The calculus of differentials for the weak Stratonovich integral

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$B = B^{1/6}$ is fractional Brownian motion:

$$E[B(s)B(t)] = \frac{1}{2}(t^{1/3} + s^{1/3} - |t-s|^{1/3}).$$

$t_k = t_{k,n} = k/n$.
$\mathcal{S}_n$ is the vector space of stochastic processes $\{L(t) : t \geq 0\}$ of the form

$$L = \sum_{k=0}^{\infty} \lambda_k 1_{[t_k, t_{k+1})}, \quad \lambda_k \in \mathcal{F}_\infty^B.$$

For example, if $X = f(B)$ and $Y = g(B)$, then

$$I_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} \frac{X(t_{j-1}) + X(t_j)}{2}(Y(t_j) - Y(t_{j-1}))$$

belongs to $\mathcal{S}_n$.

$\mathcal{S}$ is the vector space of sequences $\Lambda = \{\Lambda_n\}_{n=1}^\infty$ such that

- $\Lambda_n \in \mathcal{S}_n$,
- $\Lambda_n(0)$ converges in probability, and
- there exists $\varphi_1, \varphi_3, \varphi_5 \in C^\infty$ such that

$$\begin{aligned} \delta_j(\Lambda_n) &:= \Lambda_n(t_j) - \Lambda_n(t_{j-1}) \\ &= \varphi_1(\beta_j)\Delta B_{j,n} + \varphi_3(\beta_j)\Delta B_{j,n}^3 + \varphi_5(\beta_j)\Delta B_{j,n}^5 \\ &\qquad + O(|\Delta B_{j,n}|^7). \end{aligned} \tag{T}$$

Here, $\beta_j = \dfrac{B(t_{j-1}) + B(t_j)}{2}$ and $\Delta B_{j,n} = B(t_j) - B(t_{j-1})$.

If $X = f(B)$, where $f \in C^\infty$, then we identify X with $\Lambda^X = \{\Lambda_n^X\}$, where

$$\Lambda_n^X = \sum_{k=0}^{\infty} X(t_k) 1_{[t_k, t_{k+1})}.$$

In this case,

$$\begin{aligned} \delta_j(\Lambda_n^X) &= X(t_j) - X(t_{j-1}) \\ &= f(B(t_j)) - f(B(t_{j-1})) \\ &= f'(\beta_j)\Delta B_{j,n} + \frac{1}{24} f'''(\beta_j)\Delta B_{j,n}^3 + \frac{1}{5!2^4} f^{(5)}(\beta_j)\Delta B_{j,n}^5 \\ &\qquad O(|\Delta B_{j,n}|^7). \end{aligned}$$

Also, $\Lambda_n^X \to X$ uniformly on compacts a.s.

For $\Lambda, \Theta \in \mathcal{S}$, let $\Lambda \equiv \Theta$ if $\Lambda_n - \Theta_n \to 0$ uniformly on compacts in probability (ucp). This is an equivalence relation. Let $[\Lambda]$ denote the equivalence class of Λ. Let $[\mathcal{S}] = \{[\Lambda] : \Lambda \in \mathcal{S}\}$.

As before, we identify $X = f(B)$ with $N^X = [\Lambda^X]$.

Lemma (S, 2011)

If $\Lambda \in \mathcal{S}$, then there are unique functions $\varphi_{1,\Lambda}$ and $\varphi_{3,\Lambda}$ that satisfy (T). *Also, $\Lambda \equiv \Theta$ if and only if*

- $\Lambda_n(0) - \Theta_n(0) \to 0$ *in probability, and*
- $\varphi_{1,\Lambda} = \varphi_{1,\Theta}$ *and* $\varphi_{3,\Lambda} = \varphi_{3,\Theta}$.

If $N = [\Lambda] \in [\mathcal{S}]$, then define $\varphi_{1,N} = \varphi_{1,\Lambda}$, $\varphi_{3,N} = \varphi_{3,\Lambda}$, $\mathcal{I}_N(0) = \lim \Lambda_n(0)$, and

$$\mathcal{I}_N(t) = \mathcal{I}_N(0) + \Phi_N(B(t)) + \kappa \int_0^t \left(\varphi_{3,N} - \frac{1}{24} \varphi''_{1,N} \right) (B(s))\, dW(s),$$

where

- $\Phi'_N = \varphi_{1,N}$ and $\Phi_N(0) = 0$,
- $\kappa^2 = \frac{3}{4} \sum_{r \in \mathbb{Z}} (|r+1|^{1/3} + |r-1|^{1/3} - 2|r|^{1/3})^3 > 0$, and
- W is a standard Brownian motion, independent of B.

