A change of variable formula with Itô correction term*

Jason Swanson

Department of Mathematics
University of Central Florida

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*Joint work with Chris Burdzy (University of Washington)

Classical Itô formula for Brownian motion:

$$g(B(t)) = g(B(0)) + \int_0^t g'(B(s))\,dB(s) + \frac{1}{2}\int_0^t g''(B(s))\,ds$$

Correction term due to $E|B(t+\Delta t) - B(t)|^2 = \Delta t$.

For a process F with $E|F(t+\Delta t) - F(t)|^4 \approx \Delta t$, we construct an integral such that

$$g(F(t)) = g(F(0)) + \int_0^t g'(F(s))\,dF(s) + \frac{1}{2}\int_0^t g''(F(s))\,dB(s),$$

where B is a Brownian motion independent of F.

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For a process F with $E|F(t+\Delta t) - F(t)|^4 \approx \Delta t$, we construct an integral such that

$$g(F(t)) = g(F(0)) + \int_0^t g'(F(s))\,dF(s) + \frac{1}{2}\int_0^t g''(F(s))\,dB(s),$$

where B is a Brownian motion independent of F.

Definition of F

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \dot{W}(x,t), \quad x \in \mathbb{R}; \qquad u(x,0) \equiv 0$$

$$\boxed{F(t) := u(x,t)}$$

F is a centered Gaussian process with covariance

$$\rho(s,t) = E[F(s)F(t)] = \frac{1}{\sqrt{2\pi}}(|t+s|^{1/2} - |t-s|^{1/2})$$

F is a bifractional Brownian motion, qualitatively similar to fractional Brownian motion (fBm) B^H with $H = 1/4$.

Quartic variation of F

Let $\Pi = \{0 = t_0 < t_1 < t_2 < \cdots\}$ with $t_j \uparrow \infty$ and

$$|\Pi| := \sup_{j \in \mathbb{N}} (t_j - t_{j-1}) < \infty.$$

Define $V_\Pi(t) = \sum_{0 < t_j \le t} |F(t_j) - F(t_{j-1})|^4$.

Theorem (S 2007)

$$\lim_{|\Pi| \to 0} E\left[\sup_{0 \le t \le T} \left|V_\Pi(t) - \frac{6}{\pi} t\right|^2\right] = 0, \quad \forall T > 0.$$

F is not a semimartingale; cannot construct classical stochastic integral. We construct an integral using Riemann sums.

$$\Delta t = n^{-1} \qquad t_j = j\Delta t \qquad \Delta F_j = F(t_j) - F(t_{j-1})$$

Left-endpoint and right-endpoint Riemann sums diverge.

$$E\sum_{j=1}^{\lfloor nt \rfloor} F(t_{j-1})\Delta F_j \approx -\frac{1}{\sqrt{2\pi}}\sum_{j=1}^{\lfloor nt \rfloor} \Delta t^{1/2} \approx -t\sqrt{\frac{n}{2\pi}}$$

$$E\sum_{j=1}^{\lfloor nt \rfloor} F(t_j)\Delta F_j \approx \frac{1}{\sqrt{2\pi}}\sum_{j=1}^{\lfloor nt \rfloor} \Delta t^{1/2} \approx t\sqrt{\frac{n}{2\pi}}$$

Need a symmetric Riemann sum to generate cancellations

$$\Delta t = n^{-1} \qquad t_j = j\Delta t \qquad \Delta F_j = F(t_j) - F(t_{j-1})$$

Let $\theta(t) = g(F(t), t)$ and $\bar{t}_j = (t_{j-1} + t_j)/2$. We will consider

$$I_n(g,t) = \sum_{j=1}^{\lfloor nt \rfloor} \theta(\bar{t}_j)\Delta F_j \xrightarrow{?} \int_0^t \theta(s)\,dF(s)$$

$$I_n^T(g,t) = \sum_{j=1}^{\lfloor nt \rfloor} \frac{\theta(t_{j-1}) + \theta(t_j)}{2}\Delta F_j \xrightarrow{?} \int_0^t \theta(s)\,d^T F(s)$$

Quadratic variation of F is infinite.

Define

$$Q_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} (-1)^j \Delta F_j^2 \approx \sum_{\substack{j=1 \\ j \text{ even}}}^{\lfloor nt \rfloor} (\Delta F_j^2 - \Delta F_{j-1}^2).$$

Terms have (approximately) mean 0, variance Δt.

