard normals. Since

$$\begin{aligned} X &= \frac{\sqrt{\sigma_{ii}}}{2}\left(\sqrt{2a^+}\tilde{X}^+ + \sqrt{2a^-}\tilde{X}^-\right) \\ Y &= \frac{\sqrt{\sigma_{jj}}}{2}\left(\sqrt{2a^+}\tilde{X}^+ - \sqrt{2a^-}\tilde{X}^-\right), \end{aligned}$$

we have that $X \leq 0$ and $Y \leq 0$ if and only if $(\tilde{X}^+, \tilde{X}^-)$ lies above the lines in the plane through the origin with slopes $\pm\sqrt{a^+/a^-}$. This sector of the plane has an angle, θ, that satisfies

$$\cos\theta = \frac{1 - a^+/a^-}{1 + a^+/a^-} = -\frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}, \;\; \theta \in [0, \pi].$$

Thus,

$$\begin{aligned} P(X \leq 0, Y \leq 0) &= \frac{\theta}{2\pi} \\ &= \frac{1}{2\pi} \cos^{-1}\left(-\frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}\right) \\ &= \frac{1}{4} + \frac{1}{2\pi} \sin^{-1}\left(\frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}\right), \end{aligned}$$

where $\sin^{-1}$ takes values in $[-\pi/2, \pi/2]$. ■

We may now prove Theorem 3.1.1, after which Theorem 3.1.4 is an immediate consequence of Corollary 3.2.2.

Proof of Theorem 3.1.1. Let $\rho(s,t)$ be as in the statement of Theorem 3.1.1. Let T be the set of finite sequences $\mathbf{t} = (t_1, \ldots, t_n)$ of distinct, nonnegative numbers, where the length n of these sequences ranges over the set of positive integers. For each $\mathbf{t}$ of length n, let $\mathbf{N_t} = (N_1, \ldots, N_n)$ be a multinormal random vector with mean zero and covariance $EN_iN_j = \rho(t_i, t_j)$. (By Corollary 3.2.2, with $\xi^{(1)} \stackrel{d}{=} (B^{(1)}_{t_1}, \ldots, B^{(1)}_{t_n})$, such an $\mathbf{N_t}$ exists.) Define the measure $Q_{\mathbf{t}}$ on $\mathbb{R}^n$ by $Q_{\mathbf{t}}(A) = P(\mathbf{N} \in A)$. The family of finite-dimensional distributions, $\{Q_{\mathbf{t}}\}_{\mathbf{t}\in T}$, is clearly consistent, so there exists a real-valued process, X_t, on $[0,\infty)$ that has the desired finite-dimensional distributions. It remains only to show that this process has a continuous modification, which is locally Hölder-continuous with exponent γ for every $\gamma \in (0, 1/4)$.

By the Kolmogorov-Čentsov Theorem (Theorem 2.2.8 in [17]), it will suffice to show that for every $\alpha > 4$ and every $T > 0$,

$$E|X_t - X_s|^\alpha \le C_T|t-s|^{\alpha/4}$$

for some $C_T > 0$ (depending only on T) and all $0 \le s < t \le T$.

First, observe that $X_t - X_s$ is normal with mean zero and variance

$$\begin{aligned}\sigma^2(s,t) &= E(X_t - X_s)^2 \\ &= EX_t^2 + EX_s^2 - 2EX_tX_s \\ &= \frac{\pi}{2}t + \frac{\pi}{2}s - 2\sqrt{st}\ \sin^{-1}\left(\sqrt{\frac{s}{t}}\right).\end{aligned}$$

An application of L'Hôpitals' Rule shows that

$$\frac{\pi/2 - \sin^{-1} x}{\sqrt{1-x^2}} \to 1$$

as $x \to 1$. Hence, for some constant C, $-\sin^{-1} x \le C\sqrt{1-x^2} - \pi/2$ for all $0 \le x \le 1$. Now let $x = s/t$. Then

$$\begin{aligned}\sigma^2(s,t) &= t\sigma^2(x,1) \\ &= t\left[\frac{\pi}{2} + \frac{\pi}{2}x - 2\sqrt{x}\sin^{-1}(\sqrt{x})\right].\end{aligned}$$

Using the above observation,

$$\begin{aligned}\sigma^2(s,t) &\le t\left[\frac{\pi}{2} + \frac{\pi}{2}x + 2\sqrt{x}\left(C\sqrt{1-x} - \frac{\pi}{2}\right)\right] \\ &= t\left[\frac{\pi}{2}(1-\sqrt{x})^2 + 2C\sqrt{x}\sqrt{1-x}\right].\end{aligned}$$

Since $1-\sqrt{x} \leq \sqrt{1-x}$ for $0 \leq x \leq 1$,

$$\begin{aligned} \sigma^2(s,t) &\leq \left(\frac{\pi}{2}+2C\right) t\sqrt{1-x} \\ &= \left(\frac{\pi}{2}+2C\right) \sqrt{t}\sqrt{t-s} \\ &\leq \tilde{C}_T |t-s|^{1/2} \end{aligned}$$

where $\tilde{C}_T = \left(\frac{\pi}{2}+2C\right)\sqrt{T}$.

Now, for every $\alpha > 0$, there is a constant K_α such that if N is normal with $EN = 0$, then $E|N|^\alpha = K_\alpha (EN^2)^{\alpha/2}$. Thus, for any $\alpha > 4$,

$$\begin{aligned} E|X_t - X_s|^\alpha &= K_\alpha |\sigma^2(s,t)|^{\alpha/2} \\ &\leq K_\alpha \tilde{C}_T^{\alpha/2} |t-s|^{\alpha/4} \\ &= C_T |t-s|^{\alpha/4} \end{aligned}$$

where $C_T = K_\alpha \tilde{C}_T^{\alpha/2}$. ■

3.3 General Tightness Criteria

The following result is often used to show tightness.

Proposition 3.3.1 *Let $\{X^{(n)}\}_{n=1}^\infty$ be a sequence of continuous stochastic processes $X^{(n)} = \{X_t^{(n)} : 0 \leq t < \infty\}$ on $(\Omega, \mathcal{F}, P)$, satisfying the following conditions:*

(i) $\sup_{n\geq 1} E|X_t^{(n)} - X_s^{(n)}|^\alpha \leq C_T |t-s|^{1+\beta}, \ \forall\, T > 0$ *and* $0 \leq s,t \leq T$,

(ii) $\sup_{n\geq 1} E|X_0^{(n)}|^\nu < \infty$

for some positive constants α, β, ν (universal) and C_T (depending on $T > 0$). Then the probability measures $P_n = P(X^{(n)})^{-1}$, $n \geq 1$ induced by these processes on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ form a tight sequence.

This proposition is Problem 2.4.11 in [17], which has a worked solution. An inspection of the proof reveals that condition (i) can be weakened. The version we shall employ here is the following.

Proposition 3.3.2 *Let* $\{X^{(n)}\}_{n=1}^{\infty}$ *be a sequence of continuous stochastic processes* $X^{(n)} = \{X_t^{(n)} : 0 \le t < \infty\}$ *on* $(\Omega, \mathcal{F}, P)$*, satisfying the following conditions:*

(i) $\sup_{n\ge1} P(|X_t^{(n)} - X_s^{(n)}| \ge \varepsilon) \le C_T \varepsilon^{-\alpha}|t-s|^{1+\beta}$, $\forall\ 0 < \varepsilon < 1$, $T > 0$, *and* $0 \le s, t \le T$,

(ii) $\sup_{n\ge1} E|X_0^{(n)}|^{\nu} < \infty$

for some positive constants α, β, ν *(universal) and* C_T *(depending on* $T > 0$*). Then the probability measures* $P_n = P(X^{(n)})^{-1}$, $n \ge 1$ *induced by these processes on* $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ *form a tight sequence.*

Proof. By Theorem 2.4.10 in [17], the measures P_n form a tight sequence if and only if

(a) $\lim_{\lambda\uparrow\infty} \sup_{n\ge1} P(|X_0^{(n)}| > \lambda) = 0$, and

(b) $\lim_{\delta\downarrow0} \sup_{n\ge1} P(m^T(X_\cdot^{(n)}, \delta) > \varepsilon) = 0\ \ \forall\ T > 0$ and $\varepsilon > 0$

where, for $f \in C[0,\infty)$,

$$m^T(f(\cdot), \delta) = \sup_{\substack{|t-s|\le\delta \\ s,t\in[0,T]}} |f(t) - f(s)|.$$

Since $P(|X_0^{(n)}| > \lambda) \le E|X_0^{(n)}|^{\nu}/\lambda^{\nu}$, condition (a) follows from condition (ii).

Now fix $T = 1$, $\varepsilon > 0$, and $n \in \mathbb{N}$. Let $\eta > 0$ be arbitrary. It will be shown that there exists $m_0 \in \mathbb{N}$, independent of n, such that for all $\delta \le 2^{-m_0}$,

$$P(m^T(X_\cdot^{(n)}, \delta) > \varepsilon) \le \eta,$$

which will prove condition (b) in the case $T = 1$.

For $l \in \mathbb{N}$, let

$$\Omega_l = \bigcap_{m=l}^{\infty} \left\{ \max_{1\le k\le 2^m} |X_{k/2^m}^{(n)} - X_{(k-1)/2^m}^{(n)}| < 2^{-\gamma m} \right\},$$

where $0 < \gamma < \beta/\alpha$. By (i), since $0 < 2^{-\gamma m} < 1$,

$$\begin{aligned} P(|X_{k/2^m}^{(n)} - X_{(k-1)/2^m}^{(n)}| \ge 2^{-\gamma m}) &\le C_1 2^{\gamma m\alpha} 2^{-m(1+\beta)} \\ &= C_1 2^{-m(1+\beta-\alpha\gamma)} \end{aligned}$$

Hence,

$$P\left(\max_{1\le k\le 2^m} |X^{(n)}_{k/2^m} - X^{(n)}_{(k-1)/2^m}| \ge 2^{-\gamma m}\right) = P\left(\bigcup_{k=1}^{2^m}\left\{|X^{(n)}_{k/2^m} - X^{(n)}_{(k-1)/2^m}| \ge 2^{-\gamma m}\right\}\right) \le C_1 2^{-m(\beta-\alpha\gamma)}.$$

and thus

$$P(\Omega_l^c) \le \sum_{m=l}^{\infty} C_1 2^{-m(\beta-\alpha\gamma)}.$$

Since $0 < \gamma < \beta/\alpha$, we have $\beta - \alpha\gamma > 0$ and we may therefore choose $l \in \mathbb{N}$ such that $P(\Omega_l^c) \le \eta$.

Now, for each $m \in \mathbb{N}$, let $D_m = \{k/2^m : 0 \le k \le 2^m\}$ and $D = \bigcup_{m=1}^{\infty} D_m$.

Claim: Fix $\omega \in \Omega_l$ and $m \ge l$. Then for every $h > m$,

$$|X^{(n)}_t(\omega) - X^{(n)}_s(\omega)| \le 2\sum_{j=m+1}^{h} 2^{-\gamma j}$$

whenever $s, t \in D_h$ and $0 \le t - s < 2^{-m}$.

Proof of Claim: The claim is trivial for $t - s = 0$, so suppose $t - s > 0$. If $h = m + 1$, then $0 < t - s < 2^{-m}$ implies that $t = k/2^h$ and $s = (k-1)/2^h$ for some $1 \le k \le 2^h$. Thus, $\omega \in \Omega_l$ and $h \ge l$ imply that $\max_{1\le k\le 2^h} |X^{(n)}_{k/2^h} - X^{(n)}_{(k-1)/2^h}| < 2^{-\gamma h}$ and the claim holds.

Now suppose the claim holds for all $h \in \{m+1, m+2, \ldots, M-1\}$. Let $h = M$ and suppose $s, t \in D_h$, $0 < t - s < 2^{-m}$. Let $s_1 = \min\{u \in D_{M-1} : u \ge s\}$ and $t_1 = \max\{u \in D_{M-1} : u \le t\}$. Then

$$|X^{(n)}_t(\omega) - X^{(n)}_s(\omega)| \le |X^{(n)}_t(\omega) - X^{(n)}_{t_1}(\omega)| + |X^{(n)}_{t_1}(\omega) - X^{(n)}_{s_1}(\omega)| + |X^{(n)}_{s_1}(\omega) - X^{(n)}_s(\omega)|.$$

Now, if $t_1 = t$, then $|X^{(n)}_t(\omega) - X^{(n)}_{t_1}(\omega)| = 0 < 2^{-\gamma M}$. If $t_1 \ne t$, then $t = k/2^M$ and $t_1 = (k-1)/2^M$ for some odd k with $1 \le k \le 2^M - 1$. Thus, as above, $\omega \in \Omega_l$ and $M \ge l$ imply that $|X^{(n)}_t(\omega) - X^{(n)}_{t_1}(\omega)| < 2^{-\gamma M}$. Similarly, $|X^{(n)}_{s_1}(\omega) - X^{(n)}_s(\omega)| < 2^{-\gamma M}$. By the induction hypothesis, since $0 \le t_1 - s_1 \le t - s < 2^{-m}$, we have $|X^{(n)}_{t_1}(\omega) - X^{(n)}_{s_1}(\omega)| \le 2\sum_{j=m+1}^{M-1} 2^{-\gamma j}$, and this proves the claim.

By the claim, for each fixed $\omega \in \Omega_l$ and $m \ge l$,

$$\sup_{\substack{|t-s|<2^{-m}\\ s,t\in D}} |X^{(n)}_t(\omega) - X^{(n)}_s(\omega)| \le 2\sum_{j=m+1}^{\infty} 2^{-\gamma j} \le \xi 2^{-\gamma m}$$

where $\xi = 2/(1-2^{-\gamma})$. Since $t \mapsto X_t^{(n)}(\omega)$ is continuous and D is dense in $[0,1]$, we have

$$\sup_{\substack{|t-s|<2^{-m} \\ s,t\in[0,1]}} |X_t^{(n)}(\omega) - X_s^{(n)}(\omega)| \le \xi 2^{-\gamma m}.$$

Now choose $m_0 \ge l$ such that $\xi 2^{-\gamma m_0} < \varepsilon$ and suppose $\delta \le 2^{-m_0}$. Then

$$\begin{aligned} P(m^T(X_\cdot^{(n)}, \delta) > \varepsilon) &= P\left(\sup_{\substack{|t-s|\le\delta \\ s,t\in[0,1]}} |X_t^{(n)} - X_s^{(n)}| > \varepsilon\right) \\ &\le P\left(\sup_{\substack{|t-s|\le 2^{-m_0} \\ s,t\in[0,1]}} |X_t^{(n)} - X_s^{(n)}| > \varepsilon\right). \end{aligned}$$

By the above,

$$\Omega_l \subset \left\{ \sup_{\substack{|t-s|<2^{-m_0} \\ s,t\in[0,1]}} |X_t^{(n)} - X_s^{(n)}| \le \xi 2^{-\gamma m_0} \right\} \subset \left\{ \sup_{\substack{|t-s|<2^{-m_0} \\ s,t\in[0,1]}} |X_t^{(n)} - X_s^{(n)}| \le \varepsilon \right\},$$

so that $P(m^T(X_\cdot^{(n)}, \delta) > \varepsilon) \le P(\Omega_l^c) \le \eta$, which completes the proof for the case $T = 1$.

By observing that

$$\begin{aligned} m^T(f(\cdot), \delta) &= \sup_{\substack{|t-s|\le\delta \\ s,t\in[0,T]}} |f(t) - f(s)| \\ &= \sup_{\substack{|t-s|\le\delta/T \\ s,t\in[0,1]}} |f(Tt) - f(Ts)| \\ &= m^1(f(T\cdot), \delta/T), \end{aligned}$$

the general case now follows. ■

For any real number $c \ge 0$, $\mathcal{M}(c\mathbf{x}) = c\mathcal{M}(\mathbf{x})$, so that the median processes inherit the scaling property of Brownian motion; namely, $\{X_{ct}^{(n)} : 0 \le t < \infty\} \stackrel{d}{=} \{\sqrt{c}X_t^{(n)} : 0 \le t < \infty\}$. In applying Proposition 3.3.2, we should like to make use of this scaling property. To this end, we once again reformulate the tightness criteria as follows.

Proposition 3.3.3 *Let $\{X^{(n)}\}_{n=1}^\infty$ be a sequence of continuous stochastic processes $X^{(n)} = \{X_t^{(n)} : 0 \le t < \infty\}$ on $(\Omega, \mathcal{F}, P)$. Suppose there exists $r > 0$ such that for every $c \ge 0$, $\{X_{ct}^{(n)} : 0 \le t < \infty\} \stackrel{d}{=} \{c^r X_t^{(n)} : 0 \le t < \infty\}$. Suppose that* *(i)* $\sup_{n\geq 1} P(|X^{(n)}_{1+\delta} - X^{(n)}_1| > \varepsilon) \leq C\varepsilon^{-\alpha}\delta^{1+\beta}, \ \forall\, 0 < \varepsilon < 1$ *and* $0 < \delta < \delta_0$

for some positive constants δ_0, C, α, *and* β. *Define* $\gamma = \min\{(\alpha \wedge \beta)r, 1+\beta\}$. *If* $\gamma > 1$ *and*

(ii) $\sup_{n\geq 1} E|X^{(n)}_1|^{\gamma/r} < \infty$

then the probability measures $P_n = P(X^{(n)})^{-1}$, $n \geq 1$ *induced by these processes on* $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ *form a tight sequence.*

Proof. Since $X^{(n)}_0 \equiv 0$, it suffices by Proposition 3.3.2 to show that

$$\sup_{n\geq 1} P(|X^{(n)}_t - X^{(n)}_s| \geq \varepsilon) \leq C_T \varepsilon^{-(\alpha\vee\beta)}|t-s|^{\gamma} \tag{3.3.1}$$

for all $0 < \varepsilon < 1$, $T > 0$, and $0 \leq s, t \leq T$.

First, suppose that $\varepsilon \geq 1$ and $0 < \delta < \delta_0$. Choose $m \in \mathbb{N}$ such that $\frac{1}{2} \leq \frac{\varepsilon}{m} < 1$. For $j \in \{0, \ldots, m\}$, define $t_j = j\delta/m$. Then

$$P(|X^{(n)}_{1+\delta} - X^{(n)}_1| > \varepsilon) \leq \sum_{j=1}^{m} P(|X^{(n)}_{1+t_j} - X^{(n)}_{1+t_{j-1}}| > \varepsilon/m).$$

If $c_j = (1+t_{j-1})^{-1}$, then

$$\begin{aligned} X^{(n)}_{1+t_j} - X^{(n)}_{1+t_{j-1}} &= (1+t_{j-1})^r(c_j^r X^{(n)}_{1+t_j} - c_j^r X^{(n)}_{1+t_{j-1}}) \\ &\stackrel{d}{=} (1+t_{j-1})^r(X^{(n)}_{c_j(1+t_j)} - X^{(n)}_1) \\ &= (1+t_{j-1})^r(X^{(n)}_{1+\tau_j} - X^{(n)}_1) \end{aligned}$$

where $1+\tau_j = (1+t_j)/(1+t_{j-1})$, i.e. $\tau_j = \delta/[m(1+t_{j-1})]$. Thus,

$$\begin{aligned} P(|X^{(n)}_{1+\delta} - X^{(n)}_1| > \varepsilon) &\leq \sum_{j=1}^{m} P\left(|X^{(n)}_{1+\tau_j} - X^{(n)}_1| > \frac{\varepsilon}{m(1+t_{j-1})^r}\right) \\ &\leq \sum_{j=1}^{m} C\left(\frac{\varepsilon}{m(1+t_{j-1})^r}\right)^{-\alpha} \tau_j^{1+\beta} \\ &\leq \sum_{j=1}^{m} C\left(\frac{1}{2(1+\delta_0)^r}\right)^{-\alpha}\left(\frac{\delta}{m}\right)^{1+\beta} \end{aligned}$$

Let $\tilde{C} = C2^{\alpha}(1+\delta_0)^{r\alpha}$ and observe that $m > \varepsilon$. Hence,

$$\begin{aligned} \sum_{j=1}^{m} C\left(\frac{1}{2(1+\delta_0)^r}\right)^{-\alpha}\left(\frac{\delta}{m}\right)^{1+\beta} &= \tilde{C} m^{-\beta}\delta^{1+\beta} \\ &\leq \tilde{C}\varepsilon^{-\beta}\delta^{1+\beta}. \end{aligned}$$

We have now proven

(i)′ $\sup_{n\geq 1} P(|X^{(n)}_{1+\delta} - X^{(n)}_1| > \varepsilon) \leq \tilde{C}\varepsilon^{-\beta}\delta^{1+\beta}$, $\forall\, \varepsilon \geq 1$ and $0 < \delta < \delta_0$.

Combining (i) and (i)′ gives

(i)″ $\sup_{n\geq 1} P(|X^{(n)}_{1+\delta} - X^{(n)}_1| > \varepsilon) \leq K(\varepsilon^{-\alpha} + \varepsilon^{-\beta})\delta^{1+\beta}$, $\forall\, \varepsilon > 0$ and $0 < \delta < \delta_0$,

where $K = \max\{C, \tilde{C}\}$.

Now let $0 < \varepsilon < 1$, $T > 0$, $0 \leq s < t \leq T$, and $n \in \mathbb{N}$. Define $\rho = 1/(1+\delta_0)$. We will prove (3.3.1) by considering two separate cases.

Case 1: Suppose $s > \rho t$. Then

$$\begin{aligned} X^{(n)}_t - X^{(n)}_s &= s^r(s^{-r}X^{(n)}_t - s^{-r}X^{(n)}_s) \\ &\overset{d}{=} s^r(X^{(n)}_{t/s} - X^{(n)}_1) \\ &= s^r(X^{(n)}_{1+\delta} - X^{(n)}_1) \end{aligned}$$

where $1+\delta = t/s$, i.e. $\delta = (t-s)/s < (1-\rho)/\rho = \delta_0$. In what follows, C_T will be a constant that depends on T (but not on ε, s, t, or n) that may change value from line to line.

We now have

$$\begin{aligned} P(|X^{(n)}_t - X^{(n)}_s| > \varepsilon) &= P(|X^{(n)}_{1+\delta} - X^{(n)}_1| > \varepsilon s^{-r}) \\ &\leq K(s^{r\alpha}\varepsilon^{-\alpha} + s^{r\beta}\varepsilon^{-\beta})\delta^{1+\beta} \\ &\leq C_T s^{(\alpha\wedge\beta)r}\varepsilon^{-(\alpha\vee\beta)}\delta^{1+\beta} \\ &= C_T s^{(\alpha\wedge\beta)r-1-\beta}\varepsilon^{-(\alpha\vee\beta)}|t-s|^{1+\beta} \end{aligned} \tag{3.3.2}$$

If $(\alpha\wedge\beta)r - 1 - \beta \geq 0$, then (3.3.2) and $s < T$ imply

$$\begin{aligned} P(|X^{(n)}_t - X^{(n)}_s| > \varepsilon) &\leq C_T\varepsilon^{-(\alpha\vee\beta)}|t-s|^{1+\beta} \\ &\leq C_T\varepsilon^{-(\alpha\vee\beta)}|t-s|^{\gamma} \end{aligned}$$

If $(\alpha\wedge\beta)r - 1 - \beta < 0$, then (3.3.2) and $s > \rho t \geq \rho|t-s|$ imply

$$\begin{aligned} P(|X^{(n)}_t - X^{(n)}_s| > \varepsilon) &\leq C_T\varepsilon^{-(\alpha\vee\beta)}|t-s|^{(\alpha\wedge\beta)r} \\ &\leq C_T\varepsilon^{-(\alpha\vee\beta)}|t-s|^{\gamma}. \end{aligned}$$

Case 2: Suppose $s \le \rho t$. Then $t = (t-s) + s \le (t-s) + \rho t$ implies $t \le \tau(t-s)$, where $\tau = 1/(1-\rho) = (1/\delta_0) + 1$. Now,

$$\begin{aligned} P(|X_t^{(n)} - X_s^{(n)}| > \varepsilon) &\le P(|X_t^{(n)}| + |X_s^{(n)}| > \varepsilon) \\ &\le P\left(|X_t^{(n)}| > \frac{\varepsilon}{2}\right) + P\left(|X_s^{(n)}| > \frac{\varepsilon}{2}\right) \\ &= P\left(|X_1^{(n)}| > \frac{\varepsilon}{2t^r}\right) + P\left(|X_1^{(n)}| > \frac{\varepsilon}{2s^r}\right) \\ &\le 2P\left(|X_1^{(n)}| > \frac{\varepsilon}{2t^r}\right). \end{aligned}$$

Using Chebyshev, condition (ii), and the fact that $t < \tau(t-s)$ gives

$$\begin{aligned} P(|X_t^{(n)} - X_s^{(n)}| > \varepsilon) &\le 2\left(\frac{2t^r}{\varepsilon}\right)^{\gamma/r} E|X_1^{(n)}|^{\gamma/r} \\ &\le C_T \varepsilon^{-\gamma/r} t^\gamma \\ &\le C_T \varepsilon^{-\gamma/r} |t-s|^\gamma. \end{aligned}$$

Finally, $\gamma/r \le (\alpha \wedge \beta) \le (\alpha \vee \beta)$, so

$$P(|X_t^{(n)} - X_s^{(n)}| > \varepsilon) \le C_T \varepsilon^{-(\alpha\vee\beta)} |t-s|^\gamma,$$

and this completes the proof. ∎

3.4 Median Estimates, Part I

With the general tightness criteria in place, we are now ready to formulate specific estimates on the median processes themselves. We begin by verifying condition (ii) of Proposition 3.3.3.

Lemma 3.4.1 *For all $x > 0$, $\sqrt{2\pi}\Phi(-x) \le x^{-1}e^{-x^2/2}$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$.*

Proof. Write $\sqrt{2\pi}\Phi(-x) = \int_x^\infty t^{-1} \cdot te^{-t^2/2}\,dt$ and integrate by parts. ∎

Proposition 3.4.2 *Let $M_t^{(n)}$ be as in the discussion preceding Theorem 3.1.1 and let $k = \lfloor (n+1)/2 \rfloor$. Then $M_1^{(n)}$ has density function*

$$f_n(x) = k\binom{n}{k}\frac{1}{\sqrt{2\pi}}\Phi(x)^{k-1}\Phi(-x)^{n-k}e^{-x^2/2}.$$

Proof. This is a special case of Theorem 1.3.2 in [22]. ∎

Theorem 3.4.3 (Burkholder's inequality) *Let $\{\xi_j\}_{j=1}^\infty$ be a sequence of independent random variables with $E\xi_j = 0$ for all j. Then for each $p > 1$, there exists a constant $C_p < \infty$, depending only on p, such that for all $n \in \mathbb{N}$,*

$$E\left|\sum_{j=1}^n \xi_j\right|^p \le C_p E\left|\sum_{j=1}^n |\xi_j|^2\right|^{\frac{p}{2}}.$$

Proof. This is an immediate consequence of Theorem 6.3.10 in [25]. ∎

Proposition 3.4.4 *Let $X_t^{(n)}$ be as in the discussion preceding Theorem 3.1.1. Then for each $p > 2$, there exists a constant $C_p < \infty$ such that for all $y > 0$ and all $n \in \mathbb{N}$,*

$$P(|X_1^{(n)}| > y) \le C_p y^{-p}.$$

Proof. First, suppose $y \ge 2\sqrt{n}$. By Proposition 3.4.2,

$$\begin{aligned} P(X_1^{(n)} < -y) &= P(M_1^{(n)} < -y/\sqrt{n}) \\ &= \frac{n!}{(n-k)!(k-1)!}\int_{-\infty}^{-y/\sqrt{n}} \Phi(x)^{k-1}\Phi(-x)^{n-k}\Phi'(x)\,dx \\ &\le \frac{n^k}{(k-1)!}\int_{-\infty}^{-y/\sqrt{n}} \Phi(x)^{k-1}\Phi'(x)\,dx \\ &= \frac{n^k}{k!}\Phi(-y/\sqrt{n})^k. \end{aligned}$$

By Stirling's formula, there exists a universal constant $C < \infty$ such that $k! \ge \frac{1}{C}k^k e^{-k}$. Moreover, by Lemma 3.4.1, $\Phi(-x) \le e^{-x^2/2}$ for $x \ge 1$. Thus, since $k \ge n/2$,

$$\begin{aligned} P(X_1^{(n)} < -y) &\le C\frac{n^k}{k^k e^{-k}}(e^{-y^2/(2n)})^k \\ &\le C[(2e)(e^{-y^2/(2n)})]^k. \end{aligned}$$

Since $(2e)(e^{-y^2/(2n)}) \leq (2e)(e^{-2}) < 1$,

$$\begin{aligned} P(X_1^{(n)} < -y) &\leq C[(2e)(e^{-y^2/(2n)})]^{n/2} \\ &= C(2e)^{n/2}e^{-y^2/4} \\ &\leq C(2e)^{y^2/8}e^{-y^2/4} \\ &= C\left(\frac{e}{2}\right)^{-y^2/8} \\ &\leq C_p y^{-p}. \end{aligned}$$

Now, suppose $y < 2\sqrt{n}$. In this case,

$$\begin{aligned} P(X_1^{(n)} < -y) &= P(M_1^{(n)} < -y/\sqrt{n}) \\ &= P\left(\sum_{j=1}^{n} 1_{\{B_1^{(j)} < -y/\sqrt{n}\}} \geq k\right) \\ &= P\left(\sum_{j=1}^{n} 1_{\{B_1^{(j)} < -y/\sqrt{n}\}} \geq \frac{n}{2}\right) \\ &= P\left(\sum_{j=1}^{n} \xi_j \geq n\,(1/2 - q)\right), \end{aligned}$$

where $q = \Phi(-y/\sqrt{n})$ and $\xi_j = 1_{\{B_1^{(j)} < -y/\sqrt{n}\}} - q$. By Theorem 3.4.3 and Jensen's inequality,

$$\begin{aligned} E\left|\sum_{j=1}^{n} \xi_j\right|^p &\leq C_p E\left|\sum_{j=1}^{n} |\xi_j|^2\right|^{\frac{p}{2}} \\ &\leq C_p n^{p/2} E \sum_{j=1}^{n} \frac{1}{n}|\xi_j|^p \\ &\leq C_p n^{p/2}. \end{aligned}$$

Thus, by Chebyshev,

$$\begin{aligned} P(X_1^{(n)} < -y) &\leq C_p n^{p/2}[n(1/2 - q)]^{-p} \\ &= C_p[\sqrt{n}(1/2 - q)]^{-p}. \end{aligned}$$

Now,

$$\sqrt{n}(1/2 - q) = \frac{\sqrt{n}}{\sqrt{2\pi}} \int_0^{y/\sqrt{n}} e^{-x^2/2}\,dx \geq \frac{y}{\sqrt{2\pi}} e^{-y^2/2n} \geq \frac{y}{\sqrt{2\pi}} e^{-2}.$$

Combining these results gives that, for all $y > 0$, $P(X_1^{(n)} < -y) \le C_p y^{-p}$.