Theorem (S, 2011)

Let $N_1, \ldots, N_m \in [\mathcal{S}]$. For each $k \in \{1, \ldots, m\}$, choose $\Lambda^{(k)} \in N_k$ arbitrarily. Then

$$(B, \Lambda_n^{(1)}, \ldots, \Lambda_n^{(m)}) \to (B, \mathcal{I}_{N_1}, \ldots, \mathcal{I}_{N_m})$$

in law in $D_{\mathbb{R}^{m+1}}[0, \infty)$, the Skorohod space of càdlàg functions from $[0, \infty)$ to $\mathbb{R}^{m+1}$.

- Let $X = f(B)$, where $f \in C^\infty$. If $N^X = [\Lambda^X]$, then $I_{N^X}(t) = X(t)$.
- The mapping $N \mapsto I_N(\cdot)$ is one-to-one. We therefore sometimes write $N(t)$ instead of $I_N(t)$.
- The proof of this theorem relies heavily on results from [Nourdin, Réveillac, S, 2010].

The weak Stratonovich integral

If $\Theta_n, \Lambda_n \in \mathcal{S}_n$, then define

$$(\Theta_n \circ \Lambda_n)(t) = \sum_{j=1}^{\lfloor nt \rfloor} \frac{\Theta_n(t_{j-1}) + \Theta_n(t)}{2} \delta_j(\Lambda_n).$$

Lemma (S, 2011)

Let $X = f(B)$, where $f \in C^\infty$, and $\Lambda \in \mathcal{S}$. Then $X \circ \Lambda = \{\Lambda_n^X \circ \Lambda_n\}_{n=1}^\infty \in \mathcal{S}$. Moreover, if $\Lambda \equiv \Theta$, then $X \circ \Lambda \equiv X \circ \Theta$.

If $N = [\Lambda] \in [\mathcal{S}]$, then, by the lemma, we may define the weak Stratonovich integral of X with respect to N by $X \circ N = [X \circ \Lambda]$.

From the Lemma's proof, we obtain $\mathcal{I}_{X \circ N}(0) = 0$, $\varphi_{1,X\circ N} = f\varphi_{1,N}$, and $\varphi_{3,X\circ N} = \frac{1}{8}f''\varphi_{1,N} + f\varphi_{3,N}$. From these, we may calculate $(X \circ N)_t = \mathcal{I}_{X\circ N}(t)$.

We adopt the notation $\int X \, \mathbf{d}N = X \circ N$ and $\int_0^t X(t) \, \mathbf{d}N(t) = (X \circ N)_t$.

As an example of this definition, let $Y = g(B)$, where $g \in C^\infty$. Recall that we identify Y with $N^Y \in [\mathcal{S}]$. Then $\int X \, \mathbf{d}Y = X \circ Y = X \circ N^Y$ is the equivalence class in $\mathcal{S}$ of the sequence of Riemann sums

$$\sum_{j=1}^{\lfloor nt \rfloor} \frac{X(t_{j-1}) + X(t_j)}{2} (Y(t_j) - Y(t_{j-1})).$$

And $\int_0^t X(t) \, \mathbf{d}Y(t)$ is the stochastic process defined earlier, which is the limit in law of this sequence.

The signed cubic variation

If $\Lambda \in \mathcal{S}$, let $V^\Lambda = \{V_n^\Lambda\}$, where $V_n^\Lambda(t) = \sum_{j=1}^{\lfloor nt \rfloor} (\delta_j(\Lambda_n))^3$. Then $V^\Lambda \in \mathcal{S}$ with $\varphi_{1,V^\Lambda} = 0$ and $\varphi_{3,V^\Lambda} = \varphi_{1,\Lambda}^3$. If $\Lambda \equiv \Theta$, then $V^\Lambda \equiv V^\Theta$. Hence, if $N = [\Lambda] \in [\mathcal{S}]$, then we define the signed cubic variation of N to be $[\![N]\!] = [V^\Lambda]$. We also write $[\![N]\!]_t$ for $\mathcal{I}_{[\![N]\!]}(t)$.