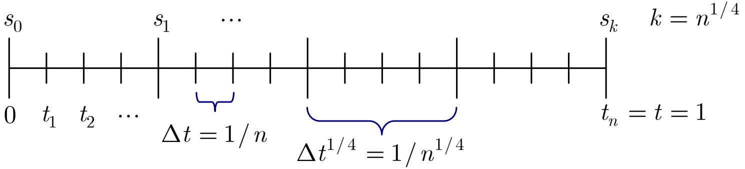
Theorem (S 2007)

If $Q_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} (-1)^j \Delta F_j^2$, then $(F, Q_n) \to (F, \kappa B)$ in law in $D_{\mathbb{R}^2}[0,\infty)$, the Skorohod space of cadlag functions from $[0,\infty)$ to $\mathbb{R}^d$, where B is a standard Brownian motion independent of F, and

$$\kappa = \left(\frac{4}{\pi} + \frac{2}{\pi} \sum_{j=1}^{\infty} (-1)^j \underbrace{(2j^{1/2} - |j-1|^{1/2} + |j+1|^{1/2})^2}_{\text{derived from the covariance of } F} \right)^{1/2} \approx 1.029.$$

We define $[\![F]\!]_t := \kappa B(t)$ to be the *alternating quadratic variation of F*.

Key idea of proof:



Let $t = 1$ and suppose $k = n^{1/4} \in \mathbb{N}$. If $s_j = j\Delta t^{1/4}$, then

$$\sum_{j=1}^{\lfloor nt \rfloor} (-1)^j \Delta F_j^2 \approx \sum_{i=1}^{k} \Bigg(\underbrace{\sum_{j=\lfloor ns_{k-1} \rfloor}^{\lfloor ns_k \rfloor} (-1)^j \Delta F_j^2}_{\text{These terms are asymptotically independent}} \Bigg)$$

Theorem (main result, informal version)

If g is "nice enough", then $I_n(g,t)$ converges in law to a process $\int_0^t g(F(s),s)\,dF(s)$ satisfying

$$g(F(t),t) = g(F(0),0) + \int_0^t \partial_t g(F(s),s)\,ds + \int_0^t \partial_x g(F(s),s)\,dF(s) + \frac{1}{2}\int_0^t \partial_x^2 g(F(s),s)\,d[\![F]\!]_s. \quad (*)$$

Definition

Let k and r be integers such that $0 \le r \le k$. We write $g \in C_r^{k,1}(\mathbb{R} \times [0, \infty))$ if

- $g : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is continuous.
- $\partial_x^j g$ exists and is cont. on $\mathbb{R} \times [0, \infty)$ for all $0 \le j \le k$.
- $\partial_t \partial_x^j g$ exists and is cont. on $\mathbb{R} \times (0, \infty)$ for all $0 \le j \le r$.
- $\overline{\lim_{t \to 0}} \sup_{x \in K} |\partial_t \partial_x^j g(x, t)| < \infty$ for all cpct K and all $0 \le j \le r$.

$C_0^{k,1} = C^{k,1} =$ functions with k spatial derivs, 1 time deriv

$g \in C_1^{k,1} \Rightarrow \partial_x g \in C^{k-1,1}$

$g \in C_2^{k,1} \Rightarrow \partial_x g \in C_1^{k-1,1}$ and $\partial_x^2 g \in C^{k-2,1}$

$\vdots$

etc.

Definition

Given $g \in C_3^{8,1}$, choose $G \in C_4^{9,1}$ such that $\partial_x G = g$. Let B be a standard Brownian motion independent of F. Define

$$\int g(F,s)\,dF = \int_0^t g(F(s),s)\,dF(s)$$
$$:= G(F(t),t) - G(F(0),0) - \int_0^t \partial_t G(F(s),s)\,ds$$
$$- \frac{\kappa}{2}\int_0^t \partial_x^2 G(F(s),s)\,dB(s).$$

By definition, then, for every $g \in C_4^{9,1}$, the Itô formula $(*)$ holds. The issue is therefore whether $I_n(g,t) \to \int g(F,s)\,dF$.