Finally, note that $\mathcal{M}(-\mathbf{x}) \le -\mathcal{M}(\mathbf{x})$. Thus, if $\tilde{M}_t^{(n)} = \mathcal{M}(-B_t^{(1)}, \ldots, -B_t^{(n)})$, then $M_1^{(n)} \overset{d}{=} \tilde{M}_1^{(n)} \le -M_1^{(n)}$. Hence,

$$\begin{aligned} P(X_1^{(n)} > y) &= P(M_1^{(n)} > y/\sqrt{n}) \\ &= P(\tilde{M}_1^{(n)} > y/\sqrt{n}) \\ &\le P(-M_1^{(n)} > y/\sqrt{n}) \\ &= P(M_1^{(n)} < -y/\sqrt{n}) \\ &= P(X_1^{(n)} < -y), \end{aligned}$$

which completes the proof. ■

Corollary 3.4.5 *For each $p > 0$, $\sup_{n\ge 1} E|X_1^{(n)}|^p < \infty$.*

Proof. For any $p > 0$,

$$E|X_1^{(n)}|^p = \int_0^\infty p y^{p-1} P(|X_1^{(n)}| > y)\, dy$$

by Lemma 1.5.7 in [10]. By Jensen's inequality, we may assume without loss of generality that $p > 1$. Thus, by Proposition 3.4.4,

$$\begin{aligned} E|X_1^{(n)}|^p &\le \int_0^1 p y^{p-1}\, dy + \int_1^\infty p y^{p-1} C_{p+1} y^{-(p+1)}\, dy \\ &= 1 + pC_{p+1} \int_1^\infty y^{-2}\, dy, \end{aligned}$$

which is finite. ■

Now that we have verified condition (ii) of Proposition 3.3.3, we must turn to condition (i), which is more challenging to establish. The ultimate goal will be to prove the following result.

Proposition 3.4.6 *Let $X_t^{(n)}$ be as in the discussion preceding Theorem 3.1.1. Then there exists a constant $\delta_0 > 0$ and a family of finite, positive constants $\{C_p\}_{p>2}$ such that for each $p > 2$,*

$$\sup_{n\ge 3} P(X_{1+\delta}^{(n)} - X_1^{(n)} > \varepsilon) \le C_p(\varepsilon^{-1}\delta^{1/6})^p \tag{3.4.1}$$

for all $0 < \varepsilon < 1$ and $0 < \delta < \delta_0$.

Remark 3.4.1 *The proof of Theorem 3.1.1 suggests that the right hand side of (3.4.1) could be replaced by* $C_p(\varepsilon^{-1}\delta^{1/4})^p$*. For technical reasons related to the method of proof, this sharp bound could not be obtained. However, the choice of* $1/6$ *as the exponent in (3.4.1) appears to be arbitrary. Presumably, with minor modifications, the right hand side of (3.4.1) could be replaced by* $C_p(\varepsilon^{-1}\delta^{\nu})^p$ *for any fixed* $\nu < 1/4$*.*

Once we establish this result, Theorem 3.1.5, and hence Theorem 3.1.2, will follow.

Proof of Theorem 3.1.5, given Proposition 3.4.6. First observe that $\{X^{(2n+1)}\}_{n=1}^{\infty}$ satisfy the initial hypothesis of Proposition 3.3.3 with $r = 1/2$. Now choose any $p > 18$. Let $\alpha = p$, $\beta = (p/6) - 1 > 2$, and check that $\gamma = \min\{(\alpha \wedge \beta)r, 1 + \beta\} = \beta/2 > 1$. Let δ_0 be as in Proposition 3.4.6 and let $0 < \varepsilon < 1$, $0 < \delta < \delta_0$ be arbitrary. Using Proposition 3.4.6 and the fact that $X_\cdot^{(2n+1)} \overset{d}{=} -X_\cdot^{(2n+1)}$,

$$
\begin{aligned}
\sup_{n\geq 1} P(|X_{1+\delta}^{(2n+1)} - X_1^{(2n+1)}| > \varepsilon) &= 2\sup_{n\geq 1} P(X_{1+\delta}^{(2n+1)} - X_1^{(2n+1)} > \varepsilon) \\
&\leq 2\sup_{n\geq 3} P(X_{1+\delta}^{(n)} - X_1^{(n)} > \varepsilon) \\
&\leq 2C_p(\varepsilon^{-1}\delta^{1/6})^p \\
&= 2C_p\varepsilon^{-\alpha}\delta^{1+\beta}.
\end{aligned}
$$

This verifies condition (i) of Proposition 3.3.3; condition (ii) is given by Corollary 3.4.5. ■

So it only remains to prove Proposition 3.4.6. The starting point in the proof is an investigation of the conditional distribution of $M_{1+\delta}^{(n)} - M_1^{(n)}$ given $M_1^{(n)}$. Informally, if $M_1^{(n)} = x$, then we know that one of $B_1^{(1)}, \ldots, B_1^{(n)}$ is equal to x and of the remaining $n-1$, $k-1$ of them (recall that $k = \lfloor (n+1)/2 \rfloor$) are less than x and $n-k$ of them are greater than x. The knowledge that $M_1^{(n)} = x$ gives us no other information than that. So we may regard $B_1^{(1)}, \ldots, B_1^{(n)}$ as consisting of one Brownian particle located at x, a group of $k-1$ iid Brownian particles conditioned to lie below x at time $t = 1$, and a group of $n-k$ iid Brownian particles conditioned to lie above x at time $t = 1$. Presumably, these heuristics could be used to derive a precise formulation of the conditional distribution of $M_{1+\delta}^{(n)} - M_1^{(n)}$ given $M_1^{(n)}$. For our purposes, however, an inequality will suffice.

To formulate this inequality, first recall that by Proposition 3.4.2, $M_1^{(n)}$ has a smooth density function, $f_n(x)$. Now, for $x, y \in \mathbb{R}$ and $\delta > 0$, define

$$\begin{aligned} p_1 = p_1(x,y,\delta) &= P\left(B_{1+\delta}^{(1)} < x + y \,|\, B_1^{(1)} < x\right) \\ &= \frac{1}{\Phi(x)} \int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right) \Phi'(t)\, dt. \end{aligned}$$

Let $p_2 = p_2(x,y,\delta) = p_1(-x,-y,\delta)$ and $q_j = 1 - p_j$. For each $k \in \mathbb{N}$, let

$$\varphi_k(x,y,\delta) = \sum_{i=0}^{k} \sum_{j=i}^{k} \binom{k}{i} \binom{k}{j} p_1^i q_1^{k-i} p_2^j q_2^{k-j}. \tag{3.4.2}$$

Proposition 3.4.7 *Let $n \geq 3$ and $k = \lfloor (n+1)/2 \rfloor \geq 2$. With the notation above, we have that for all $\delta > 0$ and all $y > 0$,*

$$P(M_{1+\delta}^{(n)} - M_1^{(n)} > y) \leq \int_{-\infty}^{\infty} \varphi_{k-1}(x,y,\delta) f_n(x)\, dx.$$

Proof. Fix $\delta > 0$ and $y > 0$. Let $M \in \mathbb{N}$ and let $h > 0$ with $M/h \in \mathbb{N}$. Then

$$\begin{aligned} P(M_{1+\delta}^{(n)} - M_1^{(n)} > y,\, |M_1^{(n)}| \leq M) &\leq \sum_{\substack{x \in h\mathbb{Z} \\ |x| \leq M}} P\left(M_{1+\delta}^{(n)} - M_1^{(n)} > y,\, M_1^{(n)} \in [x, x+h)\right) \\ &\leq \sum_{\substack{x \in h\mathbb{Z} \\ |x| \leq M}} P\left(M_{1+\delta}^{(n)} > x + y,\, M_1^{(n)} \in [x, x+h)\right). \end{aligned}$$

Let $\mathcal{S}_n = \{1, \ldots, n\}$, $\mathcal{I} = \{I \subset \mathcal{S}_n : |I| = k-1\}$, $S = \{(I,j) : I \in \mathcal{I}, j \in \mathcal{S}_n \setminus I\}$, and for $(I,j) \in S$, let $I'(j) = \mathcal{S}_n \setminus (I \cup \{j\})$.

For $(I,j) \in S$, $x \in \mathbb{R}$, and $h > 0$, define

$$A(I,j,x,h) = \{B_1^{(j)} \in [x, x+h)\} \cap \{B_1^{(i)} < B_1^{(j)},\, \forall i \in I\} \cap \{B_1^{(i)} > B_1^{(j)},\, \forall i \in I'(j)\}.$$

Note that, up to a set of measure zero,

$$\{M_1^{(n)} \in [x, x+h)\} = \bigcup_{(I,j) \in S} A(I,j,x,h),$$

and that this is a disjoint union. Thus,

$$P\left(M_{1+\delta}^{(n)} > x + y,\, M_1^{(n)} \in [x, x+h)\right) = \sum_{(I,j) \in S} P\left(M_{1+\delta}^{(n)} > x + y,\, A(I,j,x,h)\right).$$

For $(I, j) \in S$, $x \in \mathbb{R}$, and $h > 0$, define

$$\tilde{A}(I, j, x, h) = \{B_1^{(j)} \in [x, x+h)\} \cap \{B_1^{(i)} < x+h, \forall i \in I\} \cap \{B_1^{(i)} > x, \forall i \in I'(j)\}$$

and observe that $A \subset \tilde{A}$, so that

$$P\left(M_{1+\delta}^{(n)} > x+y,\, A(I, j, x, h)\right) \leq P\left(M_{1+\delta}^{(n)} > x+y,\, \tilde{A}(I, j, x, h)\right).$$

Now fix $(I, j) \in S$ and $x \in \mathbb{R}$. Define

$$\begin{aligned} N_1 &= \sum_{i \in I} 1_{\{B_{1+\delta}^{(i)} < x+y\}} \\ N_2 &= \sum_{i \in I'(j)} 1_{\{B_{1+\delta}^{(i)} > x+y\}} \\ N &= \sum_{i=1}^{n} 1_{\{B_{1+\delta}^{(i)} > x+y\}} \end{aligned}$$

and note that $\{M_{1+\delta}^{(n)} > x+y\} = \{N \geq n-k+1\}$. Also note that, up to a set of measure zero,

$$\begin{aligned} N &= N_2 + (k-1) - N_1 + 1_{\{B_{1+\delta}^{(j)} > x+y\}} \\ &\leq N_2 - N_1 + k. \end{aligned}$$

Thus, if $d(n) = n - 2k + 1 = [(-1)^n + 1]/2$, then

$$\{M_{1+\delta}^{(n)} > x+y\} \subset \{N_2 - N_1 \geq d(n)\}.$$

This gives

$$\begin{aligned} P\left(M_{1+\delta}^{(n)} > x+y,\, \tilde{A}(I, j, x, h)\right) &\leq P\left(N_2 - N_1 \geq d(n),\, \tilde{A}(I, j, x, h)\right) \\ &= \sum_{l=0}^{k-1} \sum_{m=d(n)+l}^{n-k} P(N_1 = l,\, N_2 = m,\, \tilde{A}(I, j, x, h)) \\ &= \sum_{l=0}^{k-1} \sum_{m=d(n)+l}^{n-k} P(B_1^{(j)} \in [x, x+h)) P_l^1 P_m^2, \end{aligned}$$

where

$$\begin{aligned} P_l^1 &= P\left(\{N_1 = l\} \cap \{B_1^{(i)} < x+h, \forall i \in I\}\right) \\ P_m^2 &= P\left(\{N_2 = m\} \cap \{B_1^{(i)} > x, \forall i \in I'(j)\}\right). \end{aligned}$$

By symmetry and independence,

$$P_l^1 = \binom{k-1}{l}[\psi(x+y,x+h)]^l[\Phi(x+h)-\psi(x+y,x+h)]^{k-1-l},$$

where $\psi(x,y) = P(B_{1+\delta}^{(1)} < x,\, B_1^{(1)} < y)$. Similarly,

$$P_m^2 = \binom{n-k}{m}[\psi(-x-y,-x)]^m[\Phi(-x)-\psi(-x-y,-x)]^{n-k-m}.$$

Now, observe that

$$P(\tilde{A}(I,j,x,h)) = P(B_1^{(j)} \in [x,x+h))\Phi(x+h)^{k-1}\Phi(-x)^{n-k}.$$

Also,

$$\frac{\psi(x+y,x+h)}{\Phi(x+h)} = P(B_{1+\delta}^{(1)} < x+y \,|\, B_1^{(1)} < x+h) = p_{1,h}$$

where $p_{1,h} = p_1(x+h, y-h, \delta)$. Similarly,

$$\frac{\psi(-x-y,-x)}{\Phi(-x)} = P(B_{1+\delta}^{(1)} > x+y \,|\, B_1^{(1)} > x) = p_2.$$

Thus, if $q_{1,h} = 1 - p_{1,h}$, then

$$P\left(M_{1+\delta}^{(n)} > x+y \,|\, \tilde{A}(I,j,x,h)\right) \le \sum_{l=0}^{k-1}\sum_{m=d(n)+l}^{n-k}\binom{k-1}{l}\binom{n-k}{m}p_{1,h}^l q_{1,h}^{k-1-l}p_2^m q_2^{n-k-m}.$$

If n is odd, then $d(n) = 0$ and $n-k = k-1$, so

$$P\left(M_{1+\delta}^{(n)} > x+y \,|\, \tilde{A}(I,j,x,h)\right) \le \varphi_{k-1}^h(x,y,\delta) \tag{3.4.3}$$

where

$$\varphi_k^h(x,y,\delta) = \sum_{i=0}^{k}\sum_{j=i}^{k}\binom{k}{i}\binom{k}{j}p_{1,h}^i q_{1,h}^{k-i}p_2^j q_2^{k-j}.$$

If n is even, then $d(n) = 1$ and $n-k = k$, so

$$\begin{aligned} P\left(M_{1+\delta}^{(n)} > x+y \,|\, \tilde{A}(I,j,x,h)\right) &\le \sum_{l=0}^{k-1}\sum_{m=l+1}^{k}\binom{k-1}{l}\binom{k}{m}p_{1,h}^l q_{1,h}^{k-1-l}p_2^m q_2^{k-m} \\ &= \sum_{l=0}^{k-1}\binom{k-1}{l}p_{1,h}^l q_{1,h}^{k-1-l}\sum_{m=l+1}^{k}\binom{k}{m}p_2^m q_2^{k-m}. \end{aligned}$$

To see that (3.4.3) holds in this case as well, we need simply observe that

$$\sum_{m=l+1}^{k} \binom{k}{m} p_2^m q_2^{k-m} \le \sum_{m=l}^{k-1} \binom{k-1}{m} p_2^m q_2^{k-1-m}.$$

Indeed, if $\{\xi_j\}_{j=1}^\infty$ are iid $\{0,1\}$-valued random variables with $P(\xi_1 = 1) = p_2$, then

$$\begin{aligned}
\sum_{m=l+1}^{k} \binom{k}{m} p_2^m q_2^{k-m} &= P\left(\sum_{j=1}^{k} \xi_j > l\right) \\
&= P\left(\sum_{j=1}^{k-1} \xi_j > l - \xi_k\right) \\
&\le P\left(\sum_{j=1}^{k-1} \xi_j \ge l\right) \\
&= \sum_{m=l}^{k-1} \binom{k-1}{m} p_2^m q_2^{k-1-m}.
\end{aligned}$$

Putting everything together, we have

$$\begin{aligned}
P(M_{1+\delta}^{(n)} - M_1^{(n)} > y,\, |M_1^{(n)}| \le M) &\le \sum_{\substack{x \in h\mathbb{Z} \\ |x| \le M}} \sum_{(I,j) \in S} \varphi_{k-1}^h(x, y, \delta) P(\tilde{A}(I, j, x, h)) \\
&= \sum_{\substack{x \in h\mathbb{Z} \\ |x| \le M}} \sum_{(I,j) \in S} \varphi_{k-1}^h(x, y, \delta) \frac{P(\tilde{A})}{P(A)} P(A(I, j, x, h)).
\end{aligned}$$

Note that $P(A(I, j, x, h)) \ge P(B_1^{(j)} \in [x, x+h))\Phi(x)^{k-1}\Phi(-x-h)^{n-k}$, so that $P(\tilde{A})/P(A) \le g_h(x)$, where

$$g_h(x) = \left[\frac{\Phi(x+h)}{\Phi(x)}\right]^{k-1} \left[\frac{\Phi(-x)}{\Phi(-x-h)}\right]^{n-k}.$$

Thus, by dominated convergence,

$$\begin{aligned}
P(M_{1+\delta}^{(n)} - M_1^{(n)} > y,\, |M_1^{(n)}| \le M) &\le \sum_{\substack{x \in h\mathbb{Z} \\ |x| \le M}} \varphi_{k-1}^h(x, y, \delta) g_h(x) \sum_{(I,j) \in S} P(A(I, j, x, h)) \\
&= \sum_{\substack{x \in h\mathbb{Z} \\ |x| \le M}} \varphi_{k-1}^h(x, y, \delta) g_h(x) P(M_1^{(n)} \in [x, x+h)) \\
&\to \int_{-M}^{M} \varphi_{k-1}(x, y, \delta) f_n(x)\, dx.
\end{aligned}$$

Letting $M \to \infty$ finishes the proof. ■

Now, let us fix $x \in \mathbb{R}$, $y > 0$, and $\delta > 0$. Construct two independent sequences of iid $\{0,1\}$-valued random variables $\{\xi_j^{(i)}\}_{j=1}^{\infty}$, $i \in \{1,2\}$, with $P(\xi_1^{(i)} = 1) = p_i(x,y,\delta)$. Let $Y_j = \xi_j^{(2)} - \xi_j^{(1)}$ and $S_k = \sum_{j=1}^{k} Y_j$. Then

$$\begin{aligned} P(S_k \geq 0) &= \sum_{i=0}^{k} P\left(\sum_{l=1}^{k} \xi_l^{(1)} = i, \sum_{l=1}^{k} \xi_l^{(2)} \geq i\right) \\ &= \sum_{i=0}^{k}\sum_{j=i}^{k} P\left(\sum_{l=1}^{k} \xi_l^{(1)} = i, \sum_{l=1}^{k} \xi_l^{(2)} = j\right) \\ &= \sum_{i=0}^{k}\sum_{j=i}^{k} \binom{k}{i}\binom{k}{j} p_1^i q_1^{k-i} p_2^j q_2^{k-j} \\ &= \varphi_k(x,y,\delta). \end{aligned}$$

The idea will be to estimate $\varphi_k(x,y,\delta) = P(S_k \geq 0)$ via Chebyshev's inequality. However, a straightforward application of Chebyshev to our situation will not provide sharp enough estimates, so further refinement will be necessary. For this reason, we now take a brief detour from our median processes to prove some general lemmas regarding probabilities of this sort.

3.5 Miscellaneous Lemmas

Lemma 3.5.1 *Let $\{Y_j\}_{j=1}^{\infty}$ be iid $\{-1,0,1\}$-valued random variables with $P(Y_1 = -1) = p_1$, $P(Y_1 = 1) = p_2$ and let $S_n = \sum_{j=1}^{n} Y_j$. Suppose that $\varepsilon = p_1 + p_2 > 0$ and $\mu = p_1 - p_2 > 0$. Then for each $p > 1$, there exists a finite constant C_p, depending only on p, such that*

$$P(S_n \geq 0) \leq C_p \frac{\varepsilon}{n^p \mu^{2p}}. \tag{3.5.1}$$

Proof. Since $EY_1 = -\mu$, we have

$$\begin{aligned} P(S_n \geq 0) &= P(S_n + n\mu \geq n\mu) \\ &\leq \frac{E|S_n + n\mu|^{2p}}{n^{2p}\mu^{2p}}. \end{aligned}$$

By Theorem 3.4.3 and Jensen's inequality,

$$\begin{aligned} E|S_n + n\mu|^{2p} &= E\left|\sum_{j=1}^{n}(Y_j+\mu)\right|^{2p} \\ &\leq \tilde{C}_p E\left|\sum_{j=1}^{n}|Y_j+\mu|^2\right|^{p} \\ &\leq \tilde{C}_p n^p E|Y_1+\mu|^{2p}, \end{aligned}$$

and

$$\begin{aligned} E|Y_1+\mu|^{2p} &= p_1(1-\mu)^{2p} + (1-\varepsilon)\mu^{2p} + p_2(1+\mu)^{2p} \\ &\leq 2^{2p}(p_1+p_2) + \mu^{2p} \\ &\leq (2^{2p}+1)\varepsilon \end{aligned}$$

since $\mu \leq \varepsilon$. Thus, (3.5.1) holds with $C_p = \tilde{C}_p(2^{2p}+1)$. ■

For the estimates we will need, (3.5.1) will not suffice. We will need the numerator on the right hand side of (3.5.1) to be of order ε^p, rather than of order ε. There is no way to further refine the moment estimates in the above proof since, although we haven't stated it here, Burkholder's inequality is two-sided. A different approach will be taken in the next several lemmas to refine (3.5.1) and achieve the necessary level of precision.

Lemma 3.5.2 *For $n \in \mathbb{N}$, $k \in \{0,\dots,n\}$, $p \in (0,1)$, and $x \in \mathbb{R}$, let $f(n,k,p) = \binom{n}{k}p^k q^{n-k}$, where $q = 1-p$, and let $g(n,x,p) = (2\pi npq)^{-1/2}\exp\{-(x-np)^2/2npq\}$. Then*

$$\sup_{n\in\mathbb{N}}\left(\sup_{k\in\{0,\dots,n\}}\frac{f(n,k,p)}{g(n,k,p)}\right) < \infty$$

if and only if $p = 1/2$. However, there exists a universal constant, C, independent of p, such that $f(n,k,p)/g(n,k,p) \leq C$ for all $n \in \mathbb{N}$ and all $k \in \{0,\dots,\lfloor np\rfloor\}$, provided $p \leq 1/2$.

Proof. It will first be shown that, for $p \leq 1/2$, there is a universal constant, C, such that $f(n,0,p)/g(n,0,p) \leq C$ and, if $\lfloor np\rfloor \geq 1$, $f(n,1,p)/g(n,1,p) \leq C$. We start by showing that if $\alpha > 0$, then there exists a constant, C_α, such that

$$(np)^\alpha (qe^{p/2q})^n \leq C_\alpha.$$

To prove this, first consider $2/5 \le p \le 1/2$. In this case, $qe^{p/2q} \le \frac{3}{5}e^{1/2} < 1$. Thus,

$$(np)^\alpha (qe^{p/2q})^n \le \sup_n \left[n^\alpha \left(\frac{3}{5} e^{1/2} \right)^n \right] < \infty.$$

Next, consider $0 < p < 2/5$. We claim that in this case, $q^{5/6}e^{p/2q} \le 1$. This follows from the fact that $\frac{d}{dq}[\log(q^{5/6}e^{p/2q})] = (5q-3)/6q^2 > 0$, for $q > 3/5$. Hence,

$$(np)^\alpha (qe^{p/2q})^n \le n^\alpha q^{n/6} p^\alpha.$$

Elementary calculus shows that for $x \ge 0$, $x^\alpha q^{x/6}$ attains its maximum at $x = -6\alpha/\log q$. Thus,

$$n^\alpha q^{n/6} p^\alpha \le \left(\frac{6\alpha}{e} \right)^\alpha \left(\frac{1-q}{|\log q|} \right)^\alpha .$$

Since $(q-1)/\log q \to 1$ as $q \to 1$, this proves the initial claim.

Thus, for $p \le 1/2$,

$$\begin{aligned} \frac{f(n,0,p)}{g(n,0,p)} &= \sqrt{2\pi npq}\; q^n e^{np/2q} \\ &= \sqrt{2\pi q}(np)^{1/2}(qe^{p/2q})^n \\ &\le \sqrt{2\pi} C_{1/2} \end{aligned}$$

and, if $np \ge 1$,

$$\begin{aligned} \frac{f(n,1,p)}{g(n,1,p)} &= \sqrt{2\pi npq}\; npq^{n-1} \exp\left\{ \frac{np}{2q} - \frac{1}{q} + \frac{1}{2npq} \right\} \\ &\le \sqrt{2\pi q}\; q^{n-1}(np)^{3/2} e^{np/2q} \\ &= \sqrt{\frac{2\pi}{q}}(np)^{3/2}(qe^{p/2q})^n \\ &\le \sqrt{4\pi} C_{3/2}. \end{aligned}$$

Now, for $k \in \{1, \ldots, n-1\}$, $f(n,k,p)$ is bounded above and below by universal, positive constant multiples of

$$\frac{n^{n+\frac{1}{2}}}{(n-k)^{n-k+\frac{1}{2}} k^{k+\frac{1}{2}}} p^k q^{n-k}$$

by Stirling's formula. Thus, if

$$\begin{aligned} \varphi_n(x) &= \left(n + \frac{1}{2}\right)\log n - \left(n - x + \frac{1}{2}\right)\log(n-x) - \left(x + \frac{1}{2}\right)\log x \\ &\quad + x\log p + (n-x)\log q + \frac{1}{2}\log n + \frac{1}{2}\log p + \frac{1}{2}\log q + \frac{x^2}{2npq} - \frac{x}{q} + \frac{np}{2q}, \end{aligned}$$

then there are universal, positive constants C_1 and C_2 such that

$$\log C_1 + \varphi_n(k) \le \log\left[\frac{f(n,k,p)}{g(n,k,p)}\right] \le \log C_2 + \varphi_n(k) \tag{3.5.2}$$

for all $n \in \mathbb{N}$ and all $k \in \{1, \ldots, n-1\}$.

Now,

$$\begin{aligned}
\varphi_n\left(\frac{n}{2}\right) &= \left(n+\frac{1}{2}\right)\log n - (n+1)\log\frac{n}{2} + \frac{n}{2}\log p + \frac{n}{2}\log q \\
&\quad + \frac{1}{2}\log n + \frac{1}{2}\log p + \frac{1}{2}\log q + \frac{n}{8pq} - \frac{n}{2q} + \frac{np}{2q} \\
&= n\log n + \frac{1}{2}\log n - n\log n - \log n + n\log 2 + \log 2 \\
&\quad + \frac{n}{2}\log p + \frac{n}{2}\log q + \frac{1}{2}\log n + \frac{1}{2}\log p + \frac{1}{2}\log q + \frac{n}{8pq} - \frac{n}{2} \\
&= \log 2 + \frac{1}{2}\log p + \frac{1}{2}\log q + \frac{n}{2}\left(2\log 2 + \log p + \log q + \frac{1}{4pq} - 1\right) \\
&= \frac{1}{2}\log(4pq) + \frac{n}{2}(\psi_2(p) + \psi_2(1-p)),
\end{aligned}$$

where $\psi_2(x) = \log 2 + \log x + 1/(4x) - 1/2$. Now, $\psi_2'(x) = 1/x - 1/(4x^2)$, so

$$\begin{aligned}
\psi_2'(p) - \psi_2'(1-p) &= \left(\frac{1}{p} - \frac{1}{q}\right) - \left(\frac{1}{4p^2} - \frac{1}{4q^2}\right) \\
&= \frac{q-p}{pq} - \frac{q^2-p^2}{4p^2q^2} \\
&= \left(\frac{q-p}{pq}\right)\left(1 - \frac{1}{4pq}\right).
\end{aligned}$$

Since $1 - 1/(4pq) < 0$ for all $p \neq 1/2$, the function $p \mapsto \psi_2(p) + \psi_2(1-p)$ is strictly decreasing on $(0, 1/2)$ and strictly increasing on $(1/2, 0)$. Since $\psi_2(1/2) = 0$, $\psi_2(p) + \psi_2(1-p) > 0$ for all $p \neq 1/2$. Thus, if $p \neq 1/2$, then $\varphi_n(n/2) \to \infty$ as $n \to \infty$. It now follows from (3.5.2) that if $p \neq 1/2$,

$$\sup_{n\in\mathbb{N}}\left(\sup_{k\in\{0,\ldots,n\}} \frac{f(n,k,p)}{g(n,k,p)}\right) = \infty.$$

We now compute the following:

$$\begin{aligned}
\varphi_n'(x) &= \log(n-x) + \frac{n-x+1/2}{n-x} - \log x - \frac{x+1/2}{x} + \log p - \log q + \frac{x}{npq} - \frac{1}{q} \\
&= \log(n-x) + \frac{1}{2(n-x)} - \log x - \frac{1}{2x} + \log\frac{p}{q} + \frac{x}{npq} - \frac{1}{q}
\end{aligned}$$

$$\begin{aligned}
\varphi_n''(x) &= -\frac{1}{n-x} + \frac{1}{2(n-x)^2} - \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{npq} \\
\varphi_n'''(x) &= -\frac{1}{(n-x)^2} + \frac{1}{(n-x)^3} + \frac{1}{x^2} - \frac{1}{x^3} \\
\varphi_n^{(4)}(x) &= -\frac{2}{(n-x)^3} + \frac{3}{(n-x)^4} - \frac{2}{x^3} + \frac{3}{x^4} \\
&= \frac{3-2(n-x)}{(n-x)^4} + \frac{3-2x}{x^4}
\end{aligned}$$

Since $\varphi_n^{(4)} \leq 0$ on $[2, n-2]$ and $\varphi_n'''(n/2) = 0$, it follows that $\varphi_n''' \geq 0$ on $[2, n/2]$, i.e. φ_n'' is increasing on $[2, n/2]$. Now suppose $p \leq 1/2$, let $x \in [2, np]$, and write

$$\varphi_n(x) = \varphi_n(np) - \int_x^{np} \varphi_n'(t)\, dt.$$

Note that

$$\begin{aligned}
\varphi_n(np) &= \left(n + \frac{1}{2}\right)\log n - \left(nq + \frac{1}{2}\right)\log(nq) - \left(np + \frac{1}{2}\right)\log(np) \\
&\quad + np\log p + nq\log q + \frac{1}{2}\log n + \frac{1}{2}\log p + \frac{1}{2}\log q + \frac{np}{2q} - \frac{np}{q} + \frac{np}{2q} \\
&= n\log n + \log n - nq\log n - \frac{1}{2}\log n - nq\log q - np\log n - \frac{1}{2}\log n \\
&\quad - np\log p + np\log p + nq\log q \\
&= 0
\end{aligned}$$

Next, write

$$\varphi_n'(t) = \varphi_n'(np) - \int_t^{np} \varphi_n''(s)\, ds.$$

and note that

$$\begin{aligned}
\varphi_n'(np) &= \log(nq) + \frac{1}{2nq} - \log(np) - \frac{1}{2np} + \log\frac{p}{q} \\
&= \frac{p-q}{2npq}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\varphi_n(x) &= -\int_x^{np} \left(\frac{p-q}{2npq} - \int_t^{np} \varphi_n''(s)\, ds\right) dt \\
&\leq \frac{q-p}{2q} + \int_x^{np}\int_x^{s} \varphi_n''(s)\, dt\, ds \\
&\leq \frac{1}{2} + \int_x^{np} (s-x)\varphi_n''(np)\, ds.
\end{aligned}$$

Finally, note that

$$\begin{aligned}\varphi_n''(np) &= -\frac{1}{nq} + \frac{1}{2n^2q^2} - \frac{1}{np} + \frac{1}{2n^2p^2} + \frac{1}{npq} \\ &= \frac{p^2+q^2}{2n^2p^2q^2}.\end{aligned}$$

Thus,

$$\begin{aligned}\varphi_n(x) &\leq \frac{1}{2} + \frac{p^2+q^2}{2n^2p^2q^2}\int_x^{np}(s-x)\,ds \\ &\leq \frac{1}{2} + \frac{p^2+q^2}{2n^2p^2q^2}n^2p^2 \\ &\leq \frac{3}{2}.\end{aligned}$$

It now follows from (3.5.2) that there is a universal constant, C, independent of p, such that $f(n,k,p)/g(n,k,p) \leq C$ for all $n \in \mathbb{N}$ and all $k \in \{0,\dots,\lfloor np \rfloor\}$, provided $p \leq 1/2$. Also, if $p = 1/2$, symmetry gives the same bound for $k \in \{\lfloor n/2 \rfloor + 1, \dots, n\}$, and it follows that

$$\sup_{n\in\mathbb{N}}\left(\sup_{k\in\{0,\dots,n\}}\frac{f(n,k,p)}{g(n,k,p)}\right) < \infty,$$

which completes the proof. ■

Lemma 3.5.3 *Let $\varepsilon \in (0,1/2)$ and suppose that $\{\xi_j\}_{j=1}^\infty$ are iid $\{0,1\}$-valued random variables with $P(\xi_1 = 1) = \varepsilon$. Let $T_n = \sum_{j=1}^n \xi_j$. Then for each $p > 1$, there exists a finite constant C_p, depending only on p, such that for all $n \in \mathbb{N}$,*

$$P\left(T_n \leq \frac{\varepsilon n}{2}\right) \leq C_p\frac{1}{(\varepsilon n)^p}.$$

Proof. Let f and g be as in Lemma 3.5.2 with $p = \varepsilon$, so that there exists a universal, finite constant C, independent of ε, such that $f(n,k,\varepsilon) \leq Cg(n,k,\varepsilon)$ for all $n \in \mathbb{N}$ and all $k \in \{0,\dots,\lfloor \varepsilon n \rfloor\}$.