If $X = f(B)$, then $[\![X]\!] = [\![N^X]\!]$ is the equivalence class in $\mathcal{S}$ of the sequence of sums,

$$V_n^X(t) = \sum_{j=1}^{\lfloor nt \rfloor} (X(t_j) - X(t_{j-1}))^3,$$

and

$$[\![X]\!]_t = \kappa \int_0^t (f'(B(s)))^3 \, dW(t),$$

which is the limit in law of the above sequence.

Decomposition of $N \in [\mathcal{S}]$

Each $\eta \in \mathcal{F}_\infty^B$ is identified with the equivalence class of the constant process $\eta(t) = \eta$.

Lemma (S, 2011)

Each $N \in [\mathcal{S}]$ can be written uniquely as $N = \eta + Y + V$, where $\eta \in \mathcal{F}_\infty^B$, $Y = g(B)$ with $Y(0) = 0$, and $V = \int \theta(B)\,\mathbf{d}[\![B]\!]$, where $g, \theta \in C^\infty$.

Note that $\int_0^t \theta(B(s))\,\mathbf{d}[\![B]\!]_s = \kappa \int_0^t \theta(B(s))\,dW(s)$, so every element of N can be uniquely decomposed into a smooth function of B and a Brownian martingale.

Lemma (S, 2011)

Let $X = f(B)$, where $f \in C^\infty$, and V as above. Then

$$\int X\,\mathbf{d}V = \int X\theta(B)\,\mathbf{d}[\![B]\!].$$

Change-of-variable formulas

Theorem (S, 2011)

Let $N \in [\mathcal{S}]$ and $X = f(B)$, where $f \in C^\infty$. Write $N = \eta + Y + V$, where $Y = g(B)$ and $V = \int \theta(B)\,\mathbf{d}[\![B]\!]$, with $g, \theta \in C^\infty$. Then

$$\int X\,\mathbf{d}N = \Phi(B) + \frac{1}{12}\int (f''g' - f'g'')(B)\,\mathbf{d}[\![B]\!] + \int X\,\mathbf{d}V, \quad \text{(D)}$$

where $\Phi' = fg'$ and $\Phi(0) = 0$.

- Note that $\int X\,\mathbf{d}N \in [\mathcal{S}]$, and (D) gives the decomposition of $\int X\,\mathbf{d}N$ into a smooth function of B and a Brownian martingale.
- Equation (D) expresses equality in $[\mathcal{S}]$. Hence, if we choose any sequence from the class on the left and any sequence from the class on the right, then their difference will converge to zero ucp. This is a stronger statement than simply asserting that the two sequences have the same limiting law.

Change-of-variable formulas

Corollary (S, 2011)

Let $Y = g(B)$*, where* $g \in C^\infty$*, and let* $\varphi \in C^\infty$*. Then*

$$\varphi(Y(t)) = \varphi(Y(0)) + \int_0^t \varphi'(Y(s))\,\mathbf{d}Y(s) - \frac{1}{12}\int_0^t \varphi'''(Y(s))\,\mathbf{d}[\![Y]\!].$$

Change-of-variable formulas

Corollary (S, 2011)

Let $N \in [\mathcal{S}]$, $X = f(B)$, and $Z = h(B)$, where $f, h \in C^\infty$. Let $M = \int X \, \mathbf{d}N$. Write $N = \eta + g(B) + \int \theta(B) \, \mathbf{d}[\![B]\!]$, with $g, \theta \in C^\infty$. Then

$$\int Z \, \mathbf{d}M = \int ZX \, \mathbf{d}N - \frac{1}{4} \int (f'g'h')(B) \, \mathbf{d}[\![B]\!].$$

Moreover, the above correction term is a "weak triple covariation" in the following sense: If $Y = g(B)$ and $\mathcal{V} = \{\mathcal{V}_n\}$, where

$$\mathcal{V}_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} (X(t_j) - X(t_{j-1}))(Y(t_j) - Y(t_{j-1}))(Z(t_j) - Z(t_{j-1})),$$

then $\mathcal{V} \in \mathcal{S}$ and $[\mathcal{V}] = \int (f'g'h')(B) \, \mathbf{d}[\![B]\!]$.