Fluctuations of the empirical quantiles of independent Brownian motions

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α-quantiles

ν – a probability measure

$\Phi_\nu(x) = \nu\left((-\infty, x]\right)$

$\alpha \in (0,1)$

An α **-quantile** of ν is any number q such that

$$\Phi_\nu(q-) \le \alpha \le \Phi_\nu(q)$$

- $\inf\left\{x : \alpha \le \Phi_\nu(x)\right\}$ is always an α -quantile of ν .
- If Φ_ν is continuous, then $\Phi_\nu(q) = \alpha$ for any α -quantile, q .
- If $q_1 < q_2$ are α -quantiles, then $\nu\left((q_1, q_2)\right) = 0$.

Order statistics

$X_1, X_2, \ldots, X_n$ – random variables

σ – a random permutation of $\{1, 2, \ldots, n\}$ such that

$$X_{\sigma(1)} \le X_{\sigma(2)} \le \cdots \le X_{\sigma(n)} \text{ a.s.}$$

The j **-th order statistic** of $X = (X_1, X_2, \ldots, X_n)$ is $X_{j:n} = X_{\sigma(j)}$

- $-X_{j:n} = (-X)_{(n-j+1):n}$

The model

$B(t)$ – one-dimensional Brownian motion

$B(0)$ has a density $f \in C^{\infty}(\mathbb{R})$, with $\sup_{x\in\mathbb{R}} \left(1+|x|^{n}\right)\left|f^{(m)}(x)\right| < \infty$ for all m, n.

Fix $\alpha \in (0,1)$.

Assume $f(x)dx$ has a unique α-quantile, $q(0)$, such that $f(q(0)) > 0$.

$\{B_n\}$ – iid copies of B

$B_{j:n}(t)$ – j-th order statistic of $(B_1(t), B_2(t), \ldots, B_n(t))$

Note: $B_{j:n}$ is a continuous process

$\{j(n)\}_{n=1}^{\infty}$ – integers such that $1 \leq j(n) \leq n$ and $j(n)/n = \alpha + o\left(n^{-1/2}\right)$

$$\boxed{Q_n(t) = B_{j(n):n}(t)}$$

The model

$u(x,t)$ – density of $B(t)$

$q(t)$ – unique α -quantile of $u(x,t)dx$

Lemma: $q \in C[0,\infty) \cap C^{\infty}(0,\infty)$ and $q'(t) = -\dfrac{\partial_x u(q(t),t)}{2u(q(t),t)}$ for all $t > 0$.

Proof sketch:

$$\alpha = P(B(t) \le q(t)) = \int_{-\infty}^{q(t)} u(x,t)dx$$

$$0 = u(q(t),t)q'(t) + \int_{-\infty}^{q(t)} \partial_t u(x,t)dx$$

$$0 = u(q(t),t)q'(t) + \frac{1}{2}\int_{-\infty}^{q(t)} \partial_x^2 u(x,t)dx$$

$$0 = u(q(t),t)q'(t) + \frac{1}{2}\partial_x u(q(t),t)$$

The model

$u(x,t)$ – density of $B(t)$

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Lemma: $q \in C[0,\infty) \cap C^{\infty}(0,\infty)$ and $q'(t) = -\dfrac{\partial_x u(q(t),t)}{2u(q(t),t)}$ for all $t > 0$.

Proof sketch: Suppose $q(0) > \liminf_{t\to 0} q(t)$. Then $\exists \; t_n \downarrow 0$ and $\varepsilon > 0$ s.t. $q(0) - \varepsilon > q(t_n)$, which implies

$$\alpha = P(B(t_n) \le q(t_n)) \le P(B(t_n) \le q(0) - \varepsilon)$$

$$\xrightarrow{n\to\infty} P(B(0) \le q(0) - \varepsilon) \le P(B(0) \le q(0)) = \alpha.$$

Hence, $P(B(0) \le q(0) - \varepsilon) = \alpha$, and the α -quantile is not unique, a contradiction. □

The model

$$F_n(t) = n^{1/2}\left(Q_n(t) - q(t)\right)$$

Theorem: $F_n \Rightarrow F$ in $C[0,\infty)$, where F is a continuous, centered Gaussian process with covariance

$$\rho(s,t) = \frac{P(B(s) \le q(s), B(t) \le q(t)) - \alpha^2}{u(q(s),s)u(q(t),t)}.$$

Outline

- **Convergence of finite-dimensional distributions**

  In particular, ρ defines a Gaussian process

- **Properties of the limit process**

  Comparison with fBm, $B^{1/4}$

  (for fixed $H \in (0,1)$, $B^H(\cdot)$ is a centered Gaussian process with $B^H(0) = 0$ and $E\left|B^H(t) - B^H(s)\right|^2 = |t-s|^{2H}$.)