Let $m = \lfloor \varepsilon n/2 \rfloor$, so that

$$\begin{aligned}P(T_n \leq m) &= \sum_{k=0}^m P(T_n = k) \\ &\leq C\sum_{k=0}^m g(n,k,\varepsilon).\end{aligned}$$

If $\varepsilon n \leq 4$, then $P(T_n \leq m) \leq 1 \leq 4^p/(\varepsilon n)^p$, so that we may assume without loss of generality that $\varepsilon n > 4$. Note that $x \mapsto g(n,x,\varepsilon)$ is increasing on $[0, \varepsilon n]$ and $\varepsilon n > 4$ implies $m + 1 \leq (\varepsilon n/2) + 1 < 3\varepsilon n/4$. Thus,

$$\begin{aligned} P(T_n \leq m) &\leq C \int_0^{m+1} g(n,x,\varepsilon)\, dx \\ &\leq C \int_{-\infty}^{3\varepsilon n/4} g(n,x,\varepsilon)\, dx \\ &= \frac{C}{\sqrt{2\pi t}} \int_{-\infty}^{3\varepsilon n/4} e^{-(x-\varepsilon n)^2/2t}\, dx, \end{aligned}$$

where $t = n\varepsilon(1-\varepsilon)$. By a change of variables, then

$$\begin{aligned} P(T_n \leq m) &\leq C\Phi\left(-\frac{\varepsilon n}{4\sqrt{t}}\right) \\ &\leq C\Phi\left(-\frac{\sqrt{\varepsilon n}}{4}\right). \end{aligned}$$

By Lemma 3.4.1,

$$\begin{aligned} P(T_n \leq m) &\leq \frac{C}{\sqrt{2\pi}} \cdot \frac{4}{\sqrt{\varepsilon n}} e^{-\varepsilon n/32} \\ &\leq C\sqrt{\frac{2}{\pi}} e^{-\varepsilon n/32}. \end{aligned}$$

Since there exists $K_p < \infty$ such that $x^p e^{-x/32} \leq K_p$ for all $x \in [0, \infty)$,

$$P(T_n \leq m) \leq C\sqrt{\frac{2}{\pi}} K_p \frac{1}{(\varepsilon n)^p},$$

which finishes the proof. ■

Corollary 3.5.4 *Let ε, ξ_j, and T_n be as in Lemma 3.5.3. Then for each $p > 1$, there exists a finite constant C_p, depending only on p, such that for all $n \in \mathbb{N}$,*

$$E\left[\frac{1}{T_n^p}\,;\, T_n > 0\right] \leq C_p \frac{1}{(\varepsilon n)^p}.$$

Proof. By Lemma 3.5.3,

$$\begin{aligned} E\left[\frac{1}{T_n^p}\,;\, T_n > 0\right] &= E\left[\frac{1}{T_n^p}\,;\, 1 \leq T_n \leq \frac{\varepsilon n}{2}\right] + E\left[\frac{1}{T_n^p}\,;\, T_n > \frac{\varepsilon n}{2}\right] \\ &\leq P\left(T_n \leq \frac{\varepsilon n}{2}\right) + \left(\frac{\varepsilon n}{2}\right)^{-p} \\ &\leq C_p \frac{1}{(\varepsilon n)^p} + \frac{2^p}{(\varepsilon n)^p}, \end{aligned}$$

and the proof is complete. ■

Lemma 3.5.5 *Let* $\{Y_j\}_{j=1}^{\infty}$ *be iid* $\{-1,0,1\}$*-valued random variables with* $P(Y_1=-1)=p_1$, $P(Y_1=1)=p_2$ *and let* $S_n=\sum_{j=1}^n Y_j$. *Suppose that* $\varepsilon=p_1+p_2\in(0,1/2)$ *and* $\mu=p_1-p_2>0$. *Then for each* $p>1$, *there exists a finite constant* C_p, *depending only on* p, *such that*

$$P(S_n\geq 0)\leq C_p\frac{\varepsilon^p}{n^p\mu^{2p}}. \tag{3.5.3}$$

Proof. Let $\{\tilde{Y}_j\}_{j=1}^{\infty}$ be a sequence of iid $\{-1,1\}$-valued random variables with $P(\tilde{Y}_1=-1)=p_1/\varepsilon$. Let $\{\xi_j\}_{j=1}^{\infty}$ be a sequence of iid $\{0,1\}$-valued random variables, independent of $\{\tilde{Y}_j\}_{j=1}^{\infty}$, with $P(\xi_1=1)=\varepsilon$. Then $\{\tilde{Y}_j\xi_j\}_{j=1}^{\infty}$ is an iid sequence of random variables with $\tilde{Y}_1\xi_1\stackrel{d}{=}Y_1$.

Let $\tilde{S}_n=\sum_{j=1}^n\tilde{Y}_j$ and note that by Lemma 3.5.1,

$$P(\tilde{S}_n\geq 0)\leq\tilde{C}_p\frac{1}{n^p(\mu/\varepsilon)^{2p}}=\tilde{C}_p\frac{\varepsilon^{2p}}{n^p\mu^{2p}}. \tag{3.5.4}$$

Define $T_n=\sum_{j=1}^n\xi_j$, so that

$$\begin{aligned}P(S_n\geq 0) &= P\left(\sum_{j=1}^n\tilde{Y}_j\xi_j\geq 0\right)\\ &= \sum_{k=0}^n P\left(\sum_{j=1}^n\tilde{Y}_j\xi_j\geq 0,\, T_n=k\right)\\ &= \sum_{k=0}^n\sum_{\substack{\alpha\in\{0,1\}^n\\|\alpha|=k}} P\left(\sum_{j=1}^n\tilde{Y}_j\xi_j\geq 0,\,\xi^{(n)}=\alpha\right),\end{aligned}$$

where $|\alpha|=\alpha_1+\cdots+\alpha_n$ and $\xi^{(n)}=(\xi_1,\ldots,\xi_n)$. Thus,

$$\begin{aligned}P(S_n\geq 0) &= \sum_{k=0}^n\sum_{\substack{\alpha\in\{0,1\}^n\\|\alpha|=k}} P\left(\sum_{\{j:\alpha_j=1\}}\tilde{Y}_j\geq 0,\,\xi^{(n)}=\alpha\right)\\ &= \sum_{k=0}^n\sum_{\substack{\alpha\in\{0,1\}^n\\|\alpha|=k}} P\left(\sum_{\{j:\alpha_j=1\}}\tilde{Y}_j\geq 0\right)P(\xi^{(n)}=\alpha).\end{aligned}$$

By symmetry,

$$\begin{aligned} P(S_n \geq 0) &= \sum_{k=0}^{n} \sum_{\substack{\alpha \in \{0,1\}^n \\ |\alpha|=k}} P\left(\sum_{j=1}^{k} \tilde{Y}_j \geq 0 \right) P(\xi^{(n)} = \alpha) \\ &= \sum_{k=0}^{n} P(\tilde{S}_k \geq 0) P(T_n = k). \end{aligned}$$

Using (3.5.4),

$$\begin{aligned} P(S_n \geq 0) &\leq P(T_n = 0) + \tilde{C}_p \frac{\varepsilon^{2p}}{\mu^{2p}} \sum_{k=1}^{n} \frac{1}{k^p} P(T_n = k) \\ &= (1-\varepsilon)^n + \tilde{C}_p \frac{\varepsilon^{2p}}{\mu^{2p}} E\left[\frac{1}{T_n^p} \,;\, T_n > 0 \right] \\ &\leq (1-\varepsilon)^n + \tilde{C}_p' \frac{\varepsilon^{2p}}{\mu^{2p}} \frac{1}{(\varepsilon n)^p}, \end{aligned}$$

by Corollary 3.5.4. Now note that $1 - \varepsilon \leq e^{-\varepsilon}$, so that

$$(1-\varepsilon)^n \leq e^{-\varepsilon n} \leq \tilde{C}_p'' \frac{1}{(\varepsilon n)^p} = \tilde{C}_p'' \frac{\varepsilon^p}{n^p \varepsilon^{2p}} \leq \tilde{C}_p'' \frac{\varepsilon^p}{n^p \mu^{2p}},$$

which gives (3.5.3) with $C_p = \tilde{C}_p'' + \tilde{C}_p'$. ■

3.6 Median Estimates, Part II

With Lemma 3.5.5 in place, let us return to the function $\varphi_k(x, y, \delta)$ of Proposition 3.4.7. We saw earlier that, for fixed $x \in \mathbb{R}$, $y > 0$, and $\delta > 0$, $\varphi_k(x, y, \delta) = P(S_k \geq 0)$, where $S_k = \sum_{j=1}^{k} Y_j$ and $\{Y_j\}_{j=1}^{\infty}$ is an iid sequence of $\{-1, 0, 1\}$-valued random variables with

$$\begin{aligned} P(Y_1 = -1) &= \tilde{p}_1(x, y, \delta) = p_1(x, y, \delta) q_2(x, y, \delta) \\ P(Y_1 = +1) &= \tilde{p}_2(x, y, \delta) = p_2(x, y, \delta) q_1(x, y, \delta). \end{aligned}$$

For (3.5.3) to be of use, we will need an upper bound on $\tilde{\varepsilon} = \tilde{p}_1 + \tilde{p}_2$ and a lower bound on $\tilde{\mu} = \tilde{p}_1 - \tilde{p}_2$. In particular, we will need $\tilde{\varepsilon} < 1/2$ and $\tilde{\mu} > 0$. Unfortunately, although $\tilde{\mu}$ is increasing in x, it is negative for x sufficiently close to $-\infty$. We will therefore need to make use of the fact that for any $x_0 \in \mathbb{R}$, Proposition 3.4.7 gives

$$\begin{aligned} P(M_{1+\delta}^{(n)} - M_1^{(n)} > y) &\leq \int_{x_0}^{\infty} \varphi_{k-1}(x,y,\delta) f_n(x)\,dx + \int_{-\infty}^{x_0} \varphi_{k-1}(x,y,\delta) f_n(x)\,dx \\ &\leq \int_{x_0}^{\infty} \varphi_{k-1}(x,y,\delta) f_n(x)\,dx + \int_{-\infty}^{x_0} f_n(x)\,dx \\ &= \int_{x_0}^{\infty} \varphi_{k-1}(x,y,\delta) f_n(x)\,dx + P(M_1^{(n)} \leq x_0). \end{aligned}$$

In fact, we can go one step further in obtaining a simple estimate for this probability by proving that $x \mapsto \varphi_k(x,y,\delta)$ is decreasing.

Lemma 3.6.1 *Let n, k, and φ_k be as in Proposition 3.4.7. Fix $y > 0$ and $\delta > 0$. Then for all $x_0 \in \mathbb{R}$,*

$$P(M_{1+\delta}^{(n)} - M_1^{(n)} > y) \leq \varphi_{k-1}(x_0,y,\delta) + P(M_1^{(n)} \leq x_0).$$

Proof. By the above discussion, it remains only to show that $x \mapsto \varphi_k(x,y,\delta)$ is decreasing. By our probabilistic representation of φ_k and the facts that $q_j = 1 - p_j$ and $p_2(x,y,\delta) = p_1(-x,-y,\delta)$, it will suffice to show that for fixed $y \in \mathbb{R}$ and $\delta > 0$, $x \mapsto p_1(x,y,\delta)$ is increasing.

Let

$$\begin{aligned} \psi(x,y,\delta) &= P(B_{1+\delta}^{(1)} < x + y,\ B_1^{(1)} < x) \\ &= \int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right) \Phi'(t)\,dt, \end{aligned} \tag{3.6.1}$$

so that $p_1 = \psi/\Phi(x)$. Integrating by parts gives

$$\psi(x,y,\delta) = \Phi\left(\frac{y}{\sqrt{\delta}}\right)\Phi(x) + \frac{1}{\sqrt{\delta}} \int_{-\infty}^{x} \Phi'\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi(t)\,dt,$$

so that

$$\begin{aligned} \partial_x p_1 &= -\frac{\Phi'(x)}{[\Phi(x)]^2}\psi + \frac{1}{\Phi(x)}\left[\Phi\left(\frac{y}{\sqrt{\delta}}\right)\Phi'(x) + \frac{1}{\sqrt{\delta}}\int_{-\infty}^{x} \Phi'\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi'(t)\,dt\right] \\ &= -\frac{\Phi'(x)}{[\Phi(x)]^2\sqrt{\delta}} \int_{-\infty}^{x} \Phi'\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi(t)\,dt + \frac{1}{\Phi(x)\sqrt{\delta}} \int_{-\infty}^{x} \Phi'\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi'(t)\,dt \\ &= \frac{1}{\Phi(x)\sqrt{\delta}} \int_{-\infty}^{x} \Phi'\left(\frac{x+y-t}{\sqrt{\delta}}\right)\left[\frac{\Phi'(t)}{\Phi(t)} - \frac{\Phi'(x)}{\Phi(x)}\right]\Phi(t)\,dt \end{aligned} \tag{3.6.2}$$

Now, note that

$$\begin{aligned}\frac{d}{dx}\left[\frac{\Phi'(x)}{\Phi(x)}\right] &= \frac{\Phi''(x)\Phi(x)-[\Phi'(x)]^2}{[\Phi(x)]^2}\\ &= \frac{1}{[\Phi(x)]^2}\left(-\frac{1}{\sqrt{2\pi}}xe^{-x^2/2}\Phi(x)-\frac{1}{2\pi}e^{-x^2}\right)\\ &= -\frac{e^{-x^2/2}}{\sqrt{2\pi}[\Phi(x)]^2}\left(x\Phi(x)+\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right).\end{aligned}$$

Clearly, $x\Phi(x)+\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\geq 0$ for $x\geq 0$. For $x<0$, Lemma 3.4.1 gives

$$\begin{aligned}x\Phi(x)+\frac{1}{\sqrt{2\pi}}e^{-x^2/2} &= x\Phi(-|x|)+\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\\ &\geq x\frac{1}{\sqrt{2\pi}}|x|^{-1}e^{-x^2/2}+\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\\ &= 0.\end{aligned}$$

Thus, $x\mapsto\Phi'(x)/\Phi(x)$ is decreasing, so by (3.6.2), $\partial_x p_1\geq 0$. ∎

We now turn our attention to estimating $\tilde{\varepsilon}$ and $\tilde{\mu}$. The goal of these estimates, of course, will be to provide information on $P(M_{1+\delta}^{(n)}-M_1^{(n)}>y)$. The method of approximating this probability will depend on the value of y. As it will turn out, the estimates will be quite straightforward for $y\gg\sqrt{\delta}$. We will focus our attention, primarily, on the case $y\ll\sqrt{\delta}$. These methods will then be slightly modified to cover the case $y\approx\sqrt{\delta}$. Let us first state the result we wish to prove.

Lemma 3.6.2 *Let $\tilde{\mu}=\tilde{\mu}(x,y,\delta)$ and $q_j=q_j(x,y,\delta)$ be as in the discussion preceding Lemma 3.6.1. For each $\alpha_0>0$, there exists $\delta_0>0$ such that for all $0<\delta\leq\delta_0$ and all $\alpha\geq\alpha_0$, if we set $y=\delta^{1/2+\alpha}$ and $x=-\delta^{1/4+\alpha}$, then $\tilde{\mu}\geq\frac{1}{\sqrt{2\pi}}y$ and $q_1<q_2\leq 500\delta^{1/2}$.*

This result will follow from the following Taylor-like expansion, with remainder, of $p_1(x,y,\delta)$.

Lemma 3.6.3 *Let $p_1(x,y,\delta)$ be as in the discussion preceding Proposition 3.4.7 and write*

$$p_1(x,y,\delta)=1-\frac{1}{\pi}\tan^{-1}\sqrt{\delta}+\frac{y}{\sqrt{2\pi}}+\frac{\sqrt{\delta}}{2\pi}(x+y)^2-\frac{y^2}{2\pi\sqrt{\delta}}+R_\delta(x,y).$$

Suppose $\delta \in (0,1]$, $\alpha > 0$, and $\beta \in \mathbb{R}$. Let $y = \delta^{1/2+\alpha}$, $x = -\delta^{1/4+\beta}$, and suppose that $y \le -x \le 1$ (i.e. $\beta \in [-1/4, \alpha + 1/4]$). Then

$$|R_\delta(x,y)| \le 155(\delta^{3/4+3\beta} + \delta^{3/4+\beta} + \delta^{1/2+4\alpha}),$$

and the same bound holds for $|R_\delta(-x,-y)|$.

The proofs of the above lemmas will be postponed until the end of this section. To derive the expansion in Lemma 3.6.3, we will make use of the function ψ, given by (3.6.1), and the relation $p_1 = \psi/\Phi$. So to begin, we shall derive a Taylor expansion of ψ.

Lemma 3.6.4 *Let $\psi(x,y,\delta)$ be given by (3.6.1). Then*

(i) for $i \ge 0$,

$$\partial_x^i \psi = \int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right) \Phi^{(i+1)}(t)\, dt;$$

and

(ii) for $i \ge 0$, $j \ge 1$,

$$\partial_x^i \partial_y^j \psi = -\left(\frac{1}{\sqrt{\delta}}\right)^{j-1} \Phi^{(j-1)}\left(\frac{y}{\sqrt{\delta}}\right) \Phi^{(i+1)}(x) + \partial_x^{i+1}\partial_y^{j-1}\psi.$$

Proof. For $i = 0$, part (i) is just the definition of ψ. If (i) is true for some $i \ge 0$, then

$$\begin{aligned}
\partial_x^{i+1}\psi &= \partial_x\left[\int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi^{(i+1)}(t)\, dt\right] \\
&= \Phi\left(\frac{y}{\sqrt{\delta}}\right)\Phi^{(i+1)}(x) + \frac{1}{\sqrt{\delta}}\int_{-\infty}^{x} \Phi'\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi^{(i+1)}(t)\, dt \\
&= \int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi^{(i+2)}(t)\, dt,
\end{aligned}$$

by integration by parts.

For part (ii), first consider $j = 1$. Then

$$\begin{aligned}
\partial_x^i \partial_y^j \psi &= \partial_y \left[\int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right) \Phi^{(i+1)}(t)\, dt \right] \\
&= \int_{-\infty}^{x} \partial_y \left[\Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right)\right] \Phi^{(i+1)}(t)\, dt \\
&= \int_{-\infty}^{x} \partial_x \left[\Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right)\right] \Phi^{(i+1)}(t)\, dt \\
&= \partial_x \left[\int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right) \Phi^{(i+1)}(t)\, dt \right] - \Phi\left(\frac{y}{\sqrt{\delta}}\right) \Phi^{(i+1)}(x) \\
&= -\Phi\left(\frac{y}{\sqrt{\delta}}\right) \Phi^{(i+1)}(x) + \partial_x^{i+1}\psi,
\end{aligned}$$

and part (ii) holds for all $i \geq 0$. Now suppose part (ii) holds for some $j \geq 1$ and all $i \geq 0$. Then

$$\begin{aligned}
\partial_x^i \partial_y^{j+1} \psi &= \partial_y \left[-\left(\frac{1}{\sqrt{\delta}}\right)^{j-1} \Phi^{(j-1)}\left(\frac{y}{\sqrt{\delta}}\right) \Phi^{(i+1)}(x) + \partial_x^{i+1}\partial_y^{j-1}\psi \right] \\
&= -\left(\frac{1}{\sqrt{\delta}}\right)^{j} \Phi^{(j)}\left(\frac{y}{\sqrt{\delta}}\right) \Phi^{(i+1)}(x) + \partial_x^{i+1}\partial_y^{j}\psi.
\end{aligned}$$

By induction, the proof is complete. ■

Lemma 3.6.5 *Fix $\delta > 0$ and let $\psi(x, y) = \psi(x, y, \delta)$ be given by (3.6.1). Then*

(i) $\psi(0,0) = (\pi - \tan^{-1}\sqrt{\delta}\,)/(2\pi)$;

(ii) $\psi_x(0,0) = (8\pi)^{-1/2}(1 + (1+\delta)^{-1/2})$; and

(iii) $\psi_{xx}(0,0) = \sqrt{\delta}\,/(2\pi(1+\delta))$.

Proof. Note that $\Phi''(t) = -t\Phi'(t)$ and $\Phi'''(t) = -\Phi'(t) - t\Phi''(t) = (t^2 - 1)\Phi'(t)$. Let $P_0(t) = 1$, $P_1(t) = -t$, and $P_2(t) = t^2 - 1$, so that for $i \in \{0, 1, 2\}$, Lemma 3.6.4 gives

$$\begin{aligned}
\partial_x^i \psi(0,0) &= \int_{-\infty}^{0} \Phi\left(-\frac{t}{\sqrt{\delta}}\right) \Phi^{(i+1)}(t)\, dt \\
&= \int_{-\infty}^{0} \Phi\left(-\frac{t}{\sqrt{\delta}}\right) P_i(t)\Phi'(t)\, dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{0} \int_{-\infty}^{-t/\sqrt{\delta}} P_i(t) e^{-(s^2+t^2)/2}\, ds\, dt.
\end{aligned}$$

Using the symmetries of $P_i(t)$ and changing to polar coordinates gives

$$\partial_x^i \psi(0,0) = \frac{(-1)^i}{2\pi} \int_{-\pi/2+\tan^{-1}\sqrt{\delta}}^{\pi/2} \int_0^\infty P_i(r\cos\theta) r e^{-r^2/2}\, dr\, d\theta.$$

Integrating by parts, we compute

$$\begin{aligned} \int_0^\infty r e^{-r^2/2}\, dr &= 1, \\ \int_0^\infty r^2 e^{-r^2/2}\, dr &= \sqrt{\frac{\pi}{2}}, \text{ and} \\ \int_0^\infty r^3 e^{-r^2/2}\, dr &= 2. \end{aligned}$$

Thus,

$$\psi(0,0) = \frac{1}{2\pi}(\pi - \tan^{-1}\sqrt{\delta}),$$

which proves (i). Next,

$$\begin{aligned} \psi_x(0,0) &= \frac{1}{2\pi}\sqrt{\frac{\pi}{2}} \int_{-\pi/2+\tan^{-1}\sqrt{\delta}}^{\pi/2} \cos\theta\, d\theta \\ &= \frac{1}{2\sqrt{2\pi}}\left(1 - \sin\left(\tan^{-1}\sqrt{\delta} - \frac{\pi}{2}\right)\right), \end{aligned}$$

and $\sin(\tan^{-1}\sqrt{\delta} - \frac{\pi}{2}) = -\cos(\tan^{-1}\sqrt{\delta}) = -\frac{1}{\sqrt{1+\delta}}$, which proves (ii). Finally,

$$\begin{aligned} \psi_{xx}(0,0) &= \frac{1}{2\pi} \int_{-\pi/2+\tan^{-1}\sqrt{\delta}}^{\pi/2} (2\cos^2\theta - 1)\, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi/2+\tan^{-1}\sqrt{\delta}}^{\pi/2} \cos(2\theta)\, d\theta \\ &= \frac{1}{4\pi}\sin(2\tan^{-1}\sqrt{\delta}) \\ &= \frac{\sqrt{\delta}}{2\pi(1+\delta)}, \end{aligned}$$

which proves (iii). ■

Lemma 3.6.6 *Fix* $\delta > 0$ *and let* $\psi(x,y) = \psi(x,y,\delta)$ *be given by (3.6.1). Then*

(i) $\psi_y(0,0) = (8\pi(1+\delta))^{-1/2}$*;*

(ii) $\psi_{xy}(0,0) = \sqrt{\delta}\,/(2\pi(1+\delta))$*; and*

(iii) $\psi_{yy}(0,0) = -(2\pi\sqrt{\delta}(1+\delta))^{-1}$.

Proof. By Lemma 3.6.4,

$$\psi_y = -\Phi\left(\frac{y}{\sqrt{\delta}}\right)\Phi'(x) + \psi_x.$$

Thus, by Lemma 3.6.5,

$$\psi_y(0,0) = -\frac{1}{2\sqrt{2\pi}} + \frac{1}{2\sqrt{2\pi}}\left(1 + \frac{1}{\sqrt{1+\delta}}\right),$$

which proves (i).

Similarly,

$$\begin{aligned}\psi_{xy} &= -\Phi\left(\frac{y}{\sqrt{\delta}}\right)\Phi''(x) + \psi_{xx} \\ \psi_{yy} &= -\frac{1}{\sqrt{\delta}}\Phi'\left(\frac{y}{\sqrt{\delta}}\right)\Phi'(x) + \psi_{xy}.\end{aligned}$$

Hence,

$$\begin{aligned}\psi_{xy}(0,0) &= \psi_{xx}(0,0) \\ \psi_{yy}(0,0) &= -\frac{1}{2\pi\sqrt{\delta}} + \psi_{xy}(0,0),\end{aligned}$$

which proves (ii) and (iii). ■

Lemma 3.6.7 *Fix* $\delta > 0$ *and let* $\psi(x,y) = \psi(x,y,\delta)$ *be given by (3.6.1). Then for all* $(x,y) \in \mathbb{R}^2$,

(i) $|\psi_{xxx}(x,y)| \le 10/\sqrt{2\pi}$*;*

(ii) $|\psi_{xxy}(x,y)| \le 12/\sqrt{2\pi}$*;*

(iii) $|\psi_{xyy}(x,y)| \le (|x|\delta^{-1/2} + 12\sqrt{2\pi})/(2\pi)$*; and*

(iv) $|\psi_{yyy}(x,y)| \le (|y|\delta^{-3/2} + |x|\delta^{-1/2} + 12\sqrt{2\pi})/(2\pi)$.