Theorem (S 2007)

$(F, Q_n) \to (F, \kappa B)$ in law in $D_{\mathbb{R}^2}[0, \infty)$, where B is a standard Brownian motion independent of F.

Theorem (Burdzy, S 2010)

If $g \in C_3^{8,1}$, then $(F, Q_n, I_n(g, \cdot)) \to (F, \kappa B, \int g(F, s)\, dF)$ in law in $D_{\mathbb{R}^3}[0, \infty)$, where B is a standard BM, independent of F. (Note: The B that appears in the second component of the limit is the same B used in the definition of $\int g(F, s)\, dF$.)

The method of proof (based in part on (Kurtz, Protter 1991)) actually gives something somewhat stronger.

Theorem (Burdzy, S 2010)

If $g \in C_3^{8,1}$, then $(F, Q_n, I_n(g,\cdot)) \to (F, \kappa B, \int g(F,s)\,dF)$ in law in $D_{\mathbb{R}^3}[0,\infty)$, where B is a standard BM, independent of F.

Define $\mathcal{F}_t = \sigma\{W(A) : A \subset \mathbb{R} \times [0,t], m(A) < \infty\}$, where m is Lebesgue measure. Suppose:

- $\{W_n(\cdot)\} \subset D_{\mathbb{R}^d}[0,\infty)$
- $W_n(t) \in \mathcal{F}_t \vee \mathcal{G}_t^n$, where $\mathcal{G}_t^n$ is independent of $\mathcal{F}_t$
- $\boxed{(W_n, F, Q_n) \to (W, F, \kappa B)}$ in law in $D_{\mathbb{R}^{d+2}}[0,\infty)$.

Then $\boxed{(W_n, F, Q_n, I_n(g,\cdot)) \to (W, F, \kappa B, \int g(F,s)\,ds)}$ in law in $D_{\mathbb{R}^{d+3}}[0,\infty)$.

Example

Let $W_n = I_n(\widetilde{g}, \cdot)$ and $W = \int \widetilde{g}(F, s)\, dF$.
By the main result, $(W_n, F, Q_n) \to (W, F, \kappa B)$.
Therefore, by the extended main result,

$$(W_n, F, Q_n, I_n(g, \cdot)) \to \left(W, F, \kappa B, \int g(F, s)\, ds \right),$$

i.e.

$$(F, Q_n, I_n(\widetilde{g}, \cdot), I_n(g, \cdot)) \to \left(F, \kappa B, \int \widetilde{g}(F, s)\, dF, \int g(F, s)\, dF \right).$$

The two Riemann sums converge jointly, and the same Brownian motion appears in their correction terms

Theorem (Burdzy, S 2010)

If $g \in C_2^{6,1}$, then $I_n^T(g,\cdot)$ converges ucp (uniformly on compacts in probability) to a process $\int_0^t g(F(s),s)\,d^T F(s)$.
Moreover, if $g \in C_3^{7,1}$, then

$$g(F(t),t) = g(F(0),0) + \int_0^t \partial_t g(F(s),s)\,ds + \int_0^t \partial_x g(F(s),s)\,d^T F(s).$$

This is the classical Stratonovich change of variable formula. There is no correction term.

Regularization method (Russo, Vallois, et al)

Recall that $\Delta t = n^{-1}$, $t_j = j\Delta t$, $\theta(t) = g(F(t), t)$, and $\Delta F_j = F(t_j) - F(t_{j-1})$.

$$I_n^T(g,t) := \sum_{j=1}^{\lfloor nt \rfloor} \frac{\theta(t_{j-1}) + \theta(t_j)}{2} \Delta F_j$$

$$I_{n,\varepsilon}^T(g,t) := \frac{\Delta t}{\varepsilon} \sum_{j=1}^{\lfloor nt \rfloor} \frac{\theta(t_{j-1}) + \theta(t_{j-1} + \varepsilon)}{2} (F(t_{j-1} + \varepsilon) - F(t_{j-1}))$$

Note that $\varepsilon = \Delta t = 1/n$ implies $I_{n,\varepsilon}^T = I_n^T$.

