The calculus of differentials for the weak Stratonovich integral

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$B = B^{1/6}$ is fractional Brownian motion:

$$E[B(s)B(t)] = \frac{1}{2}(t^{1/3} + s^{1/3} - |t-s|^{1/3}).$$

$t_k = t_{k,n} = k/n$.
$\mathcal{S}_n$ is the vector space of stochastic processes $\{L(t) : t \geq 0\}$ of the form

$$L = \sum_{k=0}^{\infty} \lambda_k 1_{[t_k, t_{k+1})}, \quad \lambda_k \in \mathcal{F}_\infty^B.$$

Note that $L(t_k) = \lambda_k$. Let $\delta_j(L) = L(t_j) - L(t_{j-1})$ for $j \geq 1$. Since $t \in [t_k, t_{k+1})$ iff $\lfloor nt \rfloor = k$, we may write

$$L(t) = L(0) + \sum_{j=1}^{\lfloor nt \rfloor} \delta_j(L).$$

Example 1

If $X = f(B)$, where $f \in C^\infty$, then we define $\Lambda_n^X = \sum_{k=0}^\infty X(t_k) 1_{[t_k, t_{k+1})}$, or equivalently,

$$\Lambda_n^X(t) = X(0) + \sum_{j=1}^{\lfloor nt \rfloor} (X(t_j) - X(t_{j-1})).$$

Since X is continuous a.s., we have $\Lambda_n^X \to X$ uniformly on compacts a.s. When f is the identity, we have

$$\Lambda_n^B(t) = \sum_{j=1}^{\lfloor nt \rfloor} (B(t_j) - B(t_{j-1})).$$

Example 2

If $L \in \mathcal{S}_n$, then we define $V(L) \in \mathcal{S}_n$ by $V(L)(t) = \sum_{j=1}^{\lfloor nt \rfloor} (\delta_j(L))^3$. For example, if $X = f(B)$, where $f \in C^\infty$, then

$$V(\Lambda_n^X)(t) = \sum_{j=1}^{\lfloor nt \rfloor} (X(t_j) - X(t_{j-1}))^3,$$

and

$$V(\Lambda_n^B)(t) = \sum_{j=1}^{\lfloor nt \rfloor} (B(t_j) - B(t_{j-1}))^3.$$

[Nualart and Ortiz-Latorre, 2008] show that $V(\Lambda_n^B)$ converges in law to κW, where $\kappa > 0$ is an explicit constant, and W is a standard Brownian motion, independent of B.

Example 3

If $L, T \in \mathcal{S}_n$, then we define $L \circ T \in \mathcal{S}_n$ by

$$(L \circ T)(t) = \sum_{j=1}^{\lfloor nt \rfloor} \frac{L(t_{j-1}) + L(t_j)}{2}(T(t_j) - T(t_{j-1})).$$

For example, if $X = f(B)$, where $f \in C^\infty$, then

$$\begin{aligned}(\Lambda_n^X \circ \Lambda_n^B)(t) &= \sum_{j=1}^{\lfloor nt \rfloor} \frac{X(t_{j-1}) + X(t_j)}{2}(B(t_j) - B(t_{j-1})) \\ &= \sum_{j=1}^{\lfloor nt \rfloor} \frac{f(B(t_{j-1})) + f(B(t_j))}{2}(B(t_j) - B(t_{j-1})).\end{aligned}$$

Example 3

$$(\Lambda_n^X \circ \Lambda_n^B)(t) = \sum_{j=1}^{\lfloor nt \rfloor} \frac{f(B(t_{j-1})) + f(B(t_j))}{2} (B(t_j) - B(t_{j-1}))$$

In [Nourdin, Réveillac, S, 2010], we showed that $\Lambda_n^X \circ \Lambda_n^B$ converges in law to

$$F(B(t)) - F(B(0)) + \frac{\kappa}{12} \int_0^t F'''(B(s))\,dW(s),$$

where $F' = f$.