  Related work: [Harris, 1965], [Dürr, Goldstein, Lebowitz, 1985]

- **Tightness**

  - ♦ **Connect quantile to iid sums**
  - ♦ **Estimate iid sums in terms of their parameters**
  - ♦ **Estimate those parameters in terms of our specific model**

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- **Tightness**

  - ♦ **Connect quantile to iid sums**
  - ♦ **Estimate iid sums in terms of their parameters**
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Done in generality, with future projects in mind.

Potential future projects

- **quantiles of diffusions** (with Tom Kurtz)
- **quantiles of general Gaussian processes**

fBm: limiting fluctuations of iid copies of B^H should behave locally like $B^{H/2}$.

Convergence of finite-dimensional distributions

$X = (X(1), X(2), \ldots, X(d))$ – $\mathbb{R}^d$-valued random variable

$\Phi_j(x) = P(X(j) \le x)$, $G_{ij}(x, y) = P(X(i) \le x, X(j) \le y)$, fix $\alpha \in (0,1)$

Assume $\exists$ $q = (q(1), q(2), \ldots, q(d)) \in \mathbb{R}^d$ such that $\Phi_j(q(j)) = \alpha$,

$\Phi'_j(q(j))$ exists and is strictly positive, and G_{ij} is continuous at $(q(i), q(j))$.

$\{X_n\}$ – iid copies of X

$X_{k:n}$ – component-wise order statistics of $X_1, X_2, \ldots, X_n$;

i.e. $X_{k:n}(j)$ is the k-th order statistic of $(X_1(j), X_2(j), \ldots, X_n(j))$

Quantile CLT: If $k(n)/n = \alpha + o\left(n^{-1/2}\right)$, then $n^{1/2}\left(X_{k(n):n} - q\right) \Rightarrow N$, where N is mean zero, multi-normal, with covariance

$$\sigma_{ij} = EN(i)N(j) = \frac{G_{ij}(q(i), q(j)) - \alpha^2}{\Phi'_i(q(i))\Phi'_j(q(j))}$$

Convergence of finite-dimensional distributions

$\Phi_j(x) = P(X(j) \le x)$, $G_{ij}(x, y) = P(X(i) \le x, X(j) \le y)$

$\Phi_j(q(j)) = \alpha$, $\Phi'_j(q(j)) > 0$, G_{ij} continuous at $(q(i), q(j))$.

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- Convergence of finite-dimensional distributions is an immediate corollary.

Convergence of finite-dimensional distributions

$\Phi_j(x) = P(X(j) \le x)$, $G_{ij}(x,y) = P(X(i) \le x, X(j) \le y)$

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Quantile CLT: If $k(n)/n = \alpha + o\left(n^{-1/2}\right)$, then $n^{1/2}\left(X_{k(n):n} - q\right) \Rightarrow N$, where

$$\sigma_{ij} = EN(i)N(j) = \frac{G_{ij}(q(i), q(j)) - \alpha^2}{\Phi'_i(q(i))\Phi'_j(q(j))}$$

Proof sketch: For $x, y \in \mathbb{R}^d$, $x \le y$ iff $x(j) \le y(j)$ for all j.

$$P\left(n^{1/2}\left(X_{k(n):n} - q\right) \le x\right) = P\left(X_{k(n):n}(j) \le n^{-1/2}x(j) + q(j), \forall j\right)$$

$$= P\left(\sum_{m=1}^{n} 1_{\{X_m(j) \le n^{-1/2}x(j) + q(j)\}} \ge k(n), \forall j\right)$$

$$= P\left(\sum_{m=1}^{n} Y_{m,n}(j) \ge n^{-1/2}\left(k(n) - np_n(j)\right), \forall j\right)$$

Convergence of finite-dimensional distributions

$$P\left(n^{1/2}\left(X_{k(n):n}-q\right)\le x\right)=P\left(\sum_{m=1}^{n}Y_{m,n}(j)\ge n^{-1/2}\left(k(n)-np_n(j)\right),\forall j\right),$$

where

$$p_n(j)=P\left(X(j)\le n^{-1/2}x(j)+q(j)\right)$$

$$Y_{m,n}(j)=n^{-1/2}\left(1_{\left\{X_m(j)\le n^{-1/2}x(j)+q(j)\right\}}-p_n(j)\right).$$

Lindeberg-Feller: $\sum_{m=1}^{n}Y_{m,n}\Rightarrow\tilde{N}$, centered normal with

$$E\tilde{N}(i)\tilde{N}(j)=G_{ij}(q(i),q(j))-\alpha^2.$$

Convergence of finite-dimensional distributions

$$P\left(n^{1/2}\left(X_{k(n):n}-q\right)\leq x\right)=P\left(\sum_{m=1}^{n}Y_{m,n}(j)\geq n^{-1/2}\left(k(n)-np_n(j)\right),\forall j\right),$$

$$\sum_{m=1}^{n}Y_{m,n}\Rightarrow\tilde{N},\ E\tilde{N}(i)\tilde{N}(j)=G_{ij}(q(i),q(j))-\alpha^2.$$

$$\begin{aligned}n^{-1/2}\left(k(n)-np_n(j)\right)&=n^{1/2}\left(k(n)/n-p_n(j)\right)\\&=n^{1/2}\left(\alpha-p_n(j)\right)+o(1)\\&=\frac{\Phi_j(q(j))-\Phi_j(n^{-1/2}x(j)+q(j))}{n^{-1/2}}+o(1)\\&\to-x(j)\Phi_j'(q(j))\end{aligned}$$