Regularization method (Russo, Vallois, et al)

$$I_{n,\varepsilon}^T(g,t) := \frac{\Delta t}{\varepsilon}\sum_{j=1}^{\lfloor nt\rfloor}\frac{\theta(t_{j-1})+\theta(t_{j-1}+\varepsilon)}{2}(F(t_{j-1}+\varepsilon)-F(t_{j-1}))$$

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{\Delta t\to 0} I_{n,\varepsilon}^T(g,t)\\
&\qquad = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t \frac{\theta(s)+\theta(s+\varepsilon)}{2}(F(s+\varepsilon)-F(s))\,ds\\
&\qquad =: \int_0^t \theta(s)\,d^\circ F(s) \quad \text{(the symmetric integral)}
\end{aligned}$$

Regularization method (Russo, Vallois, et al)

$$\int_0^t \theta(s)\, d^\circ F(s) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \frac{\theta(s) + \theta(s+\varepsilon)}{2} (F(s+\varepsilon) - F(s))\, ds$$

Theorem (Gradinaru, Nourdin, Russo, Vallois 2005; Cheridito, Nualart 2005)

If $g \in C^6(\mathbb{R})$, then

$$g(B^{1/4}(t)) = g(B^{1/4}(0)) + \int_0^t g'(B^{1/4}(s))\, d^\circ B^{1/4}(s).$$

In fact, this is true for B^H with any $H > 1/6$.

Regularizing the midpoint sums

Recall that $\Delta t = n^{-1}$, $t_j = j\Delta t$, $\bar{t}_j = (t_{j-1} + t_j)/2$, $\theta(t) = g(F(t), t)$, and $\Delta F_j = F(t_j) - F(t_{j-1})$.

$$I_n(g,t) := \sum_{j=1}^{\lfloor nt \rfloor} \theta(\bar{t}_j)\Delta F_j$$

$$I_{n,\varepsilon}(g,t) := \frac{\Delta t}{2\varepsilon} \sum_{j=1}^{\lfloor nt \rfloor} \theta(\bar{t}_j)(F(\bar{t}_j + \varepsilon) - F(\bar{t}_j - \varepsilon))$$

Note that $\varepsilon = \frac{1}{2}\Delta t = 1/(2n)$ implies $I_{n,\varepsilon} = I_n$.

Regularizing the midpoint sums

$$I_{n,\varepsilon}(g,t) := \frac{\Delta t}{2\varepsilon} \sum_{j=1}^{\lfloor nt \rfloor} \theta(\bar{t}_j)(F(\bar{t}_j + \varepsilon) - F(\bar{t}_j - \varepsilon))$$

$$\begin{aligned}
\lim_{\varepsilon\to 0} \lim_{\Delta t\to 0} I_{n,\varepsilon}(g,t) \\
&= \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t \theta(s)(F(s+\varepsilon) - F((s-\varepsilon) \vee 0))\,ds \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t \frac{\theta(s) + \theta(s+\varepsilon)}{2}(F(s+\varepsilon) - F(s))\,ds \\
&= \lim_{\varepsilon\to 0} \lim_{\Delta t\to 0} I_{n,\varepsilon}^T(g,t)
\end{aligned}$$

For simplicity, assume $g(x, t) = g(x)$.
Using

$$g(x + h_1) - g(x + h_2) = \sum_{p=1}^{4} \frac{1}{p!} g^{(p)}(x)(h_1^p - h_2^p) + \text{rem},$$

we have

$$\begin{aligned} g(F(t_{2j})) &- g(F(t_{2j-2})) \\ &= \sum_{p=1}^{4} \frac{1}{p!} g^{(p)}(F(t_{2j-1}))(\Delta F_{2j}^p - (-1)^p \Delta F_{2j-1}^p) + \text{rem}. \end{aligned}$$

We use the notation $X(t) \approx Y(t)$ to mean $X - Y \to 0$ ucp. Then

$$\begin{aligned} g(F(t)) &\approx g(F(0)) + \sum_{j=1}^{\lfloor nt/2 \rfloor} (g(F(t_{2j})) - g(F(t_{2j-2}))) \\ &\approx g(F(0)) + I_{n/2}(g,t) \\ &\qquad + \sum_{p=2}^{4} \sum_{j=1}^{\lfloor nt/2 \rfloor} \frac{1}{p!} g^{(p)}(F(t_{2j-1}))(\Delta F_{2j}^p - (-1)^p \Delta F_{2j-1}^p) \end{aligned}$$

We first verify that the $p = 4$ term vanishes.