$\mathcal{S}$ is the vector space of sequences $\Lambda = \{\Lambda_n\}_{n=1}^{\infty}$ such that

- $\Lambda_n \in \mathcal{S}_n$,
- $\Lambda_n(0)$ converges in probability, and
- there exists $\varphi_1, \varphi_3, \varphi_5 \in C^\infty$ such that

$$\begin{aligned}\delta_j(\Lambda_n) &:= \Lambda_n(t_j) - \Lambda_n(t_{j-1}) \\ &= \varphi_1(\beta_j)\Delta B_{j,n} + \varphi_3(\beta_j)\Delta B_{j,n}^3 + \varphi_5(\beta_j)\Delta B_{j,n}^5 \\ &\qquad + O(|\Delta B_{j,n}|^7).\end{aligned} \tag{T}$$

Here, $\beta_j = \dfrac{B(t_{j-1}) + B(t_j)}{2}$ and $\Delta B_{j,n} = B(t_j) - B(t_{j-1})$.

Example 1

If $X = f(B)$, where $f \in C^\infty$, then we define $\Lambda^X = \{\Lambda_n^X\}$.

In this case,

$$\begin{aligned}\delta_j(\Lambda_n^X) &= X(t_j) - X(t_{j-1}) \\ &= f(B(t_j)) - f(B(t_{j-1})) \\ &= f'(\beta_j)\Delta B_{j,n} + \frac{1}{24} f'''(\beta_j)\Delta B_{j,n}^3 + \frac{1}{5!2^4} f^{(5)}(\beta_j)\Delta B_{j,n}^5 \\ &\qquad + O(|\Delta B_{j,n}|^7),\end{aligned}$$

so $\Lambda^X \in \mathcal{S}$.

Recall that the sequence Λ^X converges uniformly on compacts a.s. In fact, every sequence in $\mathcal{S}$ converges, at least in law.

Lemma (S, 2011)

If $\Lambda \in \mathcal{S}$, then there are unique functions $\varphi_{1,\Lambda}$ and $\varphi_{3,\Lambda}$ that satisfy (T).

Let $\Lambda \in \mathcal{S}$. Define $\mathcal{I}_\Lambda(0) = \lim \Lambda_n(0)$, and

$$\begin{aligned}\mathcal{I}_\Lambda(t) = \mathcal{I}_\Lambda(0) + \Phi_\Lambda(B(t))\\ + \kappa \int_0^t \left(\varphi_{3,\Lambda} - \frac{1}{24}\varphi''_{1,\Lambda}\right)(B(s))\,dW(s),\end{aligned}$$

where

- $\Phi'_\Lambda = \varphi_{1,\Lambda}$ and $\Phi_\Lambda(0) = 0$,
- $\kappa^2 = \frac{3}{4}\sum_{r\in\mathbb{Z}}(|r+1|^{1/3} + |r-1|^{1/3} - 2|r|^{1/3})^3 > 0$, and
- W is a standard Brownian motion, independent of B.

Theorem (S, 2011)

If $\Lambda^{(1)}, \dots, \Lambda^{(m)} \in \mathcal{S}$, *then*

$$(B, \Lambda_n^{(1)}, \dots, \Lambda_n^{(m)}) \to (B, \mathcal{I}_{\Lambda^{(m)}}, \dots, \mathcal{I}_{\Lambda^{(m)}})$$

in law in $D_{\mathbb{R}^{m+1}}[0, \infty)$, *the Skorohod space of càdlàg functions from* $[0, \infty)$ *to* $\mathbb{R}^{m+1}$.

- Both of the previously cited convergence results can be seen as special cases of this.
- We now have a space of sequences $\mathcal{S}$. Each sequence $\Lambda \in \mathcal{S}$ is associated to a process $\mathcal{I}_\Lambda(t)$.
- If $X = f(B)$, where $f \in C^\infty$, then $\Lambda^X \in \mathcal{S}$, and it is easy to verify that $\mathcal{I}_{\Lambda^X}(t) = X(t)$. We therefore identify X with Λ^X.
- To complete the development of the weak Stratonovich integral, we will define an equivalence relation on $\mathcal{S}$.

For $\Lambda, \Theta \in \mathcal{S}$, let $\Lambda \equiv \Theta$ if $\Lambda_n - \Theta_n \to 0$ uniformly on compacts in probability (ucp). This is an equivalence relation. Let $[\Lambda]$ denote the equivalence class of Λ. Let $[\mathcal{S}] = \{[\Lambda] : \Lambda \in \mathcal{S}\}$.

As before, we identify $X = f(B)$ with $N^X = [\Lambda^X]$.

Lemma (S, 2011)

Let $\Lambda, \Theta \in \mathcal{S}$. Then $\Lambda \equiv \Theta$ if and only if

- $\Lambda_n(0) - \Theta_n(0) \to 0$ *in probability, and*
- $\varphi_{1,\Lambda} = \varphi_{1,\Theta}$ *and* $\varphi_{3,\Lambda} = \varphi_{3,\Theta}$.