Hence,

$$P\left(n^{1/2}\left(X_{k(n):n}-q\right)\leq x\right)\to P\left(-\tilde{N}(j)/\Phi_j'(q(j))\leq x(j),\forall j\right)=P\left(N\leq x\right).\quad\square$$

Properties of the limit process

$$\rho(s,t) = \frac{P(B(s) \le q(s), B(t) \le q(t)) - \alpha^2}{u(q(s),s)u(q(t),t)}$$

Theorem: For each $T > 0$, $\exists \ \delta_0, C_1, C_2 > 0$ such that for all $0 < s < t \le T$,

$$\text{(i) } C_1 |t-s|^{-1/2} \le \partial_s \rho(s,t) \le C_2 |t-s|^{-1/2}$$

$$\text{(ii) } -C_2 |t-s|^{-1/2} \le \partial_t \rho(s,t) \le -C_1 |t-s|^{-1/2}$$

$$\text{(iii) } -C_2 |t-s|^{-3/2} \le \partial_{st}^2 \rho(s,t) \le -C_1 |t-s|^{-3/2}$$

whenever $|t-s| < \delta_0$.

Heuristic: Define $\tilde{F}(t) = u(q(t),t)F(t)$.

$$\tilde{\rho}(s,t) = P(B(s) \le q(s), B(t) \le q(t)) - \alpha^2$$

Properties of the limit process

$$\begin{aligned}\tilde{\rho}(s,t) &= P(B(s) \le q(s), B(t) \le q(t)) - \alpha^2 \\ &= P(B(s) \le q(s)) - P(B(s) \le q(s), B(t) > q(t)) - \alpha^2 \\ &= \alpha - \alpha^2 - P(B(s) \le q(s), B(t) > q(t)) \\ &\approx \alpha - \alpha^2 - Cu(q(s), s)\left|t - s\right|^{1/2}\end{aligned}$$

Fix $s < t$. Let $\delta = t - s$. Then $\left|B(t) - B(s)\right| \approx \delta^{1/2}$ and $\left|q(t) - q(s)\right| \approx \delta$. Hence, $B(s) \ll q(s) - \delta^{1/2}$ implies $B(t) \ll q(t)$. Let $I = [q(s) - \delta^{1/2}, q(s)]$.

$$\begin{aligned}P(B(s) \le q(s), B(t) > q(t)) &\approx P(B(s) \ll q(s) - \delta^{1/2}, B(t) > q(t)) \\ &\quad + P(B(s) \in I)P(B(t) > q(t) \mid B(s) \in I)\end{aligned}$$

Properties of the limit process

$$\tilde{\rho}(s,t) = P(B(s) \le q(s), B(t) \le q(t)) - \alpha^2$$
$$= P(B(s) \le q(s)) - P(B(s) \le q(s), B(t) > q(t)) - \alpha^2$$
$$= \alpha - \alpha^2 - P(B(s) \le q(s), B(t) > q(t))$$
$$\approx \alpha - \alpha^2 - Cu(q(s), s)\left|t - s\right|^{1/2}$$

Fix $s < t$. Let $\delta = t - s$. Then $\left|B(t) - B(s)\right| \approx \delta^{1/2}$ and $\left|q(t) - q(s)\right| \approx \delta$.

Hence, $B(s) \ll q(s) - \delta^{1/2}$ implies $B(t) \ll q(t)$. Let $I = [q(s) - \delta^{1/2}, q(s)]$.

$$P(B(s) \le q(s), B(t) > q(t)) \approx \underbrace{P(B(s) \ll q(s) - \delta^{1/2}, B(t) > q(t))}_{0}$$
$$+ P(B(s) \in I)\underbrace{P(B(t) > q(t) \mid B(s) \in I)}_{C}$$
$$\approx C\delta^{1/2} u(q(s), s)$$

□

Properties of the limit process

Corollaries:

- $E\left|F(t)-F(s)\right|^2 \approx C\left|t-s\right|^{1/2}$
- F is locally Hölder continuous with exponent γ for all $\gamma \in (0,1/4)$.
- (local) anti-persistence: for $\Delta s, \Delta t, \left|t-s\right|$ small…
  - ♦ $E\left[F(s)\left(F(t+\Delta t)-F(t)\right)\right] \approx -C\left|t-s\right|^{-1/2}\Delta t$
  - ♦ $E\left[\left(F(s)-F(s-\Delta s)\right)\left(F(t+\Delta t)-F(t)\right)\right] \approx -C\left|t-s\right|^{-3/2}\Delta s\