Recall that $\displaystyle\sum_{0<t_j\le t} |F(t_j) - F(t_{j-1})|^4 \xrightarrow{\text{ucp}} \frac{6}{\pi} t.$

- $\displaystyle V_n(t) = \sum_{j=1}^{\lfloor nt/2 \rfloor} \Delta F_{2j}^4 \xrightarrow{\text{ucp}} \frac{3}{\pi} t.$
- $E[T_t(V_n)]$ is uniformly bounded as $\Delta t \to 0$, where $T_t(V_n)$ is the total variation of V_n on $[0, t]$.

$$\sum_{j=1}^{\lfloor nt/2 \rfloor} g^{(4)}(F(t_{2j-1}))\Delta F_{2j}^4 \xrightarrow{\text{ucp}} \frac{3}{\pi} \int_0^t g^{(4)}(F(s))\,ds$$

$$\sum_{j=1}^{\lfloor nt/2 \rfloor} g^{(4)}(F(t_{2j-1}))(\Delta F_{2j}^4 - \Delta F_{2j-1}^4) \xrightarrow{\text{ucp}} 0$$

Recall that $\sum_{0<t_j\leq t} |F(t_j) - F(t_{j-1})|^4 \xrightarrow{\text{ucp}} \frac{6}{\pi}t.$

- $V_n(t) = \sum_{j=1}^{\lfloor nt/2 \rfloor} \Delta F_{2j}^4 \xrightarrow{\text{ucp}} \frac{3}{\pi}t.$
- $E[T_t(V_n)]$ is uniformly bounded as $\Delta t \to 0$, where $T_t(V_n)$ is the total variation of V_n on $[0, t]$.

$$\sum_{j=1}^{\lfloor nt/2 \rfloor} g^{(4)}(F(t_{2j-1}))\Delta F_{2j}^4 \xrightarrow{\text{ucp}} \frac{3}{\pi}\int_0^t g^{(4)}(F(s))\,ds$$

$$\sum_{j=1}^{\lfloor nt/2 \rfloor} g^{(4)}(F(t_{2j-1}))(\Delta F_{2j}^4 - \Delta F_{2j-1}^4) \xrightarrow{\text{ucp}} 0$$

Recall that $\displaystyle\sum_{0<t_j\leq t} |F(t_j) - F(t_{j-1})|^4 \xrightarrow{\text{ucp}} \frac{6}{\pi}t.$

- $\displaystyle V_n(t) = \sum_{j=1}^{\lfloor nt/2 \rfloor} \Delta F_{2j}^4 \xrightarrow{\text{ucp}} \frac{3}{\pi}t.$
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$$\sum_{j=1}^{\lfloor nt/2 \rfloor} g^{(4)}(F(t_{2j-1}))\Delta F_{2j}^4 \xrightarrow{\text{ucp}} \frac{3}{\pi}\int_0^t g^{(4)}(F(s))\,ds$$

$$\sum_{j=1}^{\lfloor nt/2 \rfloor} g^{(4)}(F(t_{2j-1}))(\Delta F_{2j}^4 - \Delta F_{2j-1}^4) \xrightarrow{\text{ucp}} 0$$

This leads to

$$
\begin{aligned}
I_{n/2}(g', t) \approx g(F(t)) - g(F(0)) \\
- \frac{1}{2} \sum_{j=1}^{\lfloor nt/2 \rfloor} g''(F(t_{2j-1}))(\Delta F_{2j}^2 - \Delta F_{2j-1}^2) \\
- \frac{1}{6} \sum_{j=1}^{\lfloor nt/2 \rfloor} g'''(F(t_{2j-1}))(\Delta F_{2j}^3 + \Delta F_{2j-1}^3).
\end{aligned}
$$

Similar analysis gives

$$
\begin{aligned}
I_n^T(g', t) \approx g(F(t)) - g(F(0)) \\
- \frac{1}{24} \sum_{j=1}^{\lfloor nt \rfloor} g'''(F(t_j))(\Delta F_{j+1}^3 + \Delta F_j^3).
\end{aligned}
$$

Theorem (Burdzy, S 2010)