Recall that the process $\mathcal{I}_\Lambda(t)$ is defined in terms of $\lim \Lambda_n(0)$, $\varphi_{1\Lambda}$, and $\varphi_{3,\Lambda}$. Hence, if $N = [\Lambda] \in [\mathcal{S}]$, then we may define $\mathcal{I}_N(t) = \mathcal{I}_\Lambda(t)$. Again, note that $\mathcal{I}_{N^X}(t) = X(t)$.

It can be shown that the mapping $N \mapsto \mathcal{I}_N(\cdot)$ is one-to-one. We sometimes write $N(t)$ instead of $\mathcal{I}_N(t)$.

Example 2 (signed cubic variation)

If $\Lambda \in \mathcal{S}$, let $V^\Lambda = \{V(\Lambda_n)\}$, where $V(\Lambda_n)(t) = \sum_{j=1}^{\lfloor nt \rfloor} (\delta_j(\Lambda_n))^3$. Then $V^\Lambda \in \mathcal{S}$ with $\varphi_{1,V^\Lambda} = 0$ and $\varphi_{3,V^\Lambda} = \varphi_{1,\Lambda}^3$. If $\Lambda \equiv \Theta$, then $V^\Lambda \equiv V^\Theta$. Hence, if $N = [\Lambda] \in [\mathcal{S}]$, then we define the signed cubic variation of N to be $[\![N]\!] := [V^\Lambda]$. We also write $[\![N]\!]_t$ for $\mathcal{I}_{[\![N]\!]}(t)$.

If $X = f(B)$, then $X = N^X = [\Lambda^X]$, so $[\![X]\!] = [V^{\Lambda^X}]$ is the equivalence class in $\mathcal{S}$ of the sequence of sums,

$$V(\Lambda_n^X)(t) = \sum_{j=1}^{\lfloor nt \rfloor} (X(t_j) - X(t_{j-1}))^3,$$

and

$$[\![X]\!]_t = \kappa \int_0^t (f'(B(s)))^3 \, dW(s),$$

which is the limit in law of the above sequence.

Example 3 (weak Stratonovich integral)

Let $\Theta_n, \Lambda_n \in \mathcal{S}_n$. Recall that

$$(\Theta_n \circ \Lambda_n)(t) = \sum_{j=1}^{\lfloor nt \rfloor} \frac{\Theta_n(t_{j-1}) + \Theta_n(t)}{2} (\Lambda_n(t_j) - \Lambda_n(t_{j-1})).$$

Lemma (S, 2011)

Let $X = f(B)$, where $f \in C^\infty$, and $\Lambda \in \mathcal{S}$. Then $X \circ \Lambda := \{\Lambda_n^X \circ \Lambda_n\}_{n=1}^\infty \in \mathcal{S}$. Moreover, if $\Lambda \equiv \Theta$, then $X \circ \Lambda \equiv X \circ \Theta$.

If $N = [\Lambda] \in [\mathcal{S}]$, then, by the lemma, we may define the weak Stratonovich integral of X with respect to N by $X \circ N := [X \circ \Lambda]$.

Example 3 (weak Stratonovich integral)

We adopt the notation $\int X\,\mathbf{d}N = X \circ N$ and $\int_0^t X(s)\,\mathbf{d}N(s) = (X \circ N)_t = \mathcal{I}_{X \circ N}(t)$.

As an example of this definition, let $Y = g(B)$, where $g \in C^\infty$. Recall that we identify Y with $N^Y \in [\mathcal{S}]$. Then $\int X\,\mathbf{d}Y = X \circ Y = X \circ N^Y$ is the equivalence class in $\mathcal{S}$ of the sequence of Riemann sums

$$\sum_{j=1}^{\lfloor nt \rfloor} \frac{X(t_{j-1}) + X(t_j)}{2}(Y(t_j) - Y(t_{j-1})).$$

And $\int_0^t X(s)\,\mathbf{d}Y(s)$ is the stochastic process $\mathcal{I}_{X \circ Y}(t)$, which is the limit in law of this sequence.