Proof. By Lemma 3.6.4,

$$\begin{aligned}|\psi_{xxx}(x,y)| &= \left|\int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi^{(4)}(t)\,dt\right| \\ &\le \int_{-\infty}^{\infty} |\Phi^{(4)}(t)|\,dt.\end{aligned}$$

Since $\Phi^{(4)}(t) = (3t - t^3)\Phi'(t)$, we have

$$\begin{aligned}|\psi_{xxx}(x,y)| &\le 2\int_{0}^{\infty} (3t+t^3)\Phi'(t)\,dt \\ &= \frac{10}{\sqrt{2\pi}},\end{aligned}$$

which proves (i). Similarly,

$$\begin{aligned}|\psi_{xxy}(x,y)| &= \left|-\Phi\left(\frac{y}{\sqrt{\delta}}\right)\Phi'''(x) + \psi_{xxx}(x,y)\right| \\ &\le |\Phi'''(x)| + \frac{10}{\sqrt{2\pi}}.\end{aligned}$$

Elementary calculus shows that for all $x \in \mathbb{R}$, $|\Phi'''(x)| \le 2(2\pi)^{-1/2}$, which proves (ii). Likewise,

$$\begin{aligned}\psi_{xyy} &= -\frac{1}{\sqrt{\delta}}\Phi'\left(\frac{y}{\sqrt{\delta}}\right)\Phi''(x) + \psi_{xxy} \\ &= \frac{x}{\sqrt{\delta}}\Phi'\left(\frac{y}{\sqrt{\delta}}\right)\Phi'(x) + \psi_{xxy} \\ \psi_{yyy} &= -\frac{1}{\delta}\Phi''\left(\frac{y}{\sqrt{\delta}}\right)\Phi'(x) + \psi_{xyy} \\ &= \frac{y}{\delta^{3/2}}\Phi'\left(\frac{y}{\sqrt{\delta}}\right)\Phi'(x) + \psi_{xyy},\end{aligned}$$

and these yield (iii) and (iv). ■

Lemma 3.6.8 *Fix $\delta > 0$ and let $\psi(x,y) = \psi(x,y,\delta)$ be given by (3.6.1). Write*

$$\psi(x,y) = \frac{1}{2} - \frac{1}{2\pi}\tan^{-1}\sqrt{\delta} + \frac{x}{\sqrt{2\pi}} + \frac{y}{2\sqrt{2\pi}} + \frac{\sqrt{\delta}}{4\pi}(x+y)^2 - \frac{y^2}{4\pi\sqrt{\delta}} + \tilde{R}(x,y).$$

Then

$$|\tilde{R}(x,y)| \le (|x|+|y|)^3 + \frac{|x||y|^2}{\sqrt{\delta}}(|x|+|y|) + \frac{|y|^4}{\delta^{3/2}} + \delta^{3/2}(x+y)^2 + \delta(|x|+|y|).$$

Proof. By Taylor's Theorem (see, e.g., [26]), we have that

$$\begin{aligned}\psi(x,y) &= \psi(0,0) + x\psi_x(0,0) + y\psi_y(0,0) \\ &\quad + \frac{1}{2!}[x^2\psi_{xx}(0,0) + 2xy\psi_{xy}(0,0) + y^2\psi_{yy}(0,0)] + R^{(1)}(x,y),\end{aligned}$$

where

$$R^{(1)}(x,y) = \frac{1}{3!}[x^3\psi_{xxx}(\bar{x},\bar{y}) + 3x^2y\psi_{xxy}(\bar{x},\bar{y}) + 3xy^2\psi_{xyy}(\bar{x},\bar{y}) + y^3\psi_{yyy}(\bar{x},\bar{y})]$$

and $(\bar{x},\bar{y}) = (\theta x, \theta y)$ for some $\theta = \theta(x,y) \in (0,1)$. By the preceding lemmas, this gives

$$\begin{aligned}\psi(x,y) &= \frac{1}{2} - \frac{1}{2\pi}\tan^{-1}\sqrt{\delta} + \frac{x}{2\sqrt{2\pi}}\left(1 + \frac{1}{\sqrt{1+\delta}}\right) + \frac{y}{2\sqrt{2\pi}\sqrt{1+\delta}} \\ &\quad + \frac{(x+y)^2\sqrt{\delta}}{4\pi(1+\delta)} - \frac{y^2}{4\pi\sqrt{\delta}} + R^{(1)}(x,y).\end{aligned}$$

Now,

$$\begin{aligned}\frac{x}{2\sqrt{2\pi}}\left(1 + \frac{1}{\sqrt{1+\delta}}\right) &= \frac{x}{\sqrt{2\pi}} + \frac{x}{2\sqrt{2\pi}}\left(\frac{1}{\sqrt{1+\delta}} - 1\right), \\ \frac{y}{2\sqrt{2\pi}\sqrt{1+\delta}} &= \frac{y}{2\sqrt{2\pi}} + \frac{y}{2\sqrt{2\pi}}\left(\frac{1}{\sqrt{1+\delta}} - 1\right), \text{ and} \\ \frac{(x+y)^2\sqrt{\delta}}{4\pi(1+\delta)} &= \frac{\sqrt{\delta}}{4\pi}(x+y)^2 + \frac{\sqrt{\delta}}{4\pi}(x+y)^2\left(\frac{1}{1+\delta} - 1\right).\end{aligned}$$

Thus, if

$$R^{(2)}(x,y) = \frac{x+y}{2\sqrt{2\pi}}\left(\frac{1}{\sqrt{1+\delta}} - 1\right) - \frac{\delta^{3/2}(x+y)^2}{4\pi(1+\delta)},$$

then $\tilde{R} = R^{(1)} + R^{(2)}$. Since $x \mapsto \sqrt{x}$ is concave,

$$\begin{aligned}\left|\frac{1}{\sqrt{1+\delta}} - 1\right| &= \left|\frac{\sqrt{1+\delta} - 1}{\delta}\right|\left|\frac{\delta}{\sqrt{1+\delta}}\right| \\ &\leq \frac{\delta}{2\sqrt{1+\delta}}.\end{aligned}$$

Hence,

$$|R^{(2)}(x,y)| \leq \delta(|x| + |y|) + \delta^{3/2}(x+y)^2.$$

Next, by Lemma 3.6.7,

$$\begin{aligned}
|R^{(1)}(x,y)| &\le \frac{1}{3!}\left[\frac{10|x|^3}{\sqrt{2\pi}} + \frac{36|x|^2|y|}{\sqrt{2\pi}} + 3|x||y|^2\left(\frac{|x|}{2\pi\sqrt{\delta}} + \frac{12}{\sqrt{2\pi}}\right)\right. \\
&\quad \left. + |y|^3\left(\frac{|y|}{2\pi\delta^{3/2}} + \frac{|x|}{2\pi\sqrt{\delta}} + \frac{12}{\sqrt{2\pi}}\right)\right] \\
&\le \frac{1}{3!}\left[\frac{12}{\sqrt{2\pi}}(|x|+|y|)^3 + \frac{3|x||y|^2}{2\pi\sqrt{\delta}}(|x|+|y|) + \frac{|y|^4}{2\pi\delta^{3/2}}\right] \\
&\le (|x|+|y|)^3 + \frac{|x||y|^2}{\sqrt{\delta}}(|x|+|y|) + \frac{|y|^4}{\delta^{3/2}}.
\end{aligned}$$

Combined with the estimate for $R^{(2)}$, this completes the proof. ■

Proof of Lemma 3.6.3. Let $\psi(x,y) = \psi(x,y,\delta)$ be given by (3.6.1). Then $p_1(x,y,\delta) = \psi(x,y)/\Phi(x)$. Write

$$\Phi(x) = \frac{1}{2} + \frac{x}{\sqrt{2\pi}} + r_1(x),$$

where $r_1(x) = \frac{1}{2}x^2\Phi''(\bar{x})$ and $\bar{x} = \theta x$ for some $\theta = \theta(x) \in (0,1)$. Since

$$\Phi''(x) = -\frac{1}{\sqrt{2\pi}}\, xe^{-x^2/2},$$

we have $|r_1(x)| \le \frac{1}{2\sqrt{2\pi}}|x|^3$. We may now write, for $x \ne -\sqrt{\pi/2}$,

$$\frac{1}{\Phi(x)} = \frac{1}{\frac{1}{2} + \frac{x}{\sqrt{2\pi}}} + r_2(x),$$

where

$$r_2(x) = \frac{-r_1(x)}{\Phi(x)\left(\frac{1}{2} + \frac{x}{\sqrt{2\pi}}\right)}.$$

Similarly, we may write

$$\frac{1}{\Phi(x)} = 2 + r_3(x),$$

where

$$\begin{aligned}
r_3(x) &= r_2(x) + \frac{1}{\frac{1}{2} + \frac{x}{\sqrt{2\pi}}} - 2 \\
&= r_2(x) + \frac{2\sqrt{2\pi} - 2(\sqrt{2\pi} + 2x)}{\sqrt{2\pi} + 2x} \\
&= r_2(x) - \frac{4x}{\sqrt{2\pi} + 2x}.
\end{aligned}$$

Now let us assume $|x| \le 1$. Then $x \ne -\sqrt{\pi/2}$ and the above applies. Note that

$$|r_2(x)| \le \frac{|r_1(x)|}{\Phi(-1)\left(\frac{1}{2} - \frac{1}{\sqrt{2\pi}}\right)}.$$

Since $\Phi(-1) \ge \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \ge \frac{1}{10}$, we have $|r_2(x)| \le 100|r_1(x)| \le \frac{50}{\sqrt{2\pi}}|x|^3$. Also,

$$\begin{aligned} |r_3(x)| &\le |r_2(x)| + \left(\frac{4}{\sqrt{2\pi} - 2}\right)|x| \\ &= |r_2(x)| + \frac{2}{\sqrt{2\pi}}\left(\frac{1}{2} - \frac{1}{\sqrt{2\pi}}\right)^{-1}|x| \\ &\le \frac{50}{\sqrt{2\pi}}|x|^3 + \frac{20}{\sqrt{2\pi}}|x|. \end{aligned}$$

Since $|x| \le 1$, this gives $|r_3(x)| \le \frac{70}{\sqrt{2\pi}}|x|$.

We now apply Lemma 3.6.8, which yields

$$\begin{aligned} p_1(x,y,\delta) &= \frac{1}{\Phi(x)}\psi(x,y) \\ &= \left(\frac{1}{2} + \frac{x}{\sqrt{2\pi}}\right)\frac{1}{\Phi(x)} \\ &\quad + \left(-\frac{1}{2\pi}\tan^{-1}\sqrt{\delta} + \frac{y}{2\sqrt{2\pi}} + \frac{\sqrt{\delta}}{4\pi}(x+y)^2 - \frac{y^2}{4\pi\sqrt{\delta}} + \tilde{R}(x,y)\right)\frac{1}{\Phi(x)} \\ &= 1 - \frac{1}{\pi}\tan^{-1}\sqrt{\delta} + \frac{y}{\sqrt{2\pi}} + \frac{\sqrt{\delta}}{2\pi}(x+y)^2 - \frac{y^2}{2\pi\sqrt{\delta}} + R_\delta(x,y), \end{aligned}$$

where

$$\begin{aligned} R_\delta(x,y) &= \left(\frac{1}{2} + \frac{x}{\sqrt{2\pi}}\right)r_2(x) \\ &\quad + \left(-\frac{1}{2\pi}\tan^{-1}\sqrt{\delta} + \frac{y}{2\sqrt{2\pi}} + \frac{\sqrt{\delta}}{4\pi}(x+y)^2 - \frac{y^2}{4\pi\sqrt{\delta}}\right)r_3(x) \\ &\quad + \frac{\tilde{R}(x,y)}{\Phi(x)}. \end{aligned}$$

Thus,

$$\begin{aligned} |R_\delta(x,y)| &\le |r_2(x)| + \left(\frac{\tan^{-1}\sqrt{\delta}}{2\pi} + \frac{|y|}{2\sqrt{2\pi}} + \frac{\sqrt{\delta}}{4\pi}(x+y)^2 + \frac{y^2}{4\pi\sqrt{\delta}}\right)|r_3(x)| + \frac{|\tilde{R}(x,y)|}{\Phi(-1)} \\ &\le \frac{50}{\sqrt{2\pi}}|x|^3 + \left(\frac{\sqrt{\delta}}{2\pi} + \frac{|y|}{2\sqrt{2\pi}} + \frac{\sqrt{\delta}}{4\pi}(x+y)^2 + \frac{y^2}{4\pi\sqrt{\delta}}\right)\frac{70}{\sqrt{2\pi}}|x| + 10|\tilde{R}(x,y)|. \end{aligned}$$

By Lemma 3.6.8,

$$\begin{aligned}|R_\delta(x,y)| &\leq \frac{50}{\sqrt{2\pi}}|x|^3 + \left(\frac{\sqrt{\delta}}{2\pi} + \frac{|y|}{2\sqrt{2\pi}} + \frac{\sqrt{\delta}}{4\pi}(x+y)^2 + \frac{y^2}{4\pi\sqrt{\delta}}\right)\frac{70}{\sqrt{2\pi}}|x| \\ &\quad +10\left[(|x|+|y|)^3 + \frac{|x||y|^2}{\sqrt{\delta}}(|x|+|y|) + \frac{|y|^4}{\delta^{3/2}} + \delta^{3/2}(x+y)^2 + \delta(|x|+|y|)\right].\end{aligned}$$

Now let x and y be as in the statement of Lemma 3.6.3. By assumption, $0 < y \leq -x \leq 1$, so the above estimates are valid. Note also that, by symmetry, this same bound holds for $|R_\delta(-x,-y)|$. It remains only to write everything in terms of δ. We have $x = -\delta^{1/4+\beta}$ and $y = \delta^{1/2+\alpha}$. Using the fact that $|x+y| \leq |x|+|y| \leq 2|x| \leq 2$, we have that

$$\begin{aligned}|R_\delta(x,y)| &\leq 25\delta^{3/4+3\beta} + 5\delta^{3/4+\beta} + 6\delta^{3/4+\alpha+\beta} + 10\delta^{3/4+\beta} + 5\delta^{3/4+2\alpha+\beta} \\ &\quad +80\delta^{3/4+3\beta} + 20\delta^{3/4+2\alpha+\beta} + 10\delta^{1/2+4\alpha} + 40\delta^{2+2\beta} + 20\delta^{5/4+\beta} \\ &= 115\delta^{3/4+3\beta} + 15\delta^{3/4+\beta} + 6\delta^{3/4+\alpha+\beta} + 25\delta^{3/4+2\alpha+\beta} \\ &\quad +10\delta^{1/2+4\alpha} + 40\delta^{2+2\beta} + 20\delta^{5/4+\beta}. \qquad (3.6.3)\end{aligned}$$

To simplify further, note that $\alpha > 0$ and $\delta \leq 1$, so that

$$\begin{aligned}|R_\delta(x,y)| &\leq 115\delta^{3/4+3\beta} + 15\delta^{3/4+\beta} + 6\delta^{3/4+\beta} + 25\delta^{3/4+\beta} \\ &\quad +10\delta^{1/2+4\alpha} + 40\delta^{2+2\beta} + 20\delta^{3/4+\beta} \\ &= 115\delta^{3/4+3\beta} + 66\delta^{3/4+\beta} + 10\delta^{1/2+4\alpha} + 40\delta^{2+2\beta}.\end{aligned}$$

Now, if $\beta \geq 0$, then $2+2\beta > 3/4+\beta$, and

$$|R_\delta(x,y)| \leq 115\delta^{3/4+3\beta} + 106\delta^{3/4+\beta} + 10\delta^{1/2+4\alpha}.$$

Otherwise, if $\beta < 0$, then $2+2\beta > 3/4+3\beta$, and

$$|R_\delta(x,y)| \leq 155\delta^{3/4+3\beta} + 66\delta^{3/4+\beta} + 10\delta^{1/2+4\alpha}.$$

In either case,

$$|R_\delta(x,y)| \leq 155(\delta^{3/4+3\beta} + \delta^{3/4+\beta} + \delta^{1/2+4\alpha}),$$

which completes the proof. ∎

Proof of Lemma 3.6.2. First observe that

$$\begin{aligned}\tilde{\mu} &= \tilde{p}_1 - \tilde{p}_2 \\ &= p_1 q_2 - p_2 q_1 \\ &= p_1(1-p_2) - p_2(1-p_1) \\ &= p_1 - p_2.\end{aligned}$$

Note that $p_2(x,y,\delta) = p_1(-x,-y,\delta)$. Thus, if $\delta \in (0,1]$, $\alpha > 0$, $y = \delta^{1/2+\alpha}$, and $x = -\delta^{1/4+\alpha}$, then $y \leq -x \leq 1$ and by Lemma 3.6.3 with $\beta = \alpha$,

$$\tilde{\mu} = \frac{2y}{\sqrt{2\pi}} + R_\delta^{\pm}(x,y),$$

where $R_\delta^{\pm}(x,y) = R_\delta(x,y) - R_\delta(-x,-y)$; and, hence,

$$\begin{aligned}|R_\delta^{\pm}(x,y)| &\leq 310(\delta^{3/4+3\alpha} + \delta^{3/4+\alpha} + \delta^{1/2+4\alpha}) \\ &\leq 620(\delta^{3/4+\alpha} + \delta^{1/2+4\alpha}).\end{aligned}$$

Note that if $\alpha \geq \frac{1}{12}$, then $\frac{1}{2} + 4\alpha \geq \frac{3}{4} + \alpha$ and

$$\begin{aligned}|R_\delta^{\pm}(x,y)| &\leq 1240\delta^{3/4+\alpha} \\ &= 1240\delta^{1/4}y.\end{aligned}$$

On the other hand, if $\alpha < \frac{1}{12}$, then $\frac{1}{2} + 4\alpha < \frac{3}{4} + \alpha$ and

$$\begin{aligned}|R_\delta^{\pm}(x,y)| &\leq 1240\delta^{1/2+4\alpha} \\ &= 1240\delta^{3\alpha}y.\end{aligned}$$

Now let $\alpha_0 > 0$ be given and set $\gamma = \min\{1/4, \alpha_0\}$. We then have

$$\tilde{\mu} \geq \left(\frac{2}{\sqrt{2\pi}} - 1240\delta^{\gamma}\right) y.$$

Choosing $\delta_0 \in (0,1]$, small enough so that $\frac{2}{\sqrt{2\pi}} - 1240\delta_0^{\gamma} \geq \frac{1}{\sqrt{2\pi}}$, completes the proof of the first part of the lemma.

Next, note that $\tilde{\mu} > 0$ implies $p_1 > p_2$ and, hence, $q_1 < q_2$. Lemma 3.6.3 gives

$$\begin{aligned} q_2 &= 1 - p_2 \\ &= |1 - p_1(-x, -y, \delta)| \\ &\leq \frac{1}{\pi}\tan^{-1}\sqrt{\delta} + \frac{|y|}{\sqrt{2\pi}} + \frac{\sqrt{\delta}}{2\pi}(x+y)^2 + \frac{y^2}{2\pi\sqrt{\delta}} + |R_\delta(-x,-y)| \\ &\leq \delta^{1/2} + \delta^{1/2+\alpha} + \delta^{1+2\alpha} + \delta^{1/2+2\alpha} + 155(\delta^{3/4+3\alpha} + \delta^{3/4+\alpha} + \delta^{1/2+4\alpha}) \\ &\leq 500\delta^{1/2}, \end{aligned}$$

which completes the proof of the second part of the lemma. ■

Lemma 3.6.2 will be used to estimate $P(M_{1+\delta}^{(n)} - M_1^{(n)} > y)$ in the case that $y \ll \sqrt{\delta}$. As mentioned previously, a slight modification will be necessary to handle the case $y \approx \sqrt{\delta}$.

Lemma 3.6.9 *Let p_1 and R_δ be as in Lemma 3.6.3. Suppose $\delta \in (0,1]$, $\alpha \leq 0$, and $\beta \in \mathbb{R}$. Let $y = \delta^{1/2+\alpha}$, $x = -\delta^{1/4+\beta}$, and suppose that $y \leq -x \leq 1$. Then*

$$|R_\delta(x,y)| \leq 155(\delta^{3/4+3\beta} + \delta^{3/4+2\alpha+\beta} + \delta^{1/2+4\alpha}),$$

and the same bound holds for $|R_\delta(-x,-y)|$.

Proof. Everything from the proof of Lemma 3.6.3, up until (3.6.3), carries through without modification. From (3.6.3), using $\alpha \leq 0$ and $\delta \leq 1$, gives

$$|R_\delta(x,y)| \leq 115\delta^{3/4+3\beta} + 66\delta^{3/4+2\alpha+\beta} + 10\delta^{1/2+4\alpha} + 40\delta^{2+2\beta}.$$

If $\beta \geq 0$, then $2 + 2\beta > 3/4 + \beta \geq 3/4 + 2\alpha + \beta$; if $\beta < 0$, then $2 + 2\beta > 3/4 + 3\beta$. We therefore have

$$|R_\delta(x,y)| \leq 155(\delta^{3/4+3\beta} + \delta^{3/4+2\alpha+\beta} + \delta^{1/2+4\alpha}),$$

and the proof is complete. ■

Lemma 3.6.10 *Let $\tilde{\mu}$ and q_j be as in Lemma 3.6.2. Let $0 < \alpha_0 < 1/16$, choose δ_0 as in Lemma 3.6.2, and set $\beta_0 = (1-16\alpha_0)/12 > 0$. Then for all $0 < \delta \leq \delta_0$ and all $\gamma \in [-\beta_0, \alpha_0]$, if we set $y = \delta^{1/2+\gamma}$ and $x = -\delta^{1/4+\gamma}$, then $\tilde{\mu} \geq \frac{1}{\sqrt{2\pi}}\delta^{1/2+\alpha_0}$ and $q_1 < q_2 \leq 500\delta^{1/2-4\beta_0}$.*

Proof. For fixed x and δ,

$$p_1(x,y,\delta) = \frac{1}{\Phi(x)} \int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right) \Phi'(t)\,dt$$

is clearly increasing in y; therefore, so is $\tilde{\mu}(x,y,\delta)$. Let x, y, and δ be as in the statement of the lemma and set $\tilde{y} = \delta^{1/2+\alpha_0}$. Then $\tilde{y} \le y$, so

$$\begin{aligned} \tilde{\mu}(x,y,\delta) &\ge \tilde{\mu}(x,\tilde{y},\delta) \\ &= p_1(x,\tilde{y},\delta) - p_1(-x,-\tilde{y},\delta) \\ &= \frac{2\tilde{y}}{\sqrt{2\pi}} + R_\delta^{\pm}(x,\tilde{y}), \end{aligned}$$

where $R_\delta^{\pm}$ is as in the proof of Lemma 3.6.2. It is easily verified that $\delta \in (0,1]$, $\alpha_0 > 0$, and $\gamma \in [-1/4, \alpha_0 + 1/4]$, so that Lemma 3.6.3 applies, yielding

$$|R_\delta^{\pm}(x,\tilde{y})| \le 310(\delta^{3/4+3\gamma} + \delta^{3/4+\gamma} + \delta^{1/2+4\alpha_0}).$$

Since both $\frac{3}{4} + 3\gamma$ and $\frac{3}{4} + \gamma$ are bounded below by $\frac{3}{4} - 3\beta_0 = \frac{1}{2} + 4\alpha_0$, we have

$$\begin{aligned} |R_\delta^{\pm}(x,y)| &\le 1240\delta^{1/2+4\alpha_0} \\ &= 1240\delta^{3\alpha_0}\tilde{y}, \end{aligned}$$

which gives

$$\begin{aligned} \tilde{\mu} &\ge \left(\frac{2}{\sqrt{2\pi}} - 1240\delta^{3\alpha_0}\right)\tilde{y} \\ &\ge \left(\frac{2}{\sqrt{2\pi}} - 1240\delta_0^{\alpha_0}\right)\tilde{y}. \end{aligned}$$

The proof of Lemma 3.6.2 shows that $\left(\frac{2}{\sqrt{2\pi}} - 1240\delta_0^{\alpha_0}\right) \ge \frac{1}{\sqrt{2\pi}}$, so $\tilde{\mu} \ge \frac{1}{\sqrt{2\pi}}\tilde{y} = \frac{1}{\sqrt{2\pi}}\delta^{1/2+\alpha_0}$.

For the final assertion, the last part of the proof of Lemma 3.6.2 shows that

$$q_1 < q_2 \le 4\delta^{1/2-2\beta_0} + |R_\delta(-x,-y)|.$$

Thus, if $\gamma > 0$, then Lemma 3.6.3 with $\alpha = \beta = \gamma$ gives that $q_2 \le 500\delta^{1/2-2\beta_0}$. Otherwise, if $\gamma \le 0$, then Lemma 3.6.9 with $\alpha = \beta = \gamma$ gives that $q_2 \le 500\delta^{1/2-4\beta_0}$. ∎

3.7 Median Estimates, Part III

We are finally in a position to piece everything together and prove Proposition 3.4.6. Writing $P(X_{1+\delta}^{(n)} - X_1^{(n)} > \varepsilon) = P(M_{1+\delta}^{(n)} - M_1^{(n)} > y)$, where $y = \varepsilon n^{-1/2}$, puts us in a position to apply the results of the previous sections. We begin with the "trivial" case, $y \gg \sqrt{\delta}$.

Lemma 3.7.1 *Let $M_t^{(n)}$ be as in the discussion preceding Theorem 3.1.1. Let $0 < \alpha_0 < 1/2$. Then for each $p > 0$, there exists a finite constant $C = C_{p,\alpha_0}$ such that*

$$P(M_{1+\delta}^{(n)} - M_1^{(n)} > \varepsilon n^{-1/2}) \le C(\varepsilon^{-1}\delta^{1/4})^p$$

whenever $0 < \varepsilon < 1$, $\delta \in (0,1)$, and $n \in \mathbb{N}$ satisfy $\varepsilon n^{-1/2} \ge \delta^{1/2-\alpha_0}$.

Proof. Choose $\alpha \in [\alpha_0, 1/2)$ such that $\varepsilon n^{-1/2} = \delta^{1/2-\alpha}$. Then

$$\begin{aligned} P(M_{1+\delta}^{(n)} - M_1^{(n)} > \varepsilon n^{-1/2}) &\le P\left(\bigcup_{j=1}^{n}\{B_{1+\delta}^{(j)} - B_1^{(j)} > \varepsilon n^{-1/2}\}\right) \\ &\le nP(B_1^{(1)} > \varepsilon n^{-1/2}\delta^{-1/2}) \\ &= (\varepsilon\delta^{-1/2+\alpha})^2 P(B_1^{(1)} > \delta^{-\alpha}). \end{aligned}$$

Let $\tilde{p} = \frac{1}{\alpha_0}\left(\frac{p}{4}+1\right)$ and choose $C = C_{\tilde{p}} = C_{p,\alpha_0}$ such that $P(B_1^{(1)} > x) \le Cx^{-\tilde{p}}$ for all $x > 0$. Then

$$\begin{aligned} P(M_{1+\delta}^{(n)} - M_1^{(n)} > \varepsilon n^{-1/2}) &\le C\delta^{2\alpha-1+\alpha\tilde{p}} \\ &\le C\delta^{-1+\alpha\tilde{p}} \\ &\le C\delta^{-1+\alpha_0\tilde{p}} \\ &= C\delta^{p/4} \\ &\le C(\varepsilon^{-1}\delta^{1/4})^p, \end{aligned}$$

and the proof is done. ■

Next, we turn to the case $y \ll \sqrt{\delta}$.

Lemma 3.7.2 *Let $M_t^{(n)}$ be as in the discussion preceding Theorem 3.1.1. Let $\alpha_0 > 0$. Choose δ_0 as in Lemma 3.6.2. Then for each $p > 2$, there exists a finite constant $C = C_{p,\alpha_0}$ such that*

$$P(M_{1+\delta}^{(n)} - M_1^{(n)} > \varepsilon n^{-1/2}) \leq C(\varepsilon^{-1}\delta^{1/4})^p$$

whenever $\varepsilon > 0$, $\delta \in (0, \delta_0]$, and $n \in \mathbb{N}$ satisfy $n \geq 3$ and $\varepsilon n^{-1/2} \leq \delta^{1/2+\alpha_0}$.

Proof. Define $y = \varepsilon n^{-1/2}$ and choose $\alpha \geq \alpha_0$ such that $y = \delta^{1/2+\alpha}$. Define $x_0 = -\delta^{1/4+\alpha}$. By Lemma 3.6.1, since $n \geq 3$,

$$P(M_{1+\delta}^{(n)} - M_1^{(n)} > \varepsilon n^{-1/2}) \leq \varphi_{k-1}(x_0, y, \delta) + P(M_1^{(n)} \leq x_0), \tag{3.7.1}$$

where φ is given by (3.4.2) and $k = \lfloor (n+1)/2 \rfloor$.

By Proposition 3.4.4, since $p > 2$ and $M_1^{(n)}$ has a continuous density function,

$$\begin{aligned} P(M_1^{(n)} \leq x_0) &= P(X_1^{(n)} \leq -n^{1/2} y \delta^{-1/4}) \\ &\leq P(|X_1^{(n)}| > \varepsilon \delta^{-1/4}) \\ &\leq C_p(\varepsilon^{-1}\delta^{1/4})^p \end{aligned}$$

To estimate the first term on the right hand side of (3.7.1), we adopt the notation of the discussion preceding Lemma 3.6.1, and write $\varphi_{k-1}(x_0, y, \delta) = P(S_{k-1} \geq 0)$. We wish to apply Lemma 3.5.5, so we must first verify its hypotheses; namely, we must check that $0 < \tilde{\varepsilon} < 1/2$ and $\tilde{\mu} > 0$. By Lemma 3.6.2, $\tilde{\mu} \geq \frac{1}{\sqrt{2\pi}} y > 0$. Note also that, by making δ_0 smaller if necessary, $q_1 < q_2 \leq 500\delta^{1/2} < 1/4$. Thus,

$$\tilde{\varepsilon} = \tilde{p}_1 + \tilde{p}_1 = p_1 q_2 + p_2 q_1 < (p_1 + p_2) q_2 < 1000\delta^{1/2} < 1/2.$$

Clearly, $\tilde{\varepsilon} > 0$, so, since $p/2 > 1$, Lemma 3.5.5 gives

$$\varphi_{k-1}(x_0, y, \delta) \leq C'_p \frac{\tilde{\varepsilon}^{p/2}}{(k-1)^{p/2} \tilde{\mu}^p}.$$

Finally, note that for $n \geq 3$, $k - 1 \geq n/6$. Thus,

$$\begin{aligned} \varphi_{k-1}(x_0, y, \delta) &\leq C'_p \frac{1000^{p/2} \delta^{p/4}}{n^{p/2} 6^{-p/2} (2\pi)^{-p/2} y^p} \\ &= C''_p \frac{\delta^{p/4}}{n^{p/2} (\varepsilon n^{-1/2})^p} \\ &= C''_p (\varepsilon^{-1}\delta^{1/4})^p. \end{aligned}$$

Letting $C = C_p + C_p''$ completes the proof. ■

Finally, we must deal with the case $y \approx \sqrt{\delta}$. Here is where we use a modification of the technique used to prove Lemma 3.7.2. This modification causes us to lose precision and we are unable to achieve the sharp bounds we achieved in the previous two lemmas. (See Remark 3.4.1.)

Lemma 3.7.3 *Let $M_t^{(n)}$ be as in the discussion preceding Theorem 3.1.1 and let $\beta_0 = 1/108$. Then there exists $\delta_0 \in (0,1)$ and a family of finite constants $\{C_p\}_{p>2}$ such that for each $p > 2$,*

$$P(M_{1+\delta}^{(n)} - M_1^{(n)} > \varepsilon n^{-1/2}) \leq C_p(\varepsilon^{-1}\delta^{1/6})^p$$

whenever $\varepsilon > 0$, $\delta \in (0, \delta_0]$, and $n \in \mathbb{N}$ satisfy $n \geq 3$ and $\delta^{1/2+6\beta_0} \leq \varepsilon n^{-1/2} \leq \delta^{1/2-\beta_0}$.

Proof. Let $\alpha_0 = 6\beta_0$ and check that $0 < \alpha_0 < 1/16$ and $\beta_0 = (1 - 16\alpha_0)/12$ so that the hypotheses of Lemma 3.6.10 are satisfied. Choose δ_0 as in Lemma 3.6.10 and let ε, δ, and n be as above. Define $y = \varepsilon n^{-1/2}$ so that $\delta^{1/2+\alpha_0} \leq y \leq \delta^{1/2-\beta_0}$. Choose $\gamma \in [-\beta_0, \alpha_0]$ such that $y = \delta^{1/2+\gamma}$ and define $x_0 = -\delta^{1/4+\gamma}$. As before, since $n \geq 3$, Lemma 3.6.1 yields (3.7.1). Also as before, since $p > 2$ and $M_1^{(n)}$ has a continuous density function, Proposition 3.4.4 yields

$$P(M_1^{(n)} \leq x_0) \leq C_p(\varepsilon^{-1}\delta^{1/4})^p \leq C_p(\varepsilon^{-1}\delta^{1/6})^p.$$

To estimate the first term on the right hand side of (3.7.1), we again adopt the notation of the discussion preceding Lemma 3.6.1, and write $\varphi_{k-1}(x_0, y, \delta) = P(S_{k-1} \geq 0)$. We wish to apply Lemma 3.5.5, so we must first verify its hypotheses; namely, we must check that $0 < \tilde{\varepsilon} < 1/2$ and $\tilde{\mu} > 0$. By Lemma 3.6.10, $\tilde{\mu} \geq \frac{1}{\sqrt{2\pi}}\delta^{1/2+\alpha_0} > 0$. Note also that, by making δ_0 smaller if necessary, $q_1 < q_2 \leq 500\delta^{1/2-4\beta_0} < 1/4$. Thus, as before, $\tilde{\varepsilon} < 1000\delta^{1/2-4\beta_0} < 1/2$. Clearly, $\tilde{\varepsilon} > 0$, so, since $p/2 > 1$, Lemma 3.5.5 gives

$$\varphi_{k-1}(x_0, y, \delta) \leq C_p' \frac{\tilde{\varepsilon}^{p/2}}{(k-1)^{p/2}\tilde{\mu}^p}.$$

Finally, note that for $n \geq 3$, $k-1 \geq n/6$. Thus,

$$\begin{aligned}
\varphi_{k-1}(x_0, y, \delta) &\leq C_p'(1000\delta^{1/2-4\beta_0})^{p/2}\left(\frac{n}{6}\right)^{-p/2}\left(\frac{1}{\sqrt{2\pi}}\delta^{1/2+\alpha_0}\right)^{-p} \\
&= C_p''(\delta^{1/2-4\beta_0})^{p/2}(\varepsilon^2 y^{-2})^{-p/2}(\delta^{1/2+\alpha_0})^{-p} \\
&= C_p''(\varepsilon^{-2}\delta^{1/2-4\beta_0}\delta^{1+2\gamma}\delta^{-1-2\alpha_0})^{p/2} \\
&= C_p''(\varepsilon^{-2}\delta^{1/2-4\beta_0+2\gamma-2\alpha_0})^{p/2}.
\end{aligned}$$

Since $\frac{1}{2} - 4\beta_0 + 2\gamma - 2\alpha_0 \geq \frac{1}{2} - 6\beta_0 - 2\alpha_0 = \frac{1}{3}$, we have $\varphi_{k-1}(x_0, y, \delta) \leq C_p''(\varepsilon^{-1}\delta^{1/6})^p$. Letting $C = C_p + C_p''$ completes the proof. ∎

Proof of Proposition 3.4.6. Fix $p > 2$. Take $\alpha_0 = 1/108$ in Lemma 3.7.1. Let $C_p^{(1)} = C_{p,\alpha_0}$, where C_{p,α_0} is as in the conclusion of that lemma. Next, take $\alpha_0 = 1/18$ in Lemma 3.7.2. Let $\delta_0^{(2)} = \delta_0$ and $C_p^{(2)} = C_{p,\alpha_0}$, where δ_0 and C_{p,α_0} are as in the conclusion of that lemma. Finally, let $\delta_0^{(3)} = \delta_0$ and $C_p^{(3)} = C_p$, where δ_0 and C_p are as in the conclusion of Lemma 3.7.3. Define $\delta_0 = \min\{\delta_0^{(2)}, \delta_0^{(3)}\}$ and $C_p = \max\{C_p^{(1)}, C_p^{(2)}, C_p^{(3)}\}$. (Observe that δ_0 does not depend on p.)

Let $\varepsilon \in (0,1)$, $\delta \in (0, \delta_0]$, and $n \in \mathbb{N}$ with $n \geq 3$. If $\varepsilon n^{-1/2} \geq \delta^{1/2-1/108}$ or $\varepsilon n^{-1/2} \leq \delta^{1/2+1/18}$, then Lemma 3.7.1 or Lemma 3.7.2, respectively, give

$$\begin{aligned}
P(X_{1+\delta}^{(n)} - X_1^{(n)} > \varepsilon) &= P(M_{1+\delta}^{(n)} - M_1^{(n)} > \varepsilon n^{-1/2}) \\
&\leq C_p(\varepsilon^{-1}\delta^{1/4})^p \\
&\leq C_p(\varepsilon^{-1}\delta^{1/6})^p.
\end{aligned}$$

If $\delta^{1/2+1/18} \leq \varepsilon n^{-1/2} \leq \delta^{1/2-1/108}$, then Lemma 3.7.3 gives

$$\begin{aligned}
P(X_{1+\delta}^{(n)} - X_1^{(n)} > \varepsilon) &= P(M_{1+\delta}^{(n)} - M_1^{(n)} > \varepsilon n^{-1/2}) \\
&\leq C_p(\varepsilon^{-1}\delta^{1/6})^p,
\end{aligned}$$

and we are done. ∎

Chapter 4

SIGNED VARIATIONS OF AN SPDE SOLUTION

4.1 Introduction and Main Results

It is well known that for any $H \in (0,1)$, there exists a continuous, centered Gaussian process $B_H(t)$ that satisfies

(i) $B_H(0) = 0$ a.s., and

(ii) $E|B_H(t) - B_H(s)|^2 = |t-s|^{2H}$ for all $s, t \geq 0$.

This process is known as *fractional Brownian motion* (or *fBm*) and the number H is its *Hurst parameter*. When $H = 1/2$, $B_H(t)$ is simply a standard Brownian motion. (Note that in (ii), we are adopting the convention that EX^k denotes $E[X^k]$. We shall use this convention throughout this chapter.)

Fractional Brownian motion is a *self-similar process*, i.e. $B_H(ct) \stackrel{d}{=} c^H B_H(t)$. The Hurst parameter of fBm tells us several things about the process. For example, the increments of $B_H(t)$ are positively correlated when $H \in (1/2, 1)$ and negatively correlated when $H \in (0, 1/2)$. Also, for any $\beta \in (0, H)$, the sample paths of $B_H(t)$ are almost surely Hölder continuous with index β. Moreover, when $H \in (1/2, 1)$, fBm exhibits *long-range dependence*, i.e.

$$\sum_{n=1}^{\infty} E[B_H(1)(B_H(n+1) - B_H(n))] = \infty,$$

which can be verified directly by observing that

$$EB_H(s)B_H(t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

(Note that long-range dependence, according to this definition, does not occur when $H \in (0, 1/2)$.) It is the long-range dependence property of fBm that has made it an appealing alternative to Brownian motion in many applications such as economics, hydrology, and the study of fluctuations in solids. See [19] and the references therein for further detailed information about fBm and its applications.

For $H \neq 1/2$, fBm is not a semi-martingale (see [23]) and, hence, Ito's calculus cannot be used to define an integral with respect to fBm or to construct a stochastic differential equation (SDE) driven by fractional noise. Several approaches have been taken to construct an alternative stochastic calculus for fBm. Among the key ingredients in these constructions are the development of a stochastic integral and the subsequent derivation of an "Ito-like" formula for change of variables.

Before discussing some of these approaches, let us first present some situations in which one might be motivated to consider fBm, or other similar processes, and in which one would like to utilize a corresponding stochastic calculus.

The first example comes from mathematical finance. The celebrated Black-Scholes model represents the logarithm of the price of a risky asset (such as a stock) by a Brownian motion. Ito's calculus is then used to perform analysis on the model such as pricing derivative commodities. It has been suggested that fBm with $H \in (1/2, 1)$, rather than Brownian motion, is a more realistic representative of the logarithm of the price process due to its long-range dependence. Constructing an analog of Black and Scholes' method for fBm obviously requires the use of a stochastic calculus for fBm. It should be noted that there can be many examples like this, not just in finance. Such an example could occur anytime one wants to model a phenomenon with long-range dependence using SDEs.

The second example stems from the connection between probability and deterministic partial differential equations (PDEs). Consider the heat equation, $\partial_t u = \frac{1}{2}\partial_x^2 u$, $u(0, x) = f(x)$. Its solution can be represented probabilistically as $u(t, x) = E_x[f(B_t)]$, where B_t is a Brownian motion. Intuitively, we can imagine a large collection of heat "particles", initially distributed with density $f(x)$, performing Brownian motions and thereby distributing themselves according to the heat equation. In reality, of course, the heat comes from particles whose motion is not Brownian, but linear. These particles collide and (at least heuristically) the collisions produce Brownian motion in the limit as the number of particles increases. If, however, the colliding particles *were* performing Brownian motion, then the collisions ought to produce, in the limit, fBm with $H = 1/4$. In [15], Harris considered an infinite number of particles, initially placed on the real line according to a Poisson distribution, performing independent Brownian motions and undergoing "elastic" collisions. By this, it is meant simply that the particles perform independent Brownian motions, but are continuously relabelled so as to preserve their initial ordering, i.e. their trajectories are interchanged whenever their paths intersect. The trajectory of a fixed particle was then observed and it was shown that, after rescaling, this trajectory indeed converged to a fBm with $H = 1/4$. These results were strengthened and generalized in [9]. Earlier in this dissertation, we considered a similar model of colliding particles. There, we took n independent Brownian motions, $B_t^{(1)}, \ldots, B_t^{(n)}$, all starting at the origin, and studied the process $X_t^{(n)} = \sqrt{n}\, M_t^{(n)}$, where at any time t, $M_t^{(n)}$ is the median of the n numbers $B_t^{(1)}, \ldots, B_t^{(n)}$. There it was shown that $X_t^{(n)}$ converged, as $n \to \infty$, to a centered Gaussian process, X_t, with covariance

$$E[X_s X_t] = \sqrt{st}\, \sin^{-1}\left(\frac{s \wedge t}{\sqrt{st}}\right).$$

While X_t is certainly *not* a fBm, it does share many of the same properties as fBm with $H = 1/4$. In particular, the process X_t exhibits *fourth-order scaling*, i.e. $E|X_{t+\Delta t} - X_t|^2 \approx \Delta t^{1/2}$ for Δt small. These heuristics suggest a connection between fBm with $H = 1/4$, as well as other processes with fourth-order scaling, to higher order parabolic PDEs such as $\partial_t u = \partial_x^4 u$. These connections are explored in [2], [5], and [13]. Clearly, any extension to fBm of the connection between SDEs and PDEs will require a stochastic calculus for fBm (or, more generally, a stochastic calculus for processes with same order scaling as fBm).

With these examples in mind, let us now consider some of the approaches that have been taken to constructing a stochastic integral and an "Ito" rule for fBm. (For a survey of many of these approaches and for references to the remainder of this introduction, see [6].) The most natural thing to do is to define

$$\int u(s)\, dB_H(s) = \lim_{\Delta t \to 0} \sum_j u(t_j)(B_H(t_j + \Delta t) - B_H(t_j)) \tag{4.1.1}$$

and then to determine the conditions under which this definition makes sense. For example, the limit on the right-hand side of (4.1.1) exists (at least in probability) if the integrand u is

Hölder continuous with index β for some $\beta > 1 - H$. In this case, the process $t \mapsto \int_0^t u\, dB_H$ is Hölder continuous with index β for all $\beta < H$. This implies, among other things, that we can only be assured of a well-defined iterated integral of this type when $H \in (1/2, 1)$.

This is the so-called "pathwise" approach to integration against fBm. It is the regularity of the sample paths of the integrand, not the adaptedness of the integrand, that is important in this approach. In particular, there is nothing terribly special about the use of left-endpoint Riemann sums in (4.1.1). In fact, if $H \in (1/2, 1)$, then when this integral exists and when the integrand is of bounded quadratic variation, we may replace $u(t_j)$ in (4.1.1) by $u(t_j^*)$, where t_j^* is any point in the interval $[t_j, t_j + \Delta t]$, without changing the definition of the integral (see Theorem 3.16 in [8]).

The change-of-variables formula for this integral, when $H \in (1/2, 1)$, is very simple:

$$f(B_H(t)) = f(B_H(0)) + \int_0^t f'(B_H(s))\, dB_H(s).$$

For other values of H, the change of variables formula can be significantly more complicated (see, for example, [1], [3], and [21], as well as Section 3.4, Proposition 3.1, and Remark 4.3 in [6]). This integral also, unfortunately, lacks one of the key features of the Ito integral. Namely, $E \int u\, dB_H$ need not be zero. See [8] for an explicit example. From the point of view of applications, this seems to be a difficulty for modelling, since a term such as $\sigma_t\, dB_H(t)$ in an SDE ought to represent a random fluctuation about the mean. On the other hand, from a purely mathematical perspective, it is also disappointing. The zero mean of the Ito integral is a key ingredient in developing the connection between Ito diffusions and their corresponding PDEs. A nonzero mean for the integral against fBm would present a significant hurdle to developing any analog of this for higher order parabolic equations.

An alternative approach is to define the integral with respect to fBm via the Malliavin calculus, as in section 3.3 of [6] (see also [7]). For $H > 1/2$, the integral defined in this way agrees with the integral defined in [8] by means of the Wick product, i.e.

$$\int u(s)\, dB_H(s) = \lim_{\Delta t \to 0} \sum_j u(t_j) \diamond (B_H(t + \Delta t) - B_H(t)).$$

(Here, $\diamond$ denotes the Wick product; for the definition of the Wick product in this context, see [8].) This approach is motivated by the fact that, when $H = 1/2$, this definition yields the classical Ito integral. Of course, when $H = 1/2$, each of the Wick products in this definition agrees with the ordinary product. In fact, it should be noted that the Ito integral with respect to a martingale that is not Brownian motion cannot, in general, be defined as the limit of Riemann sums of Wick products.

Unlike the pathwise integral, this integral does have the property that $E \int u \, dB_H = 0$. Also, at least for $H > 1/2$, the change-of-variables formula is fairly simple:

$$f(B_H(t)) = f(B_H(0)) + \int_0^t f'(B_H(s)) \, dB_H(s) + H \int_0^t s^{2H-1} f''(B_H(s)) \, ds.$$

However, the change-of-variables formula for $f(X_t)$, where X_t is itself an integral with respect to fBm, is less simple in that it involves derivatives in the sense of Malliavin. Also, for $H < 1/2$, a change-of-variables formulas is either not known or is significantly more complicated (see Remark 4.3 in [6] and the references therein). Moreover, from a modelling perspective, it is unclear what the Wick products in the approximating Riemann sums actually represent.

There are still several other approaches to constructing an integral against fBm. The work of Terry Lyons [18] provides a general construction for developing an integral against rough paths on a manifold. Coutin and Qian [4] directly apply Lyons' results to construct an integral against fBm when $H > 1/4$. This approach, of course, is pathwise in nature. Carmona, Coutin, and Montseny [3] also cite the work of Lyons as a primary source of motivation for their approach, which is to approximate fBm by semimartingales. The integrals with respect to these semimartingales are well-defined in the Ito sense, and they use a limiting procedure to define the integral with respect to fBm. Surprisingly, however, their approach seems to bear a stronger relationship to the Malliavin construction than to the pathwise construction. For a discussion on the connections between all of these various developments, see [6].

The motivation behind the work presented in this chapter lies in the attempt to derive an Ito rule for a pathwise integral with respect to fBm when $H = 1/4$. Consider a process, F_t, with fourth-order scaling, i.e. $E|F_{t+\Delta t} - F_t|^2 \approx \Delta t^{1/2}$ for Δt small, and let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function with $g(0) = 0$. Write

$$g(x+h) = g(x) + g'(x)h + \frac{1}{2} g''(x)h^2 + \frac{1}{6} g'''(x_1)h^3$$

where x_1 lies between x and $x+h$. Plugging in h_1 and h_2 and subtracting yields

$$\begin{aligned} g(x+h_1) - g(x+h_2) &= g'(x)(h_1 - h_2) + \frac{1}{2}g''(x)(h_1^2 - h_2^2) \\ &\quad + \frac{1}{6}g'''(x_1)h_1^3 - \frac{1}{6}g'''(x_2)h_2^3 \end{aligned}$$

Now define $\Delta^+ F_t = F_{t+\Delta t} - F_t$ and $\Delta^- F_t = F_t - F_{t-\Delta t}$, and take $x = F_t$, $h_1 = \Delta^+ F_t$, and $h_2 = -\Delta^- F_t$, so that

$$\begin{aligned} g(F_{t+\Delta t}) - g(F_{t-\Delta t}) &= g'(F_t)(F_{t+\Delta t} - F_{t-\Delta t}) + \frac{1}{2}g''(F_t)(|\Delta^+ F_t|^2 - |\Delta^- F_t|^2) \\ &\quad \frac{1}{6}g'''(F_t^{(1)})(\Delta^+ F_t)^3 + \frac{1}{6}g'''(F_t^{(2)})(\Delta^- F_t)^3 \end{aligned}$$

where $F_t^{(j)}$ are random points lying between $F_{t-\Delta t}$ and $F_{t+\Delta t}$. Therefore, we can write

$$\begin{aligned} g(F_t) &= \sum_j g'(F_{t_j})(F_{t_j+\Delta t} - F_{t_j-\Delta t}) + \frac{1}{2}\sum_j g''(F_{t_j})(|\Delta^+ F_{t_j}|^2 - |\Delta^- F_{t_j}|^2) \\ &\quad + \frac{1}{6}\sum_j g'''(F_{t_j}^{(1)})(\Delta^+ F_{t_j})^3 + \frac{1}{6}\sum_j g'''(F_{t_j}^{(1)})(\Delta^- F_{t_j})^3 \end{aligned} \tag{4.1.2}$$

where the sums are over a suitable partition of the interval $[0,t]$. If, then, we wish to define a pathwise, Stratonovich-type integral with respect to F_t by

$$\int_0^t g'(F_s)\,dF_s = \lim_{\Delta t \to 0} \sum_j g'(F_{t_j})(F_{t_j+\Delta t} - F_{t_j-\Delta t}), \tag{4.1.3}$$

we must investigate the convergence of the last three sums on the right-hand side of (4.1.2).

If F_t has mean zero and a symmetric distribution, then $(\Delta^+ F_{t_j})^3$ has mean zero and an approximate variance of $\Delta t^{3/2}$. The same is also true for $(\Delta^- F_{t_j})^3$. We might therefore expect the last two sums on the right-hand side of (4.1.2) to converge to zero as $\Delta t \to 0$. As for the second sum, let us simplify things for the moment and assume that $g'' \equiv 1$. In this case, we must consider the sum

$$\sum_j (|\Delta^+ F_{t_j}|^2 - |\Delta^- F_{t_j}|^2) \tag{4.1.4}$$

Each of the random variables, $|\Delta^+ F_{t_j}|^2 - |\Delta^- F_{t_j}|^2$, has an approximate mean of zero and an approximate variance of Δt. This is reminiscent of the construction of Brownian motion as the scaling limit of random walks, and we might therefore expect (4.1.4), as a function of t, to converge as $\Delta t \to 0$ to a Brownian motion. The chief difficulty in the heuristic arguments in this paragraph is, of course, that the random variables being summed together are *not* independent. If, however, these conclusions are valid, this suggests that the integral defined by (4.1.3) obeys an Ito rule of the form

$$g(F_t) = \int_0^t g'(F_s)\, dF_s + \frac{1}{2} \int_0^t g''(F_s)\, dB_s.$$

In this chapter, we shall focus on a specific process F_t with fourth-order scaling. To simplify things, we will not study the sum in (4.1.4), but rather the sum

$$\sum_j [(\Delta^+ F_{t_j})^2 \cdot \operatorname{sgn}(\Delta^+ F_{t_j})] \tag{4.1.5}$$

We then show, despite the lack of independence, that this sum, as a function of t, converges as $\Delta t \to 0$ to a Brownian motion. Presumably, the mean and covariance structure of the summands in (4.1.5) are asymptotically similar enough to those in (4.1.4) that the results and techniques in this chapter can be used to make rigorous the heuristics outlined above.

We will now describe the process F_t and state our main results. The notation established in the remainder of this introduction shall be used throughout the entirety of this chapter.

To describe the process F_t that we shall study, first consider the stochastic heat equation

$$u_t = \frac{1}{2} u_{xx} + \dot{W}(t,x) \tag{4.1.6}$$

with boundary conditions $u(0,x) \equiv 0$, where $\dot{W}(t,x)$ is a two-dimensional white noise. For the deterministic heat equation with a potential

$$u_t = \frac{1}{2} u_{xx} + g(t,x)$$

and with the same boundary conditions, it is well known that the solution is given by

$$u(t,x) = \int_{\mathbb{R}} \int_0^t p(t-r, x-y) g(r,y)\, dr\, dy,$$

where $p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ is the heat kernel. (See, for example, Chapter 4 in [11].) We therefore regard the random process

$$u(t,x) = \int_{\mathbb{R}} \int_0^t p(t-r, x-y)\, dW(r,y) \tag{4.1.7}$$

as the solution to (4.1.6). For the precise meaning of (4.1.7), we define the Hilbert space $H = L^2(\mathbb{R}^2)$ and construct a centered Gaussian process, $I(h)$, indexed by $h \in H$, such that $E[I(g)I(h)] = \int gh$. (See, for example, [16] or [20] for details.) We then interpret (4.1.7) as $u(t,x) = I(h_{tx})$, where $h_{tx}(r,y) = 1_{[0,t]}(r)p(t-r, x-y) \in H$.

For fixed $x \in \mathbb{R}$, let $F_t = F_t^{(x)} = u(t,x)$. By (4.1.7), we see that F_t is a centered Gaussian process. As will be seen later, the process F_t shares many of the properties of fBm with Hurst parameter 1/4. It is self-similar, i.e. $F_{ct} \overset{d}{=} c^{1/4}F_t$; its increments are negatively correlated; and $E|F_{t+\Delta t} - F_t|^2 \approx \Delta t^{1/2}$ for Δt small. In fact, since fBm also has a stochastic integral representation (see [6]), i.e.

$$B_H(t) = \int_0^t K_H(t,r)\, dB(r),$$

we expect that the techniques in this chapter can be directly applied to fBm itself.

Now, for fixed $n \in \mathbb{N}$, let $\Delta t = \frac{1}{n}$ and $t_j = j\Delta t$. We will consider the three processes

$$\begin{aligned} V_t^{(n)} &= \sum_{j=1}^{\lfloor nt \rfloor} \Delta F_j^4 + (nt - \lfloor nt \rfloor)\Delta F_{\lfloor nt \rfloor + 1}^4 \\ Z_t^{(n)} &= \sum_{j=1}^{\lfloor nt \rfloor} \Delta F_j^3 + (nt - \lfloor nt \rfloor)\Delta F_{\lfloor nt \rfloor + 1}^3 \\ W_t^{(n)} &= \sum_{j=1}^{\lfloor nt \rfloor} \Delta F_j^{2\pm} + (nt - \lfloor nt \rfloor)\Delta F_{\lfloor nt \rfloor + 1}^{2\pm} \end{aligned}$$

where $\Delta F_j = F_{t_j} - F_{t_{j-1}}$ and $x^{r\pm} = x^r \text{sgn}(x)$. To keep the notation manageable, ΔF_j^k shall always denote $(\Delta F_j)^k$. The processes $V^{(n)}$ are, of course, discrete approximations to the 4-th variation of the process F. The processes $Z^{(n)}$ and $W^{(n)}$ will be referred to, respectively, as the discrete cubic and quadratic signed variations of F. We will prove the following three results regarding the above processes.

Theorem 4.1.1 *As $n \to \infty$, $V_t^{(n)} \to \frac{6}{\pi}t$, where convergence is in the weak sense on the space $C[0,\infty)$, endowed with the topology of locally uniform convergence.*

Theorem 4.1.2 *As $n \to \infty$, $Z_t^{(n)} \to 0$, with convergence as in Theorem 4.1.1.*

Theorem 4.1.3 *As $n \to \infty$, $W_t^{(n)} \to \kappa W_t$, with convergence as in Theorem 4.1.1, where W_t is a Brownian motion and κ is a positive constant. Moreover, κ is given explicitly by $\kappa = (\frac{6}{\pi} - \frac{4}{\pi}\xi)^{1/2}$, where $\xi \equiv \sum_{i=1}^{\infty} K(\frac{1}{2}\gamma_i)$, $\gamma_i = 2\sqrt{i} - \sqrt{i-1} - \sqrt{i+1}$, and*

$$K(x) = \frac{6}{\pi} x\sqrt{1-x^2} + \frac{2}{\pi}(1+2x^2)\sin^{-1}(x).$$

4.2 Proof of Theorem 4.1.1

Recall that F_t is a centered Gaussian process. We begin by computing its covariance structure.

Lemma 4.2.1 *For all $s, t \in [0, \infty)$, $E[F_s F_t] = \frac{1}{\sqrt{2\pi}}(|t+s|^{1/2} - |t-s|^{1/2})$.*

Proof. Without loss of generality, assume $s \le t$. By (4.1.7),

$$\begin{aligned}
E[F_s F_t] &= \int_{\mathbb{R}} \int_0^s p(t-r, x-y) p(s-r, x-y)\, dr\, dy \\
&= \frac{1}{2\pi} \int_0^s \frac{1}{\sqrt{(t-r)(s-r)}} \int_{\mathbb{R}} \exp\left\{ -\frac{(x-y)^2}{2(t-r)} - \frac{(x-y)^2}{2(s-r)} \right\} dy\, dr \\
&= \frac{1}{2\pi} \int_0^s \frac{1}{\sqrt{(t-r)(s-r)}} \int_{\mathbb{R}} \exp\left\{ -\frac{(x-y)^2 (t+s-2r)}{2(t-r)(s-r)} \right\} dy\, dr \\
&= \frac{1}{\sqrt{2\pi}} \int_0^s \frac{1}{\sqrt{t+s-2r}}\, dr \\
&= -\frac{1}{\sqrt{2\pi}} \sqrt{t+s-2r}\,\Big|_0^s \\
&= \frac{1}{\sqrt{2\pi}}(|t+s|^{1/2} - |t-s|^{1/2}),
\end{aligned}$$

which verifies the formula. ∎

Now, since ΔF_j is a mean zero, normal random variable, its distribution is determined by its variance, $\sigma_j^2 \equiv E|\Delta F_j|^2$. Using the covariance structure of F, we estimate the order of σ_j.

Lemma 4.2.2 *For all $j \in \mathbb{N}$,*

$$\left| \sigma_j^2 - \sqrt{\frac{2\Delta t}{\pi}} \right| \le \frac{1}{\sqrt{\pi}\,(1+\sqrt{2})} \frac{1}{t_j^{3/2}} \Delta t^2 \le \frac{1}{t_j^{3/2}} \Delta t^2.$$

Proof. Note that for $s < t$,

$$\begin{aligned} E|F_t - F_s|^2 &= \frac{1}{\sqrt{2\pi}}(\sqrt{2t} + \sqrt{2s} - 2(\sqrt{t+s} - \sqrt{t-s})) \\ &= \frac{1}{\sqrt{\pi}}(\sqrt{t} + \sqrt{s} - \sqrt{2t+2s} + \sqrt{2t-2s}). \end{aligned} \tag{4.2.1}$$

Thus,

$$\begin{aligned} \left| E|F_t - F_s|^2 - \sqrt{\frac{2(t-s)}{\pi}} \right| &= \frac{1}{\sqrt{\pi}}|\sqrt{t} + \sqrt{s} - \sqrt{2t+2s}| \\ &= \frac{1}{\sqrt{\pi}} \left| \frac{(\sqrt{t} + \sqrt{s})^2 - (2t+2s)}{\sqrt{t} + \sqrt{s} + \sqrt{2t+2s}} \right| \\ &\le \frac{1}{\sqrt{\pi}\,(1+\sqrt{2})} \left| \frac{(\sqrt{t} - \sqrt{s})^2}{\sqrt{t}} \right| \\ &= \frac{1}{\sqrt{\pi}\,(1+\sqrt{2})} \frac{|t-s|^2}{\sqrt{t}(\sqrt{t} + \sqrt{s})^2} \\ &\le \frac{1}{\sqrt{\pi}\,(1+\sqrt{2})} \frac{1}{t^{3/2}}|t-s|^2. \end{aligned} \tag{4.2.2}$$

Now, take $t = t_j$ and $s = t_{j-1}$. ■

Corollary 4.2.3 *For all $j \in \mathbb{N}$, $\pi^{-1/2}\sqrt{\Delta t} \le \sigma_j^2 \le 2\sqrt{\Delta t}$.*

Proof. The upper bound follows from Lemma 4.2.2 since $2 < \pi$ and $t_j = j/n \ge 1/n = \Delta t$. Lemma 4.2.2 also implies

$$\begin{aligned} \sigma_j^2 &\ge \sqrt{\frac{2\Delta t}{\pi}} - \frac{1}{\sqrt{\pi}\,(1+\sqrt{2})} \frac{1}{t_j^{3/2}}\Delta t^2 \\ &\ge \left(\sqrt{\frac{2}{\pi}} - \frac{1}{\sqrt{\pi}\,(1+\sqrt{2})} \right) \sqrt{\Delta t} \\ &= \pi^{-1/2}\sqrt{\Delta t}, \end{aligned}$$

which gives the lower bound. ■

In the analysis to come, we will need the following estimate on the covariance of ΔF_i and ΔF_j for $i \ne j$.

Lemma 4.2.4 *For any $i, j \in \mathbb{N}$ with $i < j$,*

$$\left| E[\Delta F_i \Delta F_j] + \sqrt{\frac{\Delta t}{2\pi}}\, \gamma_{j-i} \right| \leq \frac{\Delta t^2}{(t_i + t_j)^{3/2}}$$

where γ_i is as in Theorem 4.1.3.

Proof. First note that for any $k \geq i$,

$$\begin{aligned} E[F_{t_k} \Delta F_i] &= E[F_{t_k} F_{t_i} - F_{t_k} F_{t_{i-1}}] \\ &= \frac{1}{\sqrt{2\pi}}(\sqrt{t_k + t_i} - \sqrt{t_k - t_i} - \sqrt{t_k + t_{i-1}} + \sqrt{t_k - t_{i-1}}) \\ &= \frac{1}{\sqrt{2\pi n}}(\sqrt{k+i} - \sqrt{k-i} - \sqrt{k+i-1} + \sqrt{k-i+1}). \end{aligned}$$

Thus,

$$\begin{aligned} E[\Delta F_i \Delta F_j] &= E[F_{t_j} \Delta F_i] - E[F_{t_{j-1}} \Delta F_i] \\ &= \frac{1}{\sqrt{2\pi n}}(\sqrt{j+i} - \sqrt{j-i} - \sqrt{j+i-1} + \sqrt{j-i+1} \\ &\quad -\sqrt{j+i-1} + \sqrt{j-i-1} + \sqrt{j+i-2} - \sqrt{j-i}) \\ &= -\sqrt{\frac{\Delta t}{2\pi}}\, (\gamma_{j+i-1} + \gamma_{j-i}). \end{aligned}$$

Next, note that by the concavity of $x \mapsto \sqrt{x}$, $\gamma_k > 0$ for all $k \in \mathbb{N}$. Also,

$$\begin{aligned} \gamma_k &= (\sqrt{k} - \sqrt{k-1}) - (\sqrt{k+1} - \sqrt{k}) \\ &= f(k-1) - f(k) \end{aligned}$$

where $f(x) = \sqrt{x+1} - \sqrt{x}$. For $k \geq 2$, the Mean Value Theorem gives $\gamma_k = |f'(k-\theta)|$, for some $\theta \in [0, 1]$. Now,

$$\begin{aligned} f'(x) &= \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}} \right) \\ &= \frac{1}{2}\left(\frac{\sqrt{x} - \sqrt{x+1}}{\sqrt{x(x+1)}} \right) \\ &= -\frac{1}{2}\left(\frac{1}{\sqrt{x(x+1)}(\sqrt{x} + \sqrt{x+1})} \right). \end{aligned}$$

Thus, $|f'(x)| \leq \frac{1}{4}x^{-3/2}$ and hence, $\gamma_k \leq \frac{1}{4}(k-1)^{-3/2} \leq \frac{1}{4}(k/2)^{-3/2} = \frac{1}{\sqrt{2}}k^{-3/2}$. For $k = 1$, we have $\gamma_1 = 2 - \sqrt{2} < \frac{1}{\sqrt{2}}$. Therefore, for all $k \in \mathbb{N}$,

$$0 < \gamma_k \leq \frac{1}{\sqrt{2}\, k^{3/2}}. \tag{4.2.3}$$

We now have, since $j + i \geq 2$,

$$\begin{aligned}
\left| E[\Delta F_i \Delta F_j] + \sqrt{\frac{\Delta t}{2\pi}}\, \gamma_{j-i} \right| &\leq \sqrt{\frac{\Delta t}{2\pi}} \frac{1}{\sqrt{2}\,(j+i-1)^{3/2}} \\
&\leq \sqrt{\frac{\Delta t}{2\pi}} \frac{1}{\sqrt{2}} \left(\frac{j+i}{2} \right)^{-3/2} \\
&= \sqrt{\frac{2\Delta t}{\pi}} \frac{1}{(i+j)^{3/2}}.
\end{aligned}$$

Since $2 < \pi$, $t_k = k/n$, and $\Delta t = 1/n$, this completes the proof. ■

Corollary 4.2.5 *For any $i, j \in \mathbb{N}$ with $i < j$, $|E[\Delta F_i \Delta F_j]| \leq 2\Delta t^2 (t_j - t_i)^{-3/2}$.*

Proof. This follows from Lemma 4.2.4 and (4.2.3) since $\sqrt{\frac{\Delta t}{2\pi}}\, \gamma_{j-i} \leq \sqrt{\Delta t}(j-i)^{-3/2}$. ■

The last lemma we will need for this section regards the covariance of powers of correlated Gaussian random variables.

Lemma 4.2.6 *Let X_1, X_2 be mean zero, jointly normal random variables with variances σ_j^2. If $\rho = (\sigma_1\sigma_2)^{-1}E[X_1X_2]$, then*

(i) $E[X_1^3 X_2^3] = \sigma_1^3\sigma_2^3\rho(6\rho^2 + 9)$ and

(ii) $E[X_1^4 X_2^4] = \sigma_1^4\sigma_2^4(24\rho^4 + 72\rho^2 + 9)$.

Proof. Define $Y_j = \sigma_j^{-1}X_j$, $U = Y_1$, and $V = (1-\rho^2)^{-1/2}(Y_2 - \rho Y_1)$. Then U and V are mean zero and jointly normal. Clearly, $EU^2 = 1$. We also have

$$\begin{aligned}
EV^2 &= (1-\rho^2)^{-1}(1 - 2\rho E[Y_1Y_2] + \rho^2) \\
&= (1-\rho^2)^{-1}(1 - 2\rho^2 + \rho^2) \\
&= 1
\end{aligned}$$

and $E[UV] = (1-\rho^2)^{-1/2}(E[Y_1Y_2]-\rho) = 0$. Thus U and V are independent standard normals. Since $X_1 = \sigma_1 U$ and $X_2 = \sigma_2(\sqrt{1-\rho^2}\,V + \rho U)$, we have

$$\begin{aligned} E[X_1^3X_2^3] &= \sigma_1^3\sigma_2^3 E[U^3(\sqrt{1-\rho^2}\,V+\rho U)^3] \\ &= \sigma_1^3\sigma_2^3\left\{3\rho(1-\rho^2)E[U^4V^2] + \rho^3 EU^6\right\} \\ &= \sigma_1^3\sigma_2^3\left\{9\rho(1-\rho^2) + 15\rho^3\right\} \\ &= \sigma_1^3\sigma_2^3\rho(6\rho^2+9), \end{aligned}$$

which is part (i); and we have

$$\begin{aligned} E[X_1^4X_2^4] &= \sigma_1^4\sigma_2^4 E[U^4(\sqrt{1-\rho^2}\,V+\rho U)^4] \\ &= \sigma_1^4\sigma_2^4\left\{(1-\rho^2)^2E[U^4V^4] + 6\rho^2(1-\rho^2)E[U^6V^2] + \rho^4 EU^8\right\} \\ &= \sigma_1^4\sigma_2^4\left\{9(1-\rho^2)^2 + 6(15)\rho^2(1-\rho^2) + 105\rho^4\right\} \\ &= \sigma_1^4\sigma_2^4(9 - 18\rho^2 + 9\rho^4 + 90\rho^2 - 90\rho^4 + 105\rho^4) \\ &= \sigma_1^4\sigma_2^4(24\rho^4 + 72\rho^2 + 9), \end{aligned}$$

which is part (ii). ■

Proof of Theorem 4.1.1:

In what follows, C is a finite, nonnegative, universal constant that may change value from line to line. It will suffice to show that for all $0 \le s < t$ and all $n \in \mathbb{N}$,

$$E\left|(V_t^{(n)} - V_s^{(n)}) - \frac{6}{\pi}(t-s)\right|^2 \le \frac{C}{\sqrt{n}}|t-s|^{3/2}.$$

To see why, note that this implies

$$\begin{aligned} E|V_t^{(n)} - V_s^{(n)}|^2 &\le 2\left|\frac{6}{\pi}(t-s)\right|^2 + 2C|t-s|^{3/2} \\ &\le C(|t-s|^2 + |t-s|^{3/2}) \end{aligned}$$

so that, by the Kolmogorov-Čentsov Theorem (see Problem 2.4.11 in [17]), the sequence $V^{(n)}$ is tight. Also, if $0 \le t_1 < \cdots < t_d$ and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$, then

$$\begin{aligned} E\left\|(V_{t_1}^{(n)},\ldots,V_{t_d}^{(n)}) - \frac{6}{\pi}(t_1,\ldots,t_d)\right\|^2 &= \sum_{j=1}^d E\left|V_{t_j}^{(n)} - \frac{6}{\pi}t_j\right|^2 \\ &\le \frac{C}{\sqrt{n}}\sum_{j=1}^d t_j^{3/2} \end{aligned}$$

and, hence, the finite dimensional distributions of $V^{(n)}$ converge in L^2, and therefore in probability and in distribution, to those of the constant function $t \mapsto \frac{6}{\pi}t$.

To prove the initial estimate, we write

$$V_t^{(n)} - V_s^{(n)} = \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \Delta F_j^4 + (nt - \lfloor nt \rfloor)\Delta F_{\lfloor nt \rfloor+1}^4 - (ns - \lfloor ns \rfloor)\Delta F_{\lfloor ns \rfloor+1}^4.$$

Note that the number of terms in the above sum is $\lfloor nt \rfloor - \lfloor ns \rfloor < n(t-s)+1$. First, assume that $n(t-s) \le 1$. In this case, either $\lfloor nt \rfloor - \lfloor ns \rfloor = 0$ or $\lfloor nt \rfloor - \lfloor ns \rfloor = 1$. If $\lfloor nt \rfloor - \lfloor ns \rfloor = 0$, then

$$\begin{aligned} V_t^{(n)} - V_s^{(n)} &= (nt - \lfloor nt \rfloor)\Delta F_{\lfloor nt \rfloor+1}^4 - (ns - \lfloor ns \rfloor)\Delta F_{\lfloor ns \rfloor+1}^4 \\ &= n(t-s)\Delta F_{\lfloor ns \rfloor+1}^4 \end{aligned}$$

so by Corollary 4.2.3,

$$\begin{aligned} E\left|(V_t^{(n)} - V_s^{(n)}) - \frac{6}{\pi}(t-s)\right|^2 &= |t-s|^2 E\left|n\Delta F_{\lfloor ns \rfloor+1}^4 - \frac{6}{\pi}\right|^2 \\ &\le C|t-s|^2 \\ &\le \frac{C}{\sqrt{n}}|t-s|^{3/2}. \end{aligned}$$

On the other hand, if $\lfloor nt \rfloor - \lfloor ns \rfloor = 1$, then

$$\begin{aligned} V_t^{(n)} - V_s^{(n)} &= \Delta F_{\lfloor ns \rfloor+1}^4 + (nt - \lfloor nt \rfloor)\Delta F_{\lfloor nt \rfloor+1}^4 - (ns - \lfloor ns \rfloor)\Delta F_{\lfloor ns \rfloor+1}^4 \\ &= \Delta F_{\lfloor nt \rfloor}^4 + (nt - \lfloor nt \rfloor)\Delta F_{\lfloor nt \rfloor+1}^4 - (ns - \lfloor nt \rfloor + 1)\Delta F_{\lfloor nt \rfloor}^4 \\ &= (\lfloor nt \rfloor - ns)\Delta F_{\lfloor nt \rfloor}^4 + (nt - \lfloor nt \rfloor)\Delta F_{\lfloor nt \rfloor+1}^4. \end{aligned}$$

Since $ns < \lfloor ns \rfloor + 1 = \lfloor nt \rfloor \le nt$,

$$|V_t^{(n)} - V_s^{(n)}| \le n(t-s)\Delta F_{\lfloor nt \rfloor}^4 + n(t-s)\Delta F_{\lfloor nt \rfloor+1}^4$$

and by Corollary 4.2.3,

$$\begin{aligned} E\left|(V_t^{(n)} - V_s^{(n)}) - \frac{6}{\pi}(t-s)\right|^2 &\le C(E|V_t^{(n)} - V_s^{(n)}|^2 + |t-s|^2) \\ &\le C(n^2|t-s|^2\Delta t^2 + |t-s|^2) \\ &= C|t-s|^2 \\ &\le \frac{C}{\sqrt{n}}|t-s|^{3/2}. \end{aligned}$$

Next, assume $n(t-s) > 1$. In this case, Corollary 4.2.3 gives

$$E\left|(V_t^{(n)} - V_s^{(n)}) - \frac{6}{\pi}(t-s)\right|^2 \le C\left\{E\left|\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\left(\Delta F_j^4 - \frac{6}{\pi}\Delta t\right)\right|^2 + \Delta t^2\right\}.$$

Note that in this case, $\Delta t^2 = n^{-2} \le n^{-1/2}|t-s|^{3/2}$. Now let $\sigma_j^2 = E|\Delta F_j|^2$ and write

$$\begin{aligned} E\left|\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\left(\Delta F_j^4 - \frac{6}{\pi}\Delta t\right)\right|^2 &\le 2E\left|\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}(\Delta F_j^4 - 3\sigma_j^4)\right|^2 \\ &\quad +2\left|\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\left(3\sigma_j^4 - \frac{6}{\pi}\Delta t\right)\right|^2. \end{aligned}$$

By Lemma 4.2.2,

$$\begin{aligned} \left|3\sigma_j^4 - \frac{6}{\pi}\Delta t\right| &= 3\left|\sigma_j^2 + \sqrt{\frac{2\Delta t}{\pi}}\right|\left|\sigma_j^2 - \sqrt{\frac{2\Delta t}{\pi}}\right| \\ &\le C\sqrt{\Delta t}\left|\sigma_j^2 - \sqrt{\frac{2\Delta t}{\pi}}\right| \\ &\le \frac{C}{t_j^{3/2}}\Delta t^{5/2}. \end{aligned}$$

Thus, by Jensen's inequality,

$$\begin{aligned} \left|\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\left(3\sigma_j^4 - \frac{6}{\pi}\Delta t\right)\right|^2 &\le (\lfloor nt\rfloor - \lfloor ns\rfloor)\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\left|3\sigma_j^4 - \frac{6}{\pi}\Delta t\right|^2 \\ &\le C(n(t-s)+1)\Delta t^5\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\frac{1}{t_j^3} \\ &= C\left(\frac{t-s}{n} + \frac{1}{n^2}\right)\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\frac{1}{j^3} \\ &\le \frac{C}{\sqrt{n}}|t-s|^{3/2}. \end{aligned} \tag{4.2.4}$$

Finally, we adopt the notation $x^{\sim r} = (x^r \vee 1)$, so that we may concisely combine Corollaries 4.2.3 and 4.2.5 into the inequality

$$|E[\Delta F_i \Delta F_j]| \le \frac{2\sqrt{\Delta t}}{|i-j|^{\sim 3/2}} \tag{4.2.5}$$

for all $i, j \in \mathbb{N}$. Now, by Lemma 4.2.6 with $\rho_{ij} = (\sigma_i\sigma_j)^{-1}E[\Delta F_i \Delta F_j]$, write

$$\begin{aligned} E\left|\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}(\Delta F_j^4 - 3\sigma_j^4)\right|^2 &\leq \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}|E[(\Delta F_i^4 - 3\sigma_i^4)(\Delta F_j^4 - 3\sigma_j^4)]| \\ &= \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}|E[\Delta F_i^4 \Delta F_j^4] - 9\sigma_i^4\sigma_j^4| \\ &\leq C\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\sigma_i^4\sigma_j^4\rho_{ij}^2 \\ &= C\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\sigma_i^2\sigma_j^2|E[\Delta F_i \Delta F_j]|^2. \end{aligned}$$

Now, by (4.2.5), we have

$$\begin{aligned} E\left|\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}(\Delta F_j^4 - 3\sigma_j^4)\right|^2 &\leq C\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\sqrt{\Delta t}\sqrt{\Delta t}\left(\frac{\Delta t}{|i-j|^{\sim 3}}\right) \\ &\leq C(\lfloor nt\rfloor - \lfloor ns\rfloor)\Delta t^2 \\ &\leq C\left(\frac{t-s}{n} + \frac{1}{n^2}\right) \\ &\leq \frac{C}{\sqrt{n}}|t-s|^{3/2}, \end{aligned}$$

which completes the proof. ■

4.3 Proof of Theorem 4.1.2

This proof will rely on the lemmas of the previous section.

Proof of Theorem 4.1.2:

As in the proof of Theorem 4.1.1, it will suffice to show that for all $0 \leq s < t$ and all $n \in \mathbb{N}$,

$$E|Z_t^{(n)} - Z_s^{(n)}|^2 \leq \frac{C}{n^{1/4}}|t-s|^{5/4}.$$

As before, we write

$$Z_t^{(n)} - Z_s^{(n)} = \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\Delta F_j^3 + (nt - \lfloor nt\rfloor)\Delta F_{\lfloor nt\rfloor+1}^3 - (ns - \lfloor ns\rfloor)\Delta F_{\lfloor ns\rfloor+1}^3.$$

Note that the number of terms in the above sum is $\lfloor nt \rfloor - \lfloor ns \rfloor < n(t-s)+1$. First, assume that $n(t-s) \leq 1$. In this case, either $\lfloor nt \rfloor - \lfloor ns \rfloor = 0$ or $\lfloor nt \rfloor - \lfloor ns \rfloor = 1$. If $\lfloor nt \rfloor - \lfloor ns \rfloor = 0$, then

$$Z_t^{(n)} - Z_s^{(n)} = n(t-s)\Delta F^3_{\lfloor ns \rfloor + 1}$$

so by Corollary 4.2.3,

$$\begin{aligned} E|Z_t^{(n)} - Z_s^{(n)}|^2 &\leq Cn^2|t-s|^2\Delta t^{3/2} \\ &= C\sqrt{n}|t-s|^2 \\ &\leq \frac{C}{n^{1/4}}|t-s|^{5/4}. \end{aligned}$$

On the other hand, if $\lfloor nt \rfloor - \lfloor ns \rfloor = 1$, then

$$\begin{aligned} Z_t^{(n)} - Z_s^{(n)} &= \Delta F^3_{\lfloor ns \rfloor + 1} + (nt - \lfloor nt \rfloor)\Delta F^3_{\lfloor nt \rfloor + 1} - (ns - \lfloor ns \rfloor)\Delta F^3_{\lfloor ns \rfloor + 1} \\ &= \Delta F^3_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)\Delta F^3_{\lfloor nt \rfloor + 1} - (ns - \lfloor nt \rfloor + 1)\Delta F^3_{\lfloor nt \rfloor} \\ &= (\lfloor nt \rfloor - ns)\Delta F^3_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)\Delta F^3_{\lfloor nt \rfloor + 1}. \end{aligned}$$

Since $ns < \lfloor ns \rfloor + 1 = \lfloor nt \rfloor \leq nt$,

$$|Z_t^{(n)} - Z_s^{(n)}| \leq n|t-s||\Delta F^3_{\lfloor nt \rfloor}| + n|t-s||\Delta F^3_{\lfloor nt \rfloor + 1}|$$

and by Corollary 4.2.3,

$$\begin{aligned} E|Z_t^{(n)} - Z_s^{(n)}|^2 &\leq Cn^2|t-s|^2\Delta t^{3/2} \\ &\leq \frac{C}{n^{1/4}}|t-s|^{5/4}. \end{aligned}$$

Next, assume $n(t-s) > 1$. In this case, Corollary 4.2.3 gives

$$E|Z_t^{(n)} - Z_s^{(n)}|^2 \leq C\left\{ E\left| \sum_{j=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \Delta F_j^3 \right|^2 + \Delta t^{3/2} \right\}.$$

Note that in this case, $\Delta t^{3/2} = n^{-3/2} \leq n^{-1/4}|t-s|^{5/4}$. Now, by Lemma 4.2.6 with

$\rho_{ij} = (\sigma_i\sigma_j)^{-1}E[\Delta F_i \Delta F_j]$, write

$$\begin{aligned} E\left|\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \Delta F_j^3\right|^2 &\leq \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} |E[\Delta F_i^3 \Delta F_j^3]| \\ &\leq C\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \sigma_i^3\sigma_j^3|\rho_{ij}| \\ &= C\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \sigma_i^2\sigma_j^2|E[\Delta F_i \Delta F_j]|. \end{aligned}$$

By (4.2.5), we have

$$\begin{aligned} E\left|\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \Delta F_j^3\right|^2 &\leq C\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \sqrt{\Delta t}\sqrt{\Delta t}\left(\frac{\sqrt{\Delta t}}{|i-j|^{\sim 3/2}}\right) \\ &\leq C(\lfloor nt\rfloor - \lfloor ns\rfloor)\Delta t^{3/2} \\ &\leq C\left(\frac{t-s}{\sqrt{n}} + \frac{1}{n^{3/2}}\right) \\ &\leq \frac{C}{n^{1/4}}|t-s|^{5/4}, \end{aligned}$$

and we are done. ■

4.4 Key Estimates

In this section we will establish some important estimates needed in the proof of Theorem 4.1.3.

Lemma 4.4.1 *Let X_1, X_2 be mean zero, jointly normal random variables with $EX_j^2 = 1$ and $\rho = E[X_1X_2]$. Then*

$$E[X_1^{2\pm}X_2^{2\pm}] = K(\rho),$$

where $K(x) = \frac{6}{\pi}x\sqrt{1-x^2} + \frac{2}{\pi}(1+2x^2)\sin^{-1}(x)$ is as in Theorem 4.1.3.

Proof. Define $U = X_1$ and $V = (1-\rho^2)^{-1/2}(X_2 - \rho X_1)$, so that U and V are independent standard normals. Then $X_1 = U$ and $X_2 = \sqrt{1-\rho^2}\,V + \rho U$ and

$$\begin{aligned}
E[X_1^{2\pm} X_2^{2\pm}] &= E[U(\sqrt{1-\rho^2}\,V + \rho U)]^{2\pm} \\
&= \frac{1}{2\pi}\iint [u(\sqrt{1-\rho^2}\,v + \rho u)]^{2\pm} e^{-(u^2+v^2)/2}\,du\,dv \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} [\cos\theta(\sqrt{1-\rho^2}\,\sin\theta + \rho\cos\theta)]^{2\pm} r^5 e^{-r^2/2}\,dr\,d\theta \\
&= \frac{4}{\pi}\int_0^{2\pi} [\cos\theta(\sqrt{1-\rho^2}\,\sin\theta + \rho\cos\theta)]^{2\pm}\,d\theta
\end{aligned}$$

since $\int_0^\infty r^5 e^{-r^2/2}\,dr = 8$. Now, since the above integrand remains unchanged under the transformation $\theta \mapsto \theta + \pi$, we have

$$E[X_1^{2\pm} X_2^{2\pm}] = \frac{8}{\pi}\int_{-\pi/2}^{\pi/2} [\cos\theta(\sqrt{1-\rho^2}\,\sin\theta + \rho\cos\theta)]^{2\pm}\,d\theta.$$

Since $\cos\theta \geq 0$ on the interval of integration, we have

$$\begin{aligned}
E[X_1^{2\pm} X_2^{2\pm}] &= \frac{8}{\pi}\int_a^{\pi/2} [\cos\theta(\sqrt{1-\rho^2}\,\sin\theta + \rho\cos\theta)]^2\,d\theta \\
&\quad - \frac{8}{\pi}\int_{-\pi/2}^{a} [\cos\theta(\sqrt{1-\rho^2}\,\sin\theta + \rho\cos\theta)]^2\,d\theta
\end{aligned}$$

where $a = \tan^{-1}(-\rho(1-\rho^2)^{-1/2})$. With the change of variables $\theta \mapsto -\theta$, we then have

$$\begin{aligned}
E[X_1^{2\pm} X_2^{2\pm}] &= \frac{8}{\pi}\int_{-\pi/2}^{-a} [\cos\theta(\sqrt{1-\rho^2}\,\sin\theta - \rho\cos\theta)]^2\,d\theta \\
&\quad - \frac{8}{\pi}\int_{-\pi/2}^{a} [\cos\theta(\sqrt{1-\rho^2}\,\sin\theta + \rho\cos\theta)]^2\,d\theta.
\end{aligned}$$

Assume for the moment that $\rho \leq 0$, so that $a \geq 0$. Then

$$\begin{aligned}
E[X_1^{2\pm} X_2^{2\pm}] &= -\frac{32}{\pi}\rho\sqrt{1-\rho^2}\int_{-\pi/2}^{-a} \cos^3\theta\sin\theta\,d\theta \\
&\quad - \frac{8}{\pi}\int_{-a}^{a} [\cos\theta(\sqrt{1-\rho^2}\,\sin\theta + \rho\cos\theta)]^2\,d\theta \\
&= \frac{8}{\pi}\rho\sqrt{1-\rho^2}\,\cos^4 a - \frac{16}{\pi}(1-\rho^2)\int_0^a \cos^2\theta\sin^2\theta\,d\theta \\
&\quad - \frac{16}{\pi}\rho^2\int_0^a \cos^4\theta\,d\theta \\
&= \frac{8}{\pi}\rho\sqrt{1-\rho^2}\,\cos^4 a - \frac{16}{\pi}\int_0^a \cos^4\theta\,d\theta \\
&\quad + \frac{16}{\pi}(1-\rho^2)\int_0^a (\cos^4\theta - \cos^2\theta\sin^2\theta)\,d\theta.
\end{aligned}$$

Note that $a = \sin^{-1}(-\rho)$, so that $\rho = -\sin a$. Thus,

$$\begin{aligned} E[X_1^{2\pm}X_2^{2\pm}] &= -\frac{8}{\pi}\sin a\cos^5 a - \frac{16}{\pi}\int_0^a \cos^4\theta\, d\theta \\ &\quad +\frac{16}{\pi}\cos^2 a\int_0^a (\cos^4\theta - \cos^2\theta\sin^2\theta)\, d\theta. \end{aligned}$$

Observe that

$$\begin{aligned} \frac{d}{d\theta}(\theta + \sin\theta\cos\theta + 2\sin\theta\cos^3\theta) &= 1 + \cos^2\theta - \sin^2\theta + 2\cos^4\theta - 6\sin^2\theta\cos^2\theta \\ &= 2\cos^2\theta(1 + \cos^2\theta - 3\sin^2\theta) \\ &= 2\cos^2\theta(2\cos^2\theta - 2\sin^2\theta) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{d\theta}(3\theta + 3\sin\theta\cos\theta + 2\sin\theta\cos^3\theta) &= 3 + 3\cos^2\theta - 3\sin^2\theta + 2\cos^4\theta - 6\sin^2\theta\cos^2\theta \\ &= 2\cos^2\theta(3 + \cos^2\theta - 3\sin^2\theta) \\ &= 8\cos^4\theta. \end{aligned}$$

Thus,

$$\begin{aligned} E[X_1^{2\pm}X_2^{2\pm}] &= \frac{2}{\pi}\big[-4\sin a\cos^5 a - 3a - 3\sin a\cos a - 2\sin a\cos^3 a \\ &\quad +2a\cos^2 a + 2\sin a\cos^3 a + 4\sin a\cos^5 a\big] \\ &= \frac{2}{\pi}\left[-3a - 3\sin a\cos a + 2a\cos^2 a\right] \\ &= \frac{2}{\pi}\left[-3\sin a\cos a - a(1 + 2\sin^2 a)\right] \\ &= \frac{2}{\pi}\left[3\rho\sqrt{1-\rho^2} + (1+2\rho^2)\sin^{-1}\rho\right]. \end{aligned}$$

This proves the lemma in the case that $\rho \leq 0$. In the case that $\rho > 0$, we have, by symmetry, $E[X_1^{2\pm}X_2^{2\pm}] = -K(-\rho) = K(\rho)$. ■

Lemma 4.4.2 *Let $K(x)$ be as in Theorem 4.1.3. For all $x \in [-1, 1]$,*

$$\left|K(x) - \frac{8}{\pi}x\right| \leq 2|x|^3.$$

Proof. Note that $K \in C^\infty(-1,1)$ and

$$\begin{aligned} K'(x) &= \frac{2}{\pi}\left[3\sqrt{1-x^2} - \frac{3x^2}{\sqrt{1-x^2}} + 4x\sin^{-1}(x) + \frac{1+2x^2}{\sqrt{1-x^2}}\right] \\ &= \frac{8}{\pi}[\sqrt{1-x^2} + x\sin^{-1}(x)] \\ K''(x) &= \frac{8}{\pi}\sin^{-1}(x). \end{aligned}$$

Since K'' is increasing,

$$\left|K(x) - \frac{8}{\pi}x\right| \le \frac{1}{2}x^2 K''(|x|).$$

But for $y \in [0,\pi/2]$, $\sin y \ge 2y/\pi$. Letting $y = \pi x/2$ gives $\sin^{-1}(x) \le \pi x/2$ for $x \in [0,1]$. Thus, $K''(|x|) \le 4|x|$, which completes the proof. ∎

Corollary 4.4.3 *Let X_1, X_2 be mean zero, jointly normal random variables with $EX_j^2 = 1$ and $\rho = E[X_1X_2]$. Then*

$$|E[X_1^{2\pm}X_2^{2\pm}]| \le 5|\rho|.$$

Proof. This follows since $8/\pi < 3$ and $|\rho|^3 \le |\rho|$. ∎

Lemma 4.4.4 *Let $X_1, \dots, X_4$ be mean zero, jointly normal random variables with $EX_j^2 = 1$ and $\rho_{ij} = E[X_iX_j]$. Then*

$$\left|E\left[\prod_{j=1}^4 X_j^{2\pm}\right]\right| \le C \max_{1\le j\le 3}|\rho_{j4}|$$

for some finite, nonnegative, universal constant C.

Proof. Let $\tilde{X} = (X_1, X_2, X_3)^T$ and $v = (\rho_{14}, \rho_{24}, \rho_{34})^T$. Note that $\tilde{X} - vX_4$ and X_4 are independent. Define $F : \mathbb{R}^3 \to \mathbb{R}$ by $F(x_1,x_2,x_3) = \prod_j x_j^{2\pm}$ so that

$$\begin{aligned} \left|E\left[\prod_{j=1}^4 X_j^{2\pm}\right]\right| &= |E[F(\tilde{X})X_4^{2\pm}]| \\ &= |E[(F(\tilde{X}) - F(\tilde{X} - vX_4))X_4^{2\pm}]| \\ &\le \left(E|F(\tilde{X}) - F(\tilde{X} - vX_4)|^2\right)^{1/2} \left(E|X_4|^4\right)^{1/2}. \end{aligned}$$

Now, given $x, h \in \mathbb{R}^3$, there exists $\theta \in [0,1]$ such that $F(x+h) - F(x) = h \cdot \nabla F(x + \theta h)$. Moreover,

$$\begin{aligned} |\partial_1 F(x)| &= |2|x_1|x_2^{2\pm} x_3^{2\pm}| \\ &\leq 2\|x\|^5 \end{aligned}$$

and similarly for $\partial_2 F$ and $\partial_3 F$ where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$. Thus,

$$|F(x+h) - F(x)| \leq C\|h\|(\|x\|^5 + \|h\|^5)$$

which gives

$$\begin{aligned} \left| E\left[\prod_{j=1}^4 X_j^{2\pm}\right] \right| &\leq C\left(E[\|vX_4\|^2\|\tilde{X}\|^{10}] + E\|vX_4\|^{12}\right)^{1/2} \\ &\leq C(\|v\|^2 + \|v\|^{12})^{1/2}. \end{aligned}$$

Since $\|v\| \leq C \max_{1\leq j\leq 3} |\rho_{j4}|$ and each $|\rho_{j4}| \leq 1$, this completes the proof. ■

Now, for $k_1 \leq \cdots \leq k_4 \in \mathbb{N}$, let

$$\Delta_{k_1\cdots k_4} = \prod_{j=1}^4 \Delta F_{k_j}^{2\pm} \tag{4.4.1}$$

and recall that $x^{\sim r} = (x^r \vee 1)$.

Corollary 4.4.5 *For all $k_1 \leq \cdots \leq k_4 \in \mathbb{N}$,*

$$|E\Delta_{k_1\cdots k_4}| \leq C\frac{\Delta t^2}{(k_4 - k_3)^{\sim 3/2}}$$

and

$$|E\Delta_{k_1\cdots k_4}| \leq C\frac{\Delta t^2}{(k_2 - k_1)^{\sim 3/2}}$$

for some finite, nonnegative, universal constant C.

Proof. Define $X_j = \sigma_{k_j}^{-1}\Delta F_{k_j}$. By Lemma 4.4.4 and (4.2.5),

$$\begin{aligned} |E\Delta_{k_1\cdots k_4}| &= \left| E\left[\prod_{j=1}^{4} \Delta F_{k_j}^{2\pm}\right]\right| \\ &= \left(\prod_{j=1}^{4} \sigma_{k_j}^2\right)\left| E\left[\prod_{j=1}^{4} X_j^{2\pm}\right]\right| \\ &\leq C\left(\prod_{j=1}^{4} \sigma_{k_j}^2\right) \max_{1\leq j\leq 3} |E[X_j X_4]| \\ &\leq C\Delta t^{3/2} \max_{1\leq j\leq 3} |E[\Delta F_{k_j}\Delta F_{k_4}]| \\ &\leq C\frac{\Delta t^2}{(k_4-k_3)^{\sim 3/2}}. \end{aligned}$$

By the symmetry of Lemma 4.4.4, we also have

$$\left| E\left[\prod_{j=1}^{4} X_j^{2\pm}\right]\right| \leq C \max_{2\leq j\leq 4} |\rho_{1j}|$$

which, as above, gives

$$|E\Delta_{k_1\cdots k_4}| \leq C\frac{\Delta t^2}{(k_2-k_1)^{\sim 3/2}},$$

finishing the proof. ■

Lemma 4.4.6 *Let $X_1, \ldots, X_4$ be mean zero, jointly normal random variables with $EX_j^2 = 1$ and $\rho_{ij} = E[X_i X_j]$. Suppose that $|\rho_{12}| \leq r < 1$ and let*

$$\tilde{\rho} = \max_{\substack{i=1,2\\ j=3,4}} |\rho_{ij}|.$$

Then

$$\left| E\left[\prod_{j=1}^{4} X_j^{2\pm}\right]\right| \leq C\left(|\rho_{12}\rho_{34}| + \frac{1}{\sqrt{1-r^2}}\tilde{\rho}\right)$$

for some finite, nonnegative, universal constant C.

Proof. Let $\tilde{X}_1 = (X_1, X_2)^T$ and $\tilde{X}_2 = (X_3, X_4)^T$. Define $a = (a_1, a_2)^T$ by

$$a = \frac{1}{1-\rho_{12}^2}\begin{pmatrix} 1 & -\rho_{12} \\ -\rho_{12} & 1 \end{pmatrix}\begin{pmatrix} \rho_{13} \\ \rho_{23} \end{pmatrix} = \left(E[\tilde{X}_1\tilde{X}_1^T]\right)^{-1}\begin{pmatrix} \rho_{13} \\ \rho_{23} \end{pmatrix}$$

and note that $|a_j| \le C\tilde{\rho}(1-r^2)^{-1}$. Let $V_1 = a_1X_1 + a_2X_2$. Then

$$E\left[(X_3 - V_1)\tilde{X}_1^T\right] = (\rho_{13}, \rho_{23}) - a^T E[\tilde{X}_1\tilde{X}_1^T] = 0$$

and, hence, $X_3 - V_1$ and $\tilde{X}_1$ are independent. Also observe that

$$\begin{aligned} EV_1^2 &= a^T \left(E[\tilde{X}_1\tilde{X}_1^T]\right) a \\ &= a^T(\rho_{13}, \rho_{23})^T \\ &\le C\tilde{\rho}^2(1-r^2)^{-1}. \end{aligned}$$

Similarly, we can define V_2 with $X_4 - V_2$ and $\tilde{X}_1$ independent and $EV_2^2 \le C\tilde{\rho}^2(1-r^2)^{-1}$.

Now let $V = (V_1, V_2)^T$ so that $\tilde{X}_2 - V$ and $\tilde{X}_1$ are independent. With $F(x_1, x_2) = x_1^{2\pm}x_2^{2\pm}$, we may write

$$\begin{aligned} \left|E\left[\prod_{j=1}^4 X_j^{2\pm}\right]\right| &= |E[F(\tilde{X}_1)F(\tilde{X}_2)]| \\ &\le |E[F(\tilde{X}_1)F(\tilde{X}_2 - V)]| + |E[F(\tilde{X}_1)(F(\tilde{X}_2) - F(\tilde{X}_2 - V))]| \\ &\le |EF(\tilde{X}_1)||EF(\tilde{X}_2 - V)| \\ &\quad + \left(E|F(\tilde{X}_1)|^2\right)^{1/2}\left(E|F(\tilde{X}_2) - F(\tilde{X}_2 - V)|^2\right)^{1/2}. \end{aligned}$$

Since

$$|EF(\tilde{X}_1)||EF(\tilde{X}_2 - V)| \le |EF(\tilde{X}_1)||EF(\tilde{X}_2)| + |EF(\tilde{X}_1)||E[F(\tilde{X}_2) - F(\tilde{X}_2 - V)]|$$

and

$$|EF(\tilde{X}_1)||E[F(\tilde{X}_2) - F(\tilde{X}_2 - V)]| \le \left(E|F(\tilde{X}_1)|^2\right)^{1/2}\left(E|F(\tilde{X}_2) - F(\tilde{X}_2 - V)|^2\right)^{1/2},$$

we have

$$\left|E\left[\prod_{j=1}^4 X_j^{2\pm}\right]\right| \le |EF(\tilde{X}_1)||EF(\tilde{X}_2)| + 2\left(E|F(\tilde{X}_1)|^2\right)^{1/2}\left(E|F(\tilde{X}_2) - F(\tilde{X}_2 - V)|^2\right)^{1/2}.$$

Observe that $E|F(\tilde{X}_1)|^2 = E[X_1^4X_2^4] \le C$. Also, as in the proof of Lemma 4.4.4,

$$|F(x+h) - F(x)| \le C\|h\|(\|x\|^3 + \|h\|^3).$$

Thus,

$$\begin{aligned} E|F(\tilde{X}_2) - F(\tilde{X}_2 - V)|^2 &\leq CE\left[\|V\|^2(\|\tilde{X}_2\|^6 + \|V\|^6)\right] \\ &\leq C\left(E\|V\|^4\right)^{1/2}\left(E\|\tilde{X}_2\|^{12} + E\|V\|^{12}\right)^{1/2}. \end{aligned}$$

Note that for $2 \leq k \leq 6$,

$$\begin{aligned} E\|V\|^{2k} &= E(V_1^2 + V_2^2)^k \\ &\leq C(EV_1^{2k} + EV_2^{2k}) \\ &\leq C(|EV_1^2|^k + |EV_2^2|^k). \end{aligned}$$

Since V_j is the L^2-projection of X_{j+2} onto the span of X_1 and X_2, and since the L^2 norm of X_{j+2} is 1, $EV_j^2 \leq 1$. Hence, $E\|V\|^{12} \leq C$ and

$$\begin{aligned} E\|V\|^4 &\leq C(|EV_1^2|^2 + |EV_2^2|^2) \\ &\leq C\tilde{\rho}^4(1-r^2)^{-2}. \end{aligned}$$

Thus,

$$\begin{aligned} \left|E\left[\prod_{j=1}^{4} X_j^{2\pm}\right]\right| &\leq C\left(|EF(\tilde{X}_1)||EF(\tilde{X}_2)| + \frac{1}{\sqrt{1-r^2}}\tilde{\rho}\right) \\ &\leq C\left(|\rho_{12}\rho_{34}| + \frac{1}{\sqrt{1-r^2}}\tilde{\rho}\right) \end{aligned}$$

by Corollary 4.4.3. ■

Corollary 4.4.7 *For all $k_1 \leq \cdots \leq k_4 \in \mathbb{N}$,*

$$|E\Delta_{k_1\cdots k_4}| \leq C\left(\frac{1}{(k_4-k_3)^{\sim 3/2}(k_2-k_1)^{\sim 3/2}} + \frac{1}{(k_3-k_2)^{\sim 3/2}}\right)\Delta t^2.$$

for some finite, nonnegative, universal constant C, where $\Delta_{k_1\cdots k_4}$ is as in (4.4.1).

Proof. Define $X_j = \sigma_{k_j}^{-1}\Delta F_{k_j}$. Then by Corollary 4.2.3 and (4.2.5),

$$\begin{aligned} |\rho_{ij}| &= |E[X_{k_i}X_{k_j}]| \\ &= \sigma_i^{-1}\sigma_j^{-1}|E[\Delta F_{k_i}\Delta F_{k_j}]| \\ &\leq \frac{2\sqrt{\pi}}{|k_i - k_j|^{\sim 3/2}}. \end{aligned}$$

We will consider three different cases.

Case 1: $k_2 - k_1 \geq 4$.

In this case, $|\rho_{12}| \leq r \equiv \sqrt{\pi}/4 < 1$. By Lemma 4.4.6 and (4.2.5),

$$\begin{aligned} |E\Delta_{k_1\cdots k_4}| &= \left| E\left[\prod_{j=1}^{4} \Delta F_{k_j}^{2\pm}\right]\right| \\ &= \left(\prod_{j=1}^{4} \sigma_{k_j}^2\right) \left| E\left[\prod_{j=1}^{4} X_{k_j}^{2\pm}\right]\right| \\ &\leq C\left(\prod_{j=1}^{4} \sigma_{k_j}^2\right)\left(|\rho_{12}\rho_{34}| + \frac{1}{\sqrt{1-r^2}} \max_{\substack{i=1,2\\ j=3,4}} |\rho_{ij}|\right) \\ &\leq C\Delta t^2 \left(\frac{1}{(k_4-k_3)^{\sim 3/2}(k_2-k_1)^{\sim 3/2}} + \frac{1}{(k_3-k_2)^{\sim 3/2}}\right). \end{aligned}$$

Case 2: $k_4 - k_3 \geq 4$.

By symmetry, Lemma 4.4.6 also holds if $|\rho_{34}| \leq r < 1$, and the proof of Case 1 carries over to Case 2.

Case 3: $k_2 - k_1 \leq 3$ and $k_4 - k_3 \leq 3$.

In this case,

$$\left(\frac{1}{(k_4-k_3)^{\sim 3/2}(k_2-k_1)^{\sim 3/2}} + \frac{1}{(k_3-k_2)^{\sim 3/2}}\right) > \frac{1}{(k_4-k_3)^{\sim 3/2}(k_2-k_1)^{\sim 3/2}} \geq \frac{1}{27}$$

and Hölder's inequality, together with (4.2.5), gives

$$|E\Delta_{k_1\cdots k_4}| = \left|E\left[\prod_{j=1}^{4} \Delta F_{k_j}^{2\pm}\right]\right| \leq E\left[\prod_{j=1}^{4} \Delta F_{k_j}^{2}\right] \leq C\prod_{j=1}^{4} \sigma_{k_j}^2 \leq C\Delta t^2,$$

which completes the proof. ■

Lemma 4.4.8 *Let $X_1, \ldots, X_4$ be mean zero, jointly normal random variables with $EX_j^2 = 1$ and $\rho_{ij} = E[X_iX_j]$. Suppose that*

$$M \equiv \max_{i \neq j} |\rho_{ij}| < \frac{1}{2}$$

and let $R = (1 - 3M^2 - 2M^3)^{-1} < \infty$. Then

$$\left|E\left[\prod_{j=1}^{4} X_j^{2\pm}\right]\right| \leq CR^2M^2$$

for some finite, nonnegative, universal constant C.

Proof. Define $X = (X_1, X_2, X_3)^T$ and let

$$\Sigma = E[XX^T] = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}.$$

Note that

$$\begin{aligned} \det\Sigma &= 1 - \rho_{23}^2 - \rho_{12}(\rho_{12} - \rho_{13}\rho_{23}) + \rho_{13}(\rho_{12}\rho_{23} - \rho_{13}) \\ &= 1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\rho_{12}\rho_{13}\rho_{23} \end{aligned}$$

so that $|\det\Sigma| \geq 1 - (3M^2 + 2M^3) = R^{-1} > 0$ and Σ is invertible. If Σ_{ij} is the 2×2 submatrix of Σ obtained by removing the i-th row and j-th column, and C is the 3×3 matrix given by $C_{ij} = (-1)^{i+j}\det\Sigma_{ij}$, then $\Sigma^{-1} = (\det\Sigma)^{-1}C^T$. Since $M < 1/2$, $|C_{ij}| \leq 1$ for all i, j. Thus,

$$|(\Sigma^{-1})_{ij}| \leq R$$

for all i, j.

Now let $c = (\rho_{14}, \rho_{24}, \rho_{34})^T$ and define $a = (a_1, a_2, a_3)^T = \Sigma^{-1}c$. Note that

$$\begin{aligned} |a_j| &= |(\Sigma^{-1})_{j1}\rho_{14} + (\Sigma^{-1})_{j2}\rho_{24} + (\Sigma^{-1})_{j3}\rho_{34}| \\ &\leq 3RM \end{aligned}$$

for all j. Define $U = X_4 - a^TX$ and observe that

$$\begin{aligned} E[XU] &= E[XX_4] - \left(E[XX^T]\right)a \\ &= c - \Sigma a \\ &= 0 \end{aligned}$$

so that U and X are independent. In particular, $EX_4^2 = EU^2 + E|a^TX|^2$ and $EU^2 \leq 1$.

Now write

$$\begin{aligned} \left| E\left[\prod_{j=1}^{4} X_j^{2\pm}\right] \right| &= E\left[(X_1X_2X_3)^{2\pm}(U + a^TX)^{2\pm}\right] \\ &= f(a) \end{aligned}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is given by

$$f(x) = f(x_1, x_2, x_3) = E\left[(X_1X_2X_3)^{2\pm}(U + x_1X_1 + x_2X_2 + x_3X_3)^{2\pm}\right].$$

By Theorem 2.27 in [12], f has continuous partial derivatives given by

$$\begin{aligned}
\partial_1 f(x) &= 2E\left[|X_1|^3(X_2X_3)^{2\pm}|U + x^TX|\right] \\
\partial_2 f(x) &= 2E\left[|X_2|^3(X_1X_3)^{2\pm}|U + x^TX|\right] \\
\partial_3 f(x) &= 2E\left[|X_3|^3(X_1X_2)^{2\pm}|U + x^TX|\right].
\end{aligned}$$

First, consider $\partial_3 f$. Note that

$$\begin{aligned}
\left|\partial_3 f(x) - 2E\left[|X_3|^3(X_1X_2)^{2\pm}|U|\right]\right| &\le 2E\left[|X_3|^3|X_1X_2|^2\left||U + x^TX| - |U|\right|\right] \\
&\le 2E\left[|X_3|^3|X_1X_2|^2|x^TX|\right] \\
&\le C\|x\|.
\end{aligned}$$

Thus, since U and (X_1, X_2, X_3) are independent,

$$|\partial_3 f(x)| \le C\left|E\left[|X_3|^3(X_1X_2)^{2\pm}\right]\right| + C\|x\|.$$

As in the proof of Lemma 4.4.6, let V be a linear combination of X_1 and X_2 such that $X_3 - V$ and (X_1, X_2) are independent, and $EV^2 \le CM^2$. Then

$$\begin{aligned}
|E\left[|X_3|^3(X_1X_2)^{2\pm}\right]| &\le |E\left[|X_3 - V|^3(X_1X_2)^{2\pm}\right]| + E\left[\left||X_3|^3 - |X_3 - V|^3\right||X_1X_2|^2\right] \\
&\le \left(E|X_3 - V|^3\right)|EX_1^{2\pm}X_2^{2\pm}| + CE\left\{|V|(|X_3| + |V|)^2|X_1X_2|^2\right\} \\
&\le CM
\end{aligned}$$

by Corollary 4.4.3. Thus, $|\partial_3 f(x)| \le C(M + \|x\|)$ and, by symmetry, the same estimate holds for the other partial derivatives as well.

Now, since $f(0) = 0$, there exist $\theta = \theta(x) \in [0, 1]$ such that

$$|f(x)| = |x \cdot \nabla f(\theta x)| \le C\|x\|(M + \|x\|)$$

and, hence, since $R \ge 1$,

$$\begin{aligned}
\left|E\left[\prod_{j=1}^4 X_j^{2\pm}\right]\right| &\le C\|a\|(M + \|a\|) \\
&\le CRM(M + RM) \\
&\le CRM(2RM),
\end{aligned}$$

and the proof is complete. ∎

Corollary 4.4.9 *For all $k_1 \leq \cdots \leq k_4 \in \mathbb{N}$,*

$$|E\Delta_{k_1\cdots k_4}| \leq C\frac{\Delta t^2}{m^{\sim 3}}$$

for some finite, nonnegative, universal constant C, where $\Delta_{k_1\cdots k_4}$ is as in (4.4.1), and

$$m = \min_{1\leq i\leq 3}(k_{i+1} - k_i).$$

Proof. First, assume $m \leq 3$. Then Hölder's inequality, together with Corollary 4.2.3, gives

$$|E\Delta_{k_1\cdots k_4}| \leq \prod_{j=1}^{4} E|\Delta F_{k_j}|^2 \leq C\Delta t^2 \leq C\frac{\Delta t^2}{m^{\sim 3}}.$$

Now, assume $m \geq 4$. Define $X_j = \sigma_{k_j}^{-1}\Delta F_{k_j}$. Then by Corollary 4.2.3 and (4.2.5),

$$\begin{aligned} |\rho_{ij}| &= |E[X_i X_j]| \\ &= \sigma_i^{-1}\sigma_j^{-1}|E[\Delta F_{k_i}\Delta F_{k_j}]| \\ &\leq \frac{2\sqrt{\pi}}{|k_i - k_j|^{\sim 3/2}}, \end{aligned}$$

and, hence,

$$M \equiv \max_{i\neq j}|\rho_{ij}| \leq \frac{\sqrt{\pi}}{4} < \frac{1}{2}$$

and

$$R \equiv (1 - 3M^2 - 2M^3)^{-1} \leq (1 - 5M^2)^{-1} \leq \left(1 - \frac{5\pi}{16}\right)^{-1} < \infty.$$

By Lemma 4.4.8 and (4.2.5), we have

$$\begin{aligned} |E\Delta_{k_1\cdots k_4}| &= \left|E\left[\prod_{j=1}^{4}\Delta F_{k_j}^{2\pm}\right]\right| \\ &= \left(\prod_{j=1}^{4}\sigma_{k_j}^2\right)\left|E\left[\prod_{j=1}^{4}X_{k_j}^{2\pm}\right]\right| \\ &\leq C\left(\prod_{j=1}^{4}\sigma_{k_j}^2\right)\max_{i\neq j}|E[X_i X_j]|^2 \\ &\leq C(\Delta t)\max_{i\neq j}|E[\Delta F_{k_i}\Delta F_{k_j}]|^2 \\ &\leq C\frac{\Delta t^2}{m^{\sim 3}}, \end{aligned}$$

and we are done. ■

4.5 Moments

We now use the estimates of the previous section to analyze the second and fourth moments of the increments of the processes $W_t^{(n)}$ in Theorem 4.1.3.

Proposition 4.5.1 *There exists a finite, nonnegative, universal constant C such that,*

$$E|W_t^{(n)} - W_s^{(n)}|^4 \leq C|t-s|^2$$

for all $0 \leq s < t$ and all $n \in \mathbb{N}$, where $W_t^{(n)}$ is as in Theorem 4.1.3.

Proof. Write

$$W_t^{(n)} - W_s^{(n)} = \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \Delta F_j^{2\pm} + (nt - \lfloor nt \rfloor)\Delta F_{\lfloor nt \rfloor+1}^{2\pm} - (ns - \lfloor ns \rfloor)\Delta F_{\lfloor ns \rfloor+1}^{2\pm}.$$

Note that the number of terms in the above sum is $\lfloor nt \rfloor - \lfloor ns \rfloor < n(t-s)+1$. First, assume that $n(t-s) \leq 1$. In this case, either $\lfloor nt \rfloor - \lfloor ns \rfloor = 0$ or $\lfloor nt \rfloor - \lfloor ns \rfloor = 1$. If $\lfloor nt \rfloor - \lfloor ns \rfloor = 0$, then

$$W_t^{(n)} - W_s^{(n)} = n(t-s)\Delta F_{\lfloor ns \rfloor+1}^{2\pm}$$

so by Corollary 4.2.3,

$$\begin{aligned} E|W_t^{(n)} - W_s^{(n)}|^4 &\leq Cn^4|t-s|^4\Delta t^2 \\ &= Cn^2|t-s|^4 \\ &\leq C|t-s|^2. \end{aligned}$$

On the other hand, if $\lfloor nt \rfloor - \lfloor ns \rfloor = 1$, then

$$\begin{aligned} W_t^{(n)} - W_s^{(n)} &= \Delta F_{\lfloor ns \rfloor+1}^{2\pm} + (nt - \lfloor nt \rfloor)\Delta F_{\lfloor nt \rfloor+1}^{2\pm} - (ns - \lfloor ns \rfloor)\Delta F_{\lfloor ns \rfloor+1}^{2\pm} \\ &= \Delta F_{\lfloor nt \rfloor}^{2\pm} + (nt - \lfloor nt \rfloor)\Delta F_{\lfloor nt \rfloor+1}^{2\pm} - (ns - \lfloor nt \rfloor + 1)\Delta F_{\lfloor nt \rfloor}^{2\pm} \\ &= (\lfloor nt \rfloor - ns)\Delta F_{\lfloor nt \rfloor}^{2\pm} + (nt - \lfloor nt \rfloor)\Delta F_{\lfloor nt \rfloor+1}^{2\pm}. \end{aligned}$$

Since $ns < \lfloor ns \rfloor + 1 = \lfloor nt \rfloor \le nt$,

$$|W_t^{(n)} - W_s^{(n)}| \le n|t-s|\Delta F_{\lfloor nt \rfloor}^2 + n|t-s|\Delta F_{\lfloor nt \rfloor + 1}^2$$

and by Corollary 4.2.3,

$$\begin{aligned} E|W_t^{(n)} - W_s^{(n)}|^4 &\le Cn^4|t-s|^4\Delta t^2 \\ &\le C|t-s|^2. \end{aligned}$$

Next, assume $n(t-s) > 1$. In this case, Corollary 4.2.3 gives

$$E|W_t^{(n)} - W_s^{(n)}|^4 \le C\left\{ E\left|\sum_{j=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \Delta F_j^{2\pm}\right|^4 + \Delta t^2 \right\}.$$

Note that in this case, $\Delta t^2 = n^{-2} \le |t-s|^2$. Hence, it suffices to show that

$$E\left|\sum_{j=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \Delta F_j^{2\pm}\right|^4 \le C|t-s|^2.$$

To this end, we first make several definitions. Let

$$S = \{k \in \mathbb{N}^4 : \lfloor ns \rfloor + 1 \le k_1 \le \cdots \le k_4 \le \lfloor nt \rfloor\}.$$

For $k \in S$, define $h = h(k) \in \mathbb{Z}_+^3$ by $h_i = k_{i+1} - k_i$. Also define

$$\begin{aligned} M(k) &= \max(h_1, h_2, h_3) \\ m(k) &= \min(h_1, h_2, h_3) \\ c(k) &= \operatorname{med}(h_1, h_2, h_3) \end{aligned}$$

where "med" denotes the median function. For $i \in \{1, 2, 3\}$, let

$$S_i = \{k \in S : h_i = M\}.$$

Define $N = \lfloor nt \rfloor - (\lfloor ns \rfloor + 1)$ and for $j \in \{0, 1, \ldots, N\}$, let

$$S_i^j = \{k \in S_i : M = j\}.$$

Further define

$$T_i^\ell = T_i^{j,\ell} = \{k \in S_i^j : m = \ell\}$$

and

$$V_i^\nu = V_i^{j,\ell,\nu} = \{k \in T_i^\ell : c = \nu\}.$$

We now have

$$\begin{aligned} E\left|\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \Delta F_j^{2\pm}\right|^4 &\le 4!\sum_{k\in S} |E\Delta_{k_1\cdots k_4}| \\ &\le 4!\sum_{i=1}^{3}\sum_{k\in S_i} |E\Delta_{k_1\cdots k_4}| \end{aligned}$$

where $\Delta_{k_1\cdots k_4}$ is given by (4.4.1). Observe that

$$\sum_{k\in S_i} |E\Delta_{k_1\cdots k_4}| = \sum_{j=0}^{N}\sum_{k\in S_i^j} |E\Delta_{k_1\cdots k_4}|$$

and

$$\sum_{k\in S_i^j} |E\Delta_{k_1\cdots k_4}| = \sum_{\ell=0}^{\lfloor\sqrt{j}\rfloor}\sum_{k\in T_i^\ell} |E\Delta_{k_1\cdots k_4}| + \sum_{\ell=\lfloor\sqrt{j}\rfloor+1}^{j}\sum_{k\in T_i^\ell} |E\Delta_{k_1\cdots k_4}|.$$

First suppose $0 \le \ell \le \lfloor\sqrt{j}\rfloor$. In this case, write

$$\sum_{k\in T_i^\ell} |E\Delta_{k_1\cdots k_4}| = \sum_{\nu=\ell}^{j}\sum_{k\in V_i^\nu} |E\Delta_{k_1\cdots k_4}|.$$

Fix $k \in V_i^\nu$. If $i = 1$, then $j = M = h_1 = k_2 - k_1$. If $i = 3$, then $j = M = h_3 = k_4 - k_3$. In either case, Corollary 4.4.5 gives

$$|E\Delta_{k_1\cdots k_4}| \le C\frac{1}{j^{\sim 3/2}}\Delta t^2 \le C\left(\frac{1}{(\ell\nu)^{\sim 3/2}} + \frac{1}{j^{\sim 3/2}}\right)\Delta t^2.$$

If $i = 2$, then $j = M = h_2 = k_3 - k_2$ and $\ell\nu = h_3 h_1 = (k_4 - k_3)(k_2 - k_1)$. Hence, by Corollary 4.4.7,

$$|E\Delta_{k_1\cdots k_4}| \le C\left(\frac{1}{(\ell\nu)^{\sim 3/2}} + \frac{1}{j^{\sim 3/2}}\right)\Delta t^2.$$

Now, if $k \in V_i^\nu = V_i^{j,\ell,\nu}$, choose $i' \ne i$ such that $h_{i'} = \ell$. With i' given, k is determined by k_i. Since there are two possibilities for i' and $N+1$ possibilities for k_i, $|V_i^\nu| \le 2(N+1)$.

Thus,

$$\begin{aligned}
\sum_{\ell=0}^{\lfloor\sqrt{j}\rfloor}\sum_{k\in T_i^\ell}|E\Delta_{k_1\cdots k_4}| &\leq C(N+1)\sum_{\ell=0}^{\lfloor\sqrt{j}\rfloor}\sum_{\nu=\ell}^{j}\left(\frac{1}{(\ell\nu)^{\sim 3/2}}+\frac{1}{j^{\sim 3/2}}\right)\Delta t^2 \\
&\leq C(N+1)\sum_{\ell=0}^{\lfloor\sqrt{j}\rfloor}\left(\frac{1}{\ell^{\sim 3/2}}+\frac{1}{j^{\sim 1/2}}\right)\Delta t^2 \\
&\leq C(N+1)\Delta t^2.
\end{aligned}$$

Next suppose $\lfloor\sqrt{j}\rfloor+1 \leq \ell \leq j$. In this case, if $k \in T_i^\ell$, then $m = \min_{1\leq i\leq 3}(k_{i+1}-k_i) = \ell$, so that by Corollary 4.4.9,

$$|E\Delta_{k_1\cdots k_4}| \leq C\frac{1}{\ell^{\sim 3}}\Delta t^2.$$

Since $|T_i^\ell| = \sum_{\nu=\ell}^{j}|V_i^\nu| \leq 2(N+1)j$, we have

$$\begin{aligned}
\sum_{\ell=\lfloor\sqrt{j}\rfloor+1}^{j}\sum_{k\in T_i^\ell}|E\Delta_{k_1\cdots k_4}| &\leq C(N+1)j\sum_{\ell=\lfloor\sqrt{j}\rfloor+1}^{j}\frac{1}{\ell^{\sim 3}}\Delta t^2 \\
&\leq C(N+1)j\left(\int_{\lfloor\sqrt{j}\rfloor}^{\infty}\frac{1}{x^3}\,dx\right)\Delta t^2 \\
&\leq C(N+1)\Delta t^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
E\left|\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\Delta F_j^{2\pm}\right|^4 &\leq C\sum_{j=0}^{N}(N+1)\Delta t^2 \\
&= C(N+1)^2\Delta t^2 \\
&= C\left(\frac{\lfloor nt\rfloor-\lfloor ns\rfloor}{n}\right)^2 \\
&\leq C|t-s|^2,
\end{aligned}$$

and the proof is complete. ■

We now know, by Jensen's inequality, that $E|W_t^{(n)} - W_s^{(n)}|^2 \leq C|t-s|$. However, for convergence purposes, we need to be more precise.

Proposition 4.5.2 *Fix $0 \leq s < t$. Then*

$$\lim_{n\to\infty} E|W_t^{(n)} - W_s^{(n)}|^2 = \kappa^2(t-s)$$

where κ is as in Theorem 4.1.3.

Proof. First assume that $s > 0$. Write

$$\Delta W_n \equiv W_t^{(n)} - W_s^{(n)} = S_n + \varepsilon_n$$

where

$$S_n = \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \Delta F_j^{2\pm}$$

and

$$\varepsilon_n = (nt - \lfloor nt \rfloor)\Delta F^{2\pm}_{\lfloor nt \rfloor+1} - (ns - \lfloor ns \rfloor)\Delta F^{2\pm}_{\lfloor ns \rfloor+1}.$$

Then write

$$\begin{aligned} ES_n^2 &= \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} E\Delta F_j^4 + 2 \sum_{j=\lfloor ns \rfloor+2}^{\lfloor nt \rfloor} \sum_{i=\lfloor ns \rfloor+1}^{j-1} E\left[\Delta F_i^{2\pm} \Delta F_j^{2\pm}\right] \\ &= \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \frac{6}{\pi}\Delta t - \frac{4}{\pi} \sum_{j=\lfloor ns \rfloor+2}^{\lfloor nt \rfloor} \sum_{i=\lfloor ns \rfloor+1}^{j-1} K\left(\tfrac{1}{2}\gamma_{j-i}\right)\Delta t + R_n \end{aligned}$$

where $K(x)$ is as in Theorem 4.1.3 and

$$R_n = \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \left(E\Delta F_j^4 - \frac{6}{\pi}\Delta t\right) + 2 \sum_{j=\lfloor ns \rfloor+2}^{\lfloor nt \rfloor} \sum_{i=\lfloor ns \rfloor+1}^{j-1} \left(E\left[\Delta F_i^{2\pm}\Delta F_j^{2\pm}\right] + \frac{2}{\pi} K\left(\tfrac{1}{2}\gamma_{j-i}\right)\Delta t\right).$$

Now, observe that by Hölder's inequality and Proposition 4.5.1,

$$\begin{aligned} E\Delta W_n^2 - ES_n^2 &= E\left[2S_n\varepsilon_n + \varepsilon_n^2\right] \\ &= E\left[2\Delta W_n \varepsilon_n - \varepsilon_n^2\right] \\ &\leq C\sqrt{t-s}\,|E\varepsilon_n^2|^{1/2} + E\varepsilon_n^2. \end{aligned}$$

By (4.2.5), $E\varepsilon_n^2 \leq C\Delta t \to 0$ as $n \to \infty$. Hence, it suffices to show that $ES_n^2 \to \kappa^2(t-s)$.

Next, observe that

$$\sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \frac{6}{\pi}\Delta t = \frac{6}{\pi}\left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}\right) \to \frac{6}{\pi}(t-s)$$

and

$$\begin{aligned} \sum_{j=\lfloor ns \rfloor+2}^{\lfloor nt \rfloor} \sum_{i=\lfloor ns \rfloor+1}^{j-1} K\left(\tfrac{1}{2}\gamma_{j-i}\right)\Delta t &= \sum_{j=\lfloor ns \rfloor+2}^{\lfloor nt \rfloor} \sum_{i=1}^{j-\lfloor ns \rfloor-1} K\left(\tfrac{1}{2}\gamma_i\right)\Delta t \\ &= \sum_{j=1}^{N} \sum_{i=1}^{j} \frac{1}{n} K\left(\tfrac{1}{2}\gamma_i\right) \end{aligned}$$

where $N = \lfloor nt \rfloor - \lfloor ns \rfloor - 1$. Thus,

$$\begin{aligned}\sum_{j=\lfloor ns \rfloor+2}^{\lfloor nt \rfloor} \sum_{i=\lfloor ns \rfloor+1}^{j-1} K\left(\tfrac{1}{2}\gamma_{j-i}\right)\Delta t &= \sum_{i=1}^{N}\sum_{j=i}^{N} \frac{1}{n} K\left(\tfrac{1}{2}\gamma_i\right) \\ &= \sum_{i=1}^{N}\left(\frac{N}{n} - \frac{i}{n}\right) K\left(\tfrac{1}{2}\gamma_i\right).\end{aligned}$$

Note that by (4.2.3) and Lemma 4.4.2, $\xi \equiv \sum_{i=1}^{\infty} K(\frac{1}{2}\gamma_i) < \infty$. Thus, $\sum_{i=1}^{N} \frac{i}{n} K(\frac{1}{2}\gamma_i) \to 0$ as $n \to \infty$. Since $\frac{N}{n} \to (t-s)$, we have

$$\begin{aligned}\sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \frac{6}{\pi}\Delta t - \frac{4}{\pi} \sum_{j=\lfloor ns \rfloor+2}^{\lfloor nt \rfloor} \sum_{i=\lfloor ns \rfloor+1}^{j-1} K\left(\tfrac{1}{2}\gamma_{j-i}\right)\Delta t &\to \frac{6}{\pi}(t-s) - \frac{4}{\pi}\xi(t-s) \\ &= \kappa^2(t-s)\end{aligned}$$

and it suffices to show that $R_n \to 0$ as $n \to \infty$.

Now, by Lemma 4.4.1,

$$\begin{aligned}E\left[\Delta F_i^{2\pm}\Delta F_j^{2\pm}\right] &= \sigma_i^2\sigma_j^2 E\left[\left(\frac{\Delta F_i}{\sigma_i}\right)^{2\pm}\left(\frac{\Delta F_j}{\sigma_j}\right)^{2\pm}\right] \\ &= \sigma_i^2\sigma_j^2 K(\rho_{ij})\end{aligned}$$

where

$$\begin{aligned}\rho_{ij} &= E\left[\left(\frac{\Delta F_i}{\sigma_i}\right)\left(\frac{\Delta F_j}{\sigma_j}\right)\right] \\ &= (\sigma_i\sigma_j)^{-1}E[\Delta F_i \Delta F_j].\end{aligned}$$

Define $a = \sigma_i^2\sigma_j^2$, $b = K(\rho_{ij})$, $c = \frac{2}{\pi}\Delta t$, and $d = K(-\frac{1}{2}\gamma_{j-i})$. By (4.2.5), $|a| \le C\Delta t$. By (4.2.5) and Lemma 4.2.2,

$$\begin{aligned}|a-c| &= \left|\sigma_i^2\left(\sigma_j^2 - \sqrt{\frac{2\Delta t}{\pi}}\right) + \sqrt{\frac{2\Delta t}{\pi}}\left(\sigma_i^2 - \sqrt{\frac{2\Delta t}{\pi}}\right)\right| \\ &\le C\frac{1}{t_i^{3/2}}\Delta t^{5/2}.\end{aligned}$$

By Lemma 4.4.2, $|K(x) - K(y)| \le C|x-y|$, so that $|d| \le C$ and

$$\begin{aligned}|b-d| &\le C\left|\rho_{ij} + \frac{1}{2}\gamma_{j-i}\right| \\ &= C\left|\frac{1}{\sigma_i\sigma_j}\left(E[\Delta F_i\Delta F_j] + \sqrt{\frac{\Delta t}{2\pi}}\,\gamma_{j-i}\right) - \sqrt{\frac{\Delta t}{2\pi}}\,\gamma_{j-i}\left(\frac{1}{\sigma_i\sigma_j} - \sqrt{\frac{\pi}{2\Delta t}}\right)\right|.\end{aligned}$$

Observe that

$$\sigma_i\sigma_j\left(\frac{1}{\sigma_i\sigma_j}-\sqrt{\frac{\pi}{2\Delta t}}\right) = \sqrt{\frac{\pi}{2\Delta t}}\left[\frac{\sigma_j}{\sigma_i+\sigma_j}\left(\sqrt{\frac{2\Delta t}{\pi}}-\sigma_i^2\right)+\frac{\sigma_i}{\sigma_i+\sigma_j}\left(\sqrt{\frac{2\Delta t}{\pi}}-\sigma_j^2\right)\right]$$

so that by (4.2.5) and Lemma 4.2.2

$$\left|\sigma_i^2\sigma_j^2\left(\frac{1}{\sigma_i\sigma_j}-\sqrt{\frac{\pi}{2\Delta t}}\right)\right| \le C\frac{1}{t_i^{3/2}}\Delta t^2$$

and, hence, by Lemma 4.2.4,

$$|a||b-d| \le C\frac{1}{t_i^{3/2}}\Delta t^{5/2}.$$

Putting it all together gives

$$\begin{aligned}\left|E\left[\Delta F_i^{2\pm}\Delta F_j^{2\pm}\right]+\frac{2}{\pi}K\left(\tfrac{1}{2}\gamma_{j-i}\right)\Delta t\right| &= |ab-cd| \\ &= |a(b-d)+d(a-c)| \\ &\le |a||b-d|+|d||a-c| \\ &\le C\frac{1}{t_i^{3/2}}\Delta t^{5/2}.\end{aligned}$$

Since $t_i > s > 0$, this shows that

$$\sum_{j=\lfloor ns\rfloor+2}^{\lfloor nt\rfloor}\sum_{i=\lfloor ns\rfloor+1}^{j-1}\left(E\left[\Delta F_i^{2\pm}\Delta F_j^{2\pm}\right]+\frac{2}{\pi}K\left(\tfrac{1}{2}\gamma_{j-i}\right)\Delta t\right) \to 0.$$

Combined with (4.2.4), this shows that $R_n \to 0$.

We have now proved the proposition under the assumption that $s > 0$. Now assume $s = 0$. Let $\varepsilon \in (0, t)$ be arbitrary. Then by Hölder's inequality and Proposition 4.5.1,

$$\begin{aligned}\left|E|W_t^{(n)}|^2-\kappa^2 t\right| &= \left|E|W_t^{(n)}-W_\varepsilon^{(n)}|^2-\kappa^2(t-\varepsilon)+2E\left[W_t^{(n)}W_\varepsilon^{(n)}\right]-E|W_\varepsilon^{(n)}|^2-\kappa^2\varepsilon\right| \\ &\le \left|E|W_t^{(n)}-W_\varepsilon^{(n)}|^2-\kappa^2(t-\varepsilon)\right|+C(\sqrt{t\varepsilon}+\varepsilon).\end{aligned}$$

First let $n \to \infty$, then let $\varepsilon \to 0$ to complete the proof. ■

The final lemma in this section shows that the limiting object in Theorem 4.1.3 is nontrivial.

Lemma 4.5.3 *Let κ be as in Theorem 4.1.3. Then $\kappa > 0$.*

Proof. By Lemma 4.4.2,

$$\begin{aligned}\sum_{i=1}^{\infty} K(\tfrac{1}{2}\gamma_i) &\leq \frac{8}{\pi}\sum_{i=1}^{\infty}\frac{1}{2}\gamma_i + 2\sum_{i=1}^{\infty}\left(\frac{1}{2}\gamma_i\right)^3 \\ &= \frac{4}{\pi}\sum_{i=1}^{\infty}\gamma_i + \frac{1}{4}\sum_{i=1}^{\infty}\gamma_i^3.\end{aligned}$$

Since $\gamma_i = f(i-1) - f(i)$, where $f(x) = \sqrt{x+1} - \sqrt{x}$, we have that $\sum_{i=1}^{\infty}\gamma_i = f(0) = 1$. Moreover, by (4.2.3),

$$\begin{aligned}\sum_{i=1}^{\infty}\gamma_i^3 &\leq \sum_{i=1}^{\infty}\frac{1}{(\sqrt{2}\, i^{3/2})^3} \\ &= \frac{1}{2\sqrt{2}}\sum_{i=1}^{\infty}\frac{1}{i^{9/2}} \\ &\leq \frac{1}{2\sqrt{2}}\left(1 + \int_1^{\infty}\frac{1}{x^{9/2}}\,dx\right) = \frac{9\sqrt{2}}{28}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{i=1}^{\infty} K(\tfrac{1}{2}\gamma_i) &\leq \frac{4}{\pi} + \frac{9\sqrt{2}}{112} \\ &< \frac{4}{3} + \frac{9}{54} = \frac{3}{2}.\end{aligned}$$

Since

$$\kappa^2 = \frac{4}{\pi}\left(\frac{3}{2} - \sum_{i=1}^{\infty} K(\tfrac{1}{2}\gamma_i)\right) > 0,$$

we see that $\kappa > 0$. ■

4.6 Proof of Theorem 4.1.3

By Proposition 4.5.1 and the Kolmogorov-Čentsov Theorem (see Problem 2.4.11 in [17]), we know that the processes $W^{(n)}$ are tight; it only remains to show the convergence of the finite dimensional distributions to those of Brownian motion. We will accomplish this in two steps: (1) we will show that the increments of these processes converge to normal random variables with the appropriate variances; and, (2) we will show that any process, which is a subsequential limit of these processes, has independent increments. The primary tool in the proofs of these facts will be conditioning and we will make frequent use of the filtration of σ-algebras

$$\mathcal{F}_t = \sigma\left\{\iint_A dW(r,y) : A \subset \mathbb{R} \times [0,t], m(A) < \infty\right\}$$

where m denotes Lebesgue measure on $\mathbb{R}^2$. Note that the process F_t is adapted to $\mathcal{F}_t$.

Lemma 4.6.1 *If* $t_{j-1} \geq u$, *then*

$$E|E[\Delta F_j|\mathcal{F}_u]|^2 \leq \frac{2}{(t_j - u)^{3/2}}\Delta t^2.$$

Proof. Since the distribution of the process $F = F^{(x)}$ does not depend on x, we may assume that $x = 0$. Thus, by (4.1.7), if $t \geq u$, then

$$E[F_t|\mathcal{F}_u] = \int_{\mathbb{R}}\int_0^u p(t-r,y)\,dW(r,y).$$

Therefore, if $t \geq s \geq u$, then

$$E|E[F_t - F_s|\mathcal{F}_u]|^2 = \int_{\mathbb{R}}\int_0^u |p(t-r,y) - p(s-r,y)|^2\,dr\,dy.$$

As in the proof of Lemma 4.2.1,

$$\begin{aligned}\int_{\mathbb{R}}\int_0^u p(t-r,y)p(s-r,y)\,dr\,dy &= \frac{1}{\sqrt{2\pi}}\int_0^u \frac{1}{\sqrt{t+s-2r}}\,dr\\ &= \frac{1}{\sqrt{2\pi}}(|t+s|^{1/2} - |(t-u)+(s-u)|^{1/2}).\end{aligned}$$

Thus,

$$\begin{aligned}E|E[F_t - F_s|\mathcal{F}_u]|^2 &= \pi^{-1/2}\left[\sqrt{t} - \sqrt{t-u}\right]\\ &\quad +\pi^{-1/2}\left[\sqrt{s} - \sqrt{s-u}\right]\\ &\quad -\pi^{-1/2}\left[\sqrt{2}\left(\sqrt{t+s} - \sqrt{(t-u)+(s-u)}\right)\right]\\ &= \pi^{-1/2}\left(\sqrt{t} + \sqrt{s} - \sqrt{2t+2s} + \sqrt{2t-2s}\right)\\ &\quad -\pi^{-1/2}\left(\sqrt{t-u} + \sqrt{s-u}\right.\\ &\quad \left.-\sqrt{2(t-u)+2(s-u)} + \sqrt{2(t-u)-2(s-u)}\right).\end{aligned}$$

Hence, by (4.2.1),

$$E|E[F_t - F_s|\mathcal{F}_u]|^2 = E|F_t - F_s|^2 - E|F_{t-u} - F_{s-u}|^2 \tag{4.6.1}$$

and therefore, by (4.2.2),

$$E|E[F_t - F_s|\mathcal{F}_u]|^2 \leq \left(\frac{1}{t^{3/2}} + \frac{1}{(t-u)^{3/2}}\right)|t-s|^2 \leq \frac{2}{(t-u)^{3/2}}|t-s|^2,$$

which completes the proof. ■

Proposition 4.6.2 *Fix $0 \leq s < t$ and let $W^{(n)}$ and κ be as in Theorem 4.1.3. Then*

$$W_t^{(n)} - W_s^{(n)} \xrightarrow{d} \kappa|t-s|^{1/2}\chi$$

as $n \to \infty$, where χ is a standard normal random variable.

Proof. We will prove this proposition by showing that every subsequence has a subsequence converging in distribution to the given random variable.

Let $\{n_j\}$ be any subsequence. For each $n \in \mathbb{N}$, choose $m = m_n \in \{n_j\}$ such that $m_n > m_{n-1}$ and $m_n \geq n^4/(t-s)$. Write

$$\Delta W_m \equiv W_t^{(m_n)} - W_s^{(m_n)} = S_m + \varepsilon_m$$

where

$$S_m = \sum_{j=\lfloor ms \rfloor+1}^{\lfloor mt \rfloor} \Delta F_j^{2\pm}$$

and

$$\varepsilon_m = (mt - \lfloor mt \rfloor)\Delta F_{\lfloor mt \rfloor+1}^{2\pm} - (ms - \lfloor ms \rfloor)\Delta F_{\lfloor ms \rfloor+1}^{2\pm}.$$

Note that by (4.2.5), $E\varepsilon_m^2 \leq C\Delta t = C/m$. Hence, $\varepsilon_m \to 0$ in probability and by the Converging Together Lemma (see Exercise 2.2.10 in [10]), it will suffice to show that $S_m \xrightarrow{d} \kappa|t-s|^{1/2}\chi$ as $n \to \infty$.

Now fix $n \in \mathbb{N}$ and let $\mu = m(t-s)/n$. For $0 \leq k < n$, define $u_k = \lfloor ms \rfloor + \lfloor k\mu \rfloor$, and let $u_n = \lfloor mt \rfloor$. Finally, define

$$X_k^{(n)} = \sum_{j=u_{k-1}+1}^{u_k} \Delta F_j^{2\pm}$$

so that $S_m = X_1^{(n)} + \cdots + X_n^{(n)}$. We wish to apply the Lindeberg-Feller Theorem (see Theorem 2.4.1 in [10]) to this triangular array; the problem, of course, is that the $X_k^{(n)}$'s are not independent.

Recall $\mathcal{F}_t$ from the beginning of this section and let $\mathcal{F}^j$ denote $\mathcal{F}_{t_j}$. For $u_{k-1}+1 \le j \le u_k$, let $\Delta\widetilde{F}_j = E[\Delta F_j|\mathcal{F}^{u_{k-1}}]$ and $\Delta\overline{F}_j = \Delta F_j - \Delta\widetilde{F}_j$. Note that by (4.1.7), $\Delta\overline{F}_j$ and $\mathcal{F}^{u_{k-1}}$ are independent. Now define

$$\overline{X}_k^{(n)} = \sum_{j=u_{k-1}+1}^{u_k} \Delta\overline{F}_j^{2\pm}$$

so that $X_k^{(n)} = \overline{X}_k^{(n)} + Y_k^{(n)}$, where

$$Y_k^{(n)} = \sum_{j=u_{k-1}+1}^{u_k} \left\{(\Delta\overline{F}_j + \Delta\widetilde{F}_j)^{2\pm} - \Delta\overline{F}_j^{2\pm}\right\}.$$

We then have $S_m = \overline{S}_n + T_n$, where

$$\begin{aligned}\overline{S}_n &= \sum_{k=1}^{n} \overline{X}_k^{(n)} \\ T_n &= \sum_{k=1}^{n} Y_k^{(n)}.\end{aligned}$$

Note that by the Mean Value Theorem, for all $x, y \in \mathbb{R}$, there exists $\theta = \theta(x,y) \in [0,1]$ such that $(x+y)^{2\pm} - x^{2\pm} = 2|x+\theta y|y$. Hence,

$$|(x+y)^{2\pm} - x^{2\pm}| \le C(|xy| + |y|^2), \tag{4.6.2}$$

which gives

$$E|Y_k^{(n)}| \le C \sum_{j=u_{k-1}+1}^{u_k} \left\{E|\Delta\overline{F}_j\Delta\widetilde{F}_j| + E|\Delta\widetilde{F}_j|^2\right\}.$$

By independence, $E|\Delta F_j|^2 = E|\Delta\overline{F}_j|^2 + E|\Delta\widetilde{F}_j|^2$, so that $E|\Delta\overline{F}_j|^2 \le E|\Delta F_j|^2 \le C\sqrt{\Delta t}$ by (4.2.5). Therefore, by Lemma 4.6.1,

$$\begin{aligned}E|Y_k^{(n)}| &\le C \sum_{j=u_{k-1}+1}^{u_k} \frac{1}{(t_j - t_{u_{k-1}})^{3/4}} \Delta t^{5/4} \\ &= \frac{C}{\sqrt{m}} \sum_{j=u_{k-1}+1}^{u_k} \frac{1}{(j-u_{k-1})^{3/4}} \\ &= \frac{C}{\sqrt{m}} \sum_{j=1}^{u_k - u_{k-1}} \frac{1}{j^{3/4}}.\end{aligned}$$

Now, for $1 \leq k < n$,

$$\begin{aligned} u_k - u_{k-1} &= \lfloor k\mu \rfloor - \lfloor (k-1)\mu \rfloor \\ &\leq k\mu - (k-1)\mu + 1 \\ &= \mu + 1. \end{aligned}$$

Also,

$$\begin{aligned} u_n - u_{n-1} &= \lfloor mt \rfloor - \lfloor ms \rfloor - \lfloor (n-1)\mu \rfloor \\ &\leq m(t-s) - n\mu + \mu + 2 \\ &= \mu + 2. \end{aligned}$$

Since $\mu \geq 1$, this gives $u_k - u_{k-1} \leq C\mu$ for all $1 \leq k \leq n$. Since, for $a \in \mathbb{N}$, $\sum_{j=1}^{a} j^{-3/4} \leq \int_0^a x^{-3/4}\,dx = Ca^{1/4}$, we have

$$\begin{aligned} E|Y_k^{(n)}| &\leq \frac{C}{\sqrt{m}}(u_k - u_{k-1})^{1/4} \\ &\leq \frac{C}{\sqrt{m}}\mu^{1/4}, \end{aligned}$$

and, therefore,

$$\begin{aligned} E|T_n| &\leq C\frac{\mu^{1/4}n}{\sqrt{m}} \\ &= C\frac{(t-s)^{1/4}n^{3/4}}{m^{1/4}}. \end{aligned}$$

But since $m = m_n$ was chosen so that $m \geq n^4/(t-s)$, we have that $E|T_n| \leq C\sqrt{t-s}\,/n^{1/4}$. Thus, $T_n \to 0$ in probability and, again by the Converging Together Lemma, it will suffice to show that $\overline{S}_n \xrightarrow{d} \kappa|t-s|^{1/2}\chi$ as $n \to \infty$.

We now wish to apply the Lindeberg-Feller Theorem to the triangular array $\overline{S}_n = \overline{X}_1^{(n)} + \cdots + \overline{X}_n^{(n)}$, noting that $\overline{X}_1^{(n)}, \ldots, \overline{X}_n^{(n)}$ are independent and $E\overline{X}_k^{(n)} = 0$ for all n and k. The key to checking the hypotheses of Lindeberg-Feller will be to show that

$$E\left|\sum_{j=u_{k-1}+1}^{u_k} \Delta\overline{F}_j^{2\pm}\right|^4 \leq \frac{C}{m^2}|u_k - u_{k-1}|^2. \tag{4.6.3}$$

(Compare this with Proposition 4.5.1.) To prove (4.6.3), let $\tau = u_{k-1}\Delta t = u_{k-1}/m$ and define the process

$$G_t = F_{t+\tau} - E[F_{t+\tau}|\mathcal{F}_\tau].$$

Since G_t and $\mathcal{F}_\tau$ are independent, we have for $t, s \geq 0$

$$\begin{aligned} E|G_t - G_s|^2 &= E|F_{t+\tau} - F_{s+\tau}|^2 - E|E[F_{t+\tau} - F_{s+\tau}|\mathcal{F}_\tau]|^2 \\ &= E|F_t - F_s|^2 \end{aligned}$$

by (4.6.1). Since $G_0 = 0$, it follows that, as processes, $G \stackrel{d}{=} F$. Now observe that

$$\Delta\overline{F}_j = \Delta F_j - E[\Delta F_j|\mathcal{F}_\tau] = \Delta G_{j-u_{k-1}}.$$

Thus, by Proposition 4.5.1,

$$\begin{aligned} E\left|\sum_{j=u_{k-1}+1}^{u_k} \Delta\overline{F}_j^{2\pm}\right|^4 &= E\left|\sum_{j=u_{k-1}+1}^{u_k} \Delta G_{j-u_{k-1}}^{2\pm}\right|^4 \\ &= E\left|\sum_{j=1}^{u_k-u_{k-1}} \Delta F_j^{2\pm}\right|^4 \\ &\leq C|(u_k - u_{k-1})\Delta t|^2 \end{aligned}$$

which proves (4.6.3).

We may now verify the conditions of the Lindeberg-Feller Theorem. We check that

$$\begin{aligned} \sum_{k=1}^n E|\overline{X}_k^{(n)}|^2 &\leq \frac{C}{m}\sum_{k=1}^n (u_k - u_{k-1}) \\ &= C\frac{\lfloor mt\rfloor - \lfloor ms\rfloor}{m} \\ &\leq C(t - s + 1) \end{aligned}$$

so that by passing to a subsequence, we may assume that $\sum_{k=1}^n E|X_k^{(n)}|^2 \to \sigma^2$ as $n \to \infty$ for some $\sigma \geq 0$. We also check that for all $\varepsilon > 0$,

$$\begin{aligned} \sum_{k=1}^n E\left[|\overline{X}_k^{(n)}|^2; |\overline{X}_k^{(n)}| > \varepsilon\right] &\leq \frac{1}{\varepsilon^2}\sum_{k=1}^n E|\overline{X}_k^{(n)}|^4 \leq \frac{C}{m^2\varepsilon^2}\sum_{k=1}^n |u_k - u_{k-1}|^2 \\ &\leq C\frac{n\mu^2}{m^2\varepsilon^2} = C\frac{(t-s)^2}{n\varepsilon^2} \to 0 \end{aligned}$$

as $n \to \infty$. Thus, it follows that $\overline{S}_n \xrightarrow{d} \sigma\chi$ as $n \to \infty$ and it remains only to show that $\sigma = \kappa|t-s|^{1/2}$; but this is immediate from Propositions 4.5.1 and 4.5.2. ■

This concludes step (1) of our method of proof as outlined at the beginning of this section. To accomplish step (2), we begin with a lemma.

Lemma 4.6.3 *Let $d \geq 2$ and $X^{(n)} = (X_1^{(n)}, \ldots, X_d^{(n)}) \in \mathbb{R}^d$ with $X^{(n)} \xrightarrow{d} X$. Suppose that for all $1 \leq j \leq d$,*

$$X_j \stackrel{d}{=} \sigma_j \chi \tag{4.6.4}$$

where $\sigma_j > 0$ and χ is a standard normal random variable. If $X_d^{(n)} = \overline{X}^{(n)} + Y^{(n)}$, where $Y^{(n)} \to 0$ in probability and, for all n, $\overline{X}^{(n)}$ and $(X_1^{(n)}, \ldots, X_{d-1}^{(n)})$ are independent, then X_d and $(X_1, \ldots, X_{d-1})$ are independent.

Proof. Fix $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Let

$$\begin{aligned} A_d &= \{y \in \mathbb{R}^d : y_i \leq x_i \text{ for all } 1 \leq i \leq d\} \\ A_{d-1} &= \{y \in \mathbb{R}^{d-1} : y_i \leq x_i \text{ for all } 1 \leq i \leq d-1\} \\ A_0 &= (\infty, x_d]. \end{aligned}$$

Note that by (4.6.4),

$$P[X \in \partial A_d] = P[(X_1, \ldots, X_{d-1}) \in \partial A_{d-1}] = P[X_d \in \partial A_0] = 0.$$

Thus, since $(X_1^{(n)}, \ldots, X_{d-1}^{(n)}, \overline{X}^{(n)}) = X^{(n)} - (0, \ldots, 0, Y^{(n)}) \xrightarrow{d} X$ by the Converging Together Lemma, we have

$$\begin{aligned} P[X \in A_d] &= \lim_{n\to\infty} P[(X_1^{(n)}, \ldots, X_{d-1}^{(n)}, \overline{X}^{(n)}) \in A_d] \\ &= \lim_{n\to\infty} P[(X_1^{(n)}, \ldots, X_{d-1}^{(n)}) \in A_{d-1}] P[\overline{X}^{(n)} \in A_0] \\ &= P[(X_1, \ldots, X_{d-1}) \in A_{d-1}] P[X_d \in A_0] \end{aligned}$$

which shows that X_d and $(X_1, \ldots, X_{d-1})$ are independent. ■

Proposition 4.6.4 *If $n = n_j$ is a subsequence such that $\{W_t^{(n)}\}_{t\geq 0} \xrightarrow{d} \{X_t\}_{t\geq 0}$, then X_t has independent increments.*

Proof. Fix $0 \le t_1 < t_2 < \cdots < t_d < s < t$. It will be shown that $X_t - X_s$ and $(X_{t_1}, \ldots, X_{t_d})$ are independent. Let

$$\overline{X}^{(n)} = \sum_{j=\lfloor ns \rfloor + 2}^{\lfloor nt \rfloor} \Delta \overline{F}_j^{2\pm}$$

where $\Delta \overline{F}_j = \Delta F_j - \Delta \widetilde{F}_j$, $\Delta \widetilde{F}_j = E[\Delta F_j | \mathcal{F}^{\lfloor ns \rfloor + 1}]$ and let

$$Y^{(n)} = W_t^{(n)} - W_s^{(n)} - \overline{X}^{(n)}.$$

Note that $\overline{X}^{(n)}$ and $\mathcal{F}_{\lfloor ns \rfloor + 1}$ are independent. Hence, $\overline{X}^{(n)}$ and $(W_{t_1}^{(n)}, \ldots, W_{t_d}^{(n)})$ are independent. Thus, by Lemma 4.6.3, it will suffice to show that $Y^{(n)} \to 0$ in probability.

To see this write

$$Y^{(n)} = \sum_{j=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \Delta F_j^{2\pm} + (nt - \lfloor nt \rfloor) \Delta F_{\lfloor nt \rfloor + 1}^{2\pm} - (ns - \lfloor ns \rfloor) \Delta F_{\lfloor ns \rfloor + 1}^{2\pm} - \sum_{j=\lfloor ns \rfloor + 2}^{\lfloor nt \rfloor} \Delta \overline{F}_j^{2\pm}$$

so that by (4.2.5), (4.6.2), and Lemma 4.6.1,

$$\begin{aligned}
E|Y^{(n)}| &\le C \left\{ \sqrt{\Delta t} + \sum_{j=\lfloor ns \rfloor + 2}^{\lfloor nt \rfloor} E \left| (\Delta \overline{F}_j + \Delta \widetilde{F}_j)^{2\pm} - \Delta \overline{F}_j^{2\pm} \right| \right\} \\
&\le C \left\{ \sqrt{\Delta t} + \sum_{j=\lfloor ns \rfloor + 2}^{\lfloor nt \rfloor} \frac{1}{(t_j - t_{\lfloor ns \rfloor + 1})^{3/4}} \Delta t^{5/4} \right\} \\
&= \frac{C}{\sqrt{n}} \left\{ 1 + \sum_{j=1}^{\lfloor nt \rfloor - \lfloor ns \rfloor - 1} \frac{1}{j^{3/4}} \right\} \\
&\le \frac{C}{\sqrt{n}} \left\{ 1 + n^{1/4} (t-s)^{1/4} \right\} \to 0.
\end{aligned}$$

and we are done. ∎

We now formally summarize the preceding argument.

Proof of Theorem 4.1.3:

By Lemma 4.5.3, $\kappa > 0$. By Proposition 4.5.1 and the Kolmogorov-Čentsov Theorem (see Problem 2.4.11 in [17]), the family of processes $\{W_t^{(n)}\}_{n \in \mathbb{N}}$ is tight. Thus, every subsequence has a subsequence that converges in distribution to a continuous stochastic process. For a given subsequence, call this process X_t. By Propositions 4.6.2 and 4.6.4, $X_t \overset{d}{=} \kappa W_t$, where W_t is a Brownian motion. Since every subsequence has a subsequence converging to the same limit process, the entire sequence converges to this limit. ∎

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