If $g \in C_0^{4,1}$, *then*

$$\sum_{j=1}^{\lfloor nt \rfloor} g(F(t_{j-1}), t_{j-1}) \Delta F_j^3 \xrightarrow{ucp} -\frac{3}{\pi} \int_0^t g'(F(s), s)\, ds,$$

$$\sum_{j=1}^{\lfloor nt \rfloor} g(F(t_j), t_j) \Delta F_j^3 \xrightarrow{ucp} \frac{3}{\pi} \int_0^t g'(F(s), s)\, ds.$$

- Note that $Z_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} \Delta F_j^3 \xrightarrow{\text{ucp}} 0$, but $E[T_t(Z_n)]$ explodes.
- A corollary is that $\sum_{j=1}^{\lfloor nt \rfloor} g'''(F(t_j))(\Delta F_{j+1}^3 + \Delta F_j^3) \xrightarrow{\text{ucp}} 0$.

Applying this to our Taylor expansions gives

$$I_n^T(g', t) \xrightarrow{\text{ucp}} g(F(t)) - g(F(0)),$$

and

$$I_{n/2}(g', t) \approx g(F(t)) - g(F(0)) - \frac{1}{2} \sum_{j=1}^{\lfloor nt/2 \rfloor} g''(F(t_{2j-1}))(\Delta F_{2j}^2 - \Delta F_{2j-1}^2).$$

Proof sketch for third order integrals
Backward integral

Third order backward Riemann sum:

$$X_n = \sum_{j=1}^{\lfloor nt \rfloor} g(F(t_j))\Delta F_j^3 \qquad X = \frac{3}{\pi}\int_0^t g'(F(s))\,ds$$

We prove $E|X_n - X|^2 \to 0$ by showing:

(1) $EX_n^2 \to EX^2$, and

(2) $E[X_n X] \to EX^2$,

What follows is a sketch of the proof of (1).

Proof sketch for third order integrals

Backward integral

$$EX_n^2 = \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor nt \rfloor} E[g(F(t_i))\Delta F_i^3 g(F(t_j))\Delta F_j^3].$$

The 4-tuple $(F(t_i), \Delta F_i, F(t_j), \Delta F_j)$ is Gaussian with mean zero. The expectation is a function of the variances and covariances:

$$E[g(F(t_i))\Delta F_i^3 g(F(t_j))\Delta F_j^3] = f(\sigma_1, \dots, \sigma_4, \rho_{12}, \dots, \rho_{34}).$$

Differentiate under the expectation and expand in a Taylor series. Then the above becomes

$$\approx C\Delta t \cdot E[g(F(t_i))\Delta F_i^3 g'(F(t_j))]$$
$$\approx (C\Delta t)^2 \cdot E[g'(F(t_i))g'(F(t_j))].$$

Proof sketch for third order integrals
Backward integral

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The 4-tuple $(F(t_i), \Delta F_i, F(t_j), \Delta F_j)$ is Gaussian with mean zero. The expectation is a function of the variances and covariances:

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The 4-tuple $(F(t_i), \Delta F_i, F(t_j), \Delta F_j)$ is Gaussian with mean zero. The expectation is a function of the variances and covariances:

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Proof sketch for third order integrals

Backward integral

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The 4-tuple $(F(t_i), \Delta F_i, F(t_j), \Delta F_j)$ is Gaussian with mean zero. The expectation is a function of the variances and covariances:

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Differentiate under the expectation and expand in a Taylor series. Then the above becomes

$$\approx C\Delta t \cdot E[g(F(t_i))\Delta F_i^3 g'(F(t_j))]$$
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Proof sketch for third order integrals

Backward integral

$$
\begin{aligned}
EX_n^2 &= \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor nt \rfloor} E[g(F(t_i))\Delta F_i^3 g(F(t_j))\Delta F_j^3] \\
&\approx \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor nt \rfloor} (C\Delta t)^2 \cdot E[g'(F(t_i))g'(F(t_j))] \\
&= E\left| C \sum_{j=1}^{\lfloor nt \rfloor} g'(F(t_j))\Delta t \right|^2 \to E\left| C \int_0^t g'(F(s))\,ds \right|^2.
\end{aligned}
$$

Calculation reveals that $C = 3/\pi$.