Example 3 (weak Stratonovich integral)

If $X = f(B)$, then from the proof of the previous lemma, we obtain

$$
\begin{aligned}
\mathcal{I}_{X \circ N}(0) &= 0, \\
\varphi_{1,X \circ N} &= f \varphi_{1,N}, \\
\varphi_{3,X \circ N} &= \frac{1}{8} f'' \varphi_{1,N} + f \varphi_{3,N}.
\end{aligned}
$$

From these, we can compute $\int_0^t X(s)\, \mathbf{d}N(s)$.

It can also be verified that if $X = f(B)$ and $Y = g(B)$, where $f, g \in C^\infty$, and $M, N \in [\mathcal{S}]$, then

$$
\begin{aligned}
(X + Y) \circ N &= X \circ N + Y \circ N, \\
X \circ (M + N) &= X \circ M + X \circ N.
\end{aligned}
$$

Decomposition of $N \in [\mathcal{S}]$

Each $\eta \in \mathcal{F}_\infty^B$ is identified with the equivalence class of the constant process $\eta(t) = \eta$.

Lemma (S, 2011)

Each $N \in [\mathcal{S}]$ can be written uniquely as $N = \eta + Y + V$, where $\eta \in \mathcal{F}_\infty^B$, $Y = g(B)$ with $Y(0) = 0$, and $V = \int \theta(B)\,\mathbf{d}[\![B]\!]$, where $g, \theta \in C^\infty$.

Note that $\int_0^t \theta(B(s))\,\mathbf{d}[\![B]\!]_s = \kappa \int_0^t \theta(B(s))\,dW(s)$, so every element of N can be uniquely decomposed into a smooth function of B and a Brownian martingale.

Lemma (S, 2011)

Let $X = f(B)$, where $f \in C^\infty$, and V as above. Then

$$\int X\,\mathbf{d}V = \int X\theta(B)\,\mathbf{d}[\![B]\!].$$

Change-of-variable formulas

Theorem (S, 2011)

Let $N \in [\mathcal{S}]$ and $X = f(B)$, where $f \in C^\infty$. Write $N = \eta + Y + V$, where $Y = g(B)$ and $V = \int \theta(B)\,\mathbf{d}[\![B]\!]$, with $g, \theta \in C^\infty$. Then

$$\int X\,\mathbf{d}N = \Phi(B) + \frac{1}{12}\int (f''g' - f'g'')(B)\,\mathbf{d}[\![B]\!] + \int X\,\mathbf{d}V, \quad \text{(D)}$$

where $\Phi' = fg'$ and $\Phi(0) = 0$.

- Note that $\int X\,\mathbf{d}N \in [\mathcal{S}]$, and (D) gives the decomposition of $\int X\,\mathbf{d}N$ into a smooth function of B and a Brownian martingale.
- Equation (D) expresses equality in $[\mathcal{S}]$. Hence, if we choose any sequence from the class on the left and any sequence from the class on the right, then their difference will converge to zero ucp. This is a stronger statement than simply asserting that the two sequences have the same limiting law.

Change-of-variable formulas

Corollary (S, 2011)

Let $Y = g(B)$, where $g \in C^\infty$, and let $\varphi \in C^\infty$. Then

$$\varphi(Y(t)) = \varphi(Y(0)) + \int_0^t \varphi'(Y(s))\, \mathbf{d}Y(s) - \frac{1}{12}\int_0^t \varphi'''(Y(s))\, \mathbf{d}[\![Y]\!].$$

The case $Y = B$ was proved in [Nourdin, Réveillac, S, 2010]

Change-of-variable formulas

Corollary (S, 2011)

Let $N \in [\mathcal{S}]$, $X = f(B)$, and $Z = h(B)$, where $f, h \in C^\infty$. Let $M = \int X \, \mathbf{d}N$. Write $N = \eta + g(B) + \int \theta(B) \, \mathbf{d}[\![B]\!]$, with $g, \theta \in C^\infty$. Then

$$\int Z \, \mathbf{d}M = \int ZX \, \mathbf{d}N - \frac{1}{4} \int (f'g'h')(B) \, \mathbf{d}[\![B]\!].$$

Moreover, the above correction term is a "weak triple covariation" in the following sense: If $Y = g(B)$ and $\mathcal{V} = \{\mathcal{V}_n\}$, where

$$\mathcal{V}_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} (X(t_j) - X(t_{j-1}))(Y(t_j) - Y(t_{j-1}))(Z(t_j) - Z(t_{j-1})),$$

then $\mathcal{V} \in \mathcal{S}$ and $[\mathcal{V}] = \int (f'g'h')(B) \, \mathbf{d}[\![B]\!]$.