Proof sketch for third order integrals
Forward integral

$$\sum_{j=1}^{\lfloor nt \rfloor} g(F(t_{j-1}))\Delta F_j^3 \approx \sum_{j=1}^{\lfloor nt \rfloor} g(F(t_j))\Delta F_j^3 - \sum_{j=1}^{\lfloor nt \rfloor} g'(F(t_j))\Delta F_j^4$$

$$\to \frac{3}{\pi}\int_0^t g'(F(s))\,ds - \frac{6}{\pi}\int_0^t g'(F(s))\,ds$$

$$= -\frac{3}{\pi}\int_0^t g'(F(s))\,ds$$

Recall that third order integrals give us

$$I_{n/2}(g', t) \approx g(F(t)) - g(F(0)) - \frac{1}{2} \sum_{j=1}^{\lfloor nt/2 \rfloor} g''(F(t_{2j-1}))(\Delta F_{2j}^2 - \Delta F_{2j-1}^2).$$

Also recall that

$$B_n(t) := \sum_{j=1}^{\lfloor nt/2 \rfloor} (\Delta F_{2j}^2 - \Delta F_{2j-1}^2) \xrightarrow[\text{in law}]{} [\![F]\!]_t = \kappa B(t)$$

Define

$$B_n(t) := \sum_{j=1}^{\lfloor nt/2 \rfloor} (\Delta F_{2j}^2 - \Delta F_{2j-1}^2) \xrightarrow[\text{in law}]{} [\![F]\!]_t = \kappa B(t),$$

$$F_n(t) := F(\lfloor nt \rfloor / n).$$

Then

$$\sum_{j=1}^{\lfloor nt/2 \rfloor} g''(F(t_{2j-1}))(\Delta F_{2j}^2 - \Delta F_{2j-1}^2)$$

$$= \int_0^t g''(F_n(s-))\, dB_n(s) \xrightarrow[\text{in law}]{?} \kappa \int_0^t g''(F(s))\, dB(s).$$

$$\int_0^t g''(F_n(s-))\,dB_n(s) \xrightarrow[\text{in law}]{?} \kappa \int_0^t g''(F(s))\,dB(s)$$

(Kurtz, Protter 1991): Yes, if...

- B_n is a martingale, and
- $E[B_n]_t$ is uniformly bounded as $\Delta t \to 0$, where $[B_n]$ is the quadratic variation of B_n.

$$B_n(t) = \sum_{j=1}^{\lfloor nt/2 \rfloor} (\Delta F_{2j}^2 - \Delta F_{2j-1}^2) \text{ jumps at every time } 2j\Delta t.$$

B_n is not “close enough” to a martingale to apply the tools in (Kurtz, Protter 1991).

$$\overline{B}_n(t) := \sum_{j=1}^{m^3 \lfloor mt/2 \rfloor} (\Delta F_{2j}^2 - \Delta F_{2j-1}^2), \text{ where } m = \lfloor n^{1/4} \rfloor.$$

- If $n^{1/4} \in \mathbb{N}$, then $\overline{B}_n$ jumps at times $2j\Delta t^{1/4}$.
- $\overline{B}_n(t) = B_n(t)$ at the jump times.
- $B_n - \overline{B}_n \xrightarrow{\text{ucp}} 0$.
- $\overline{B}_n$ is “close enough” to a martingale.

We therefore have

$$\int_0^t g''(F_n(s-))\, d\overline{B}_n(s) \xrightarrow[\text{in law}]{} \kappa \int_0^t g''(F(s))\, dB(s).$$

But we still need

$$\int_0^t g''(F_n(s-))\, d(B_n - \overline{B}_n)(s) \xrightarrow{?} 0.$$

Complicated by the fact that $B_n - \overline{B}_n \xrightarrow{\text{ucp}} 0$, but $E[T_t(B_n - \overline{B}_n)]$ explodes.

This is similar to the situation with third order integrals:

- $Z_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} \Delta F_j^3 \xrightarrow{\text{ucp}} 0$, but $E[T_t(Z_n)]$ explodes.
- $\displaystyle\int_0^t g(F_n(s-))\,dZ_n(s) \xrightarrow{\text{ucp}} -\frac{3}{\pi}\int_0^t g'(F(s))\,ds$

The same methods (but much more complicated calculations) are used to analyze the remainder term for the second order sums and show that

$$\int_0^t g''(F_n(s-))\,d(B_n - \overline{B}_n)(s) \to 0.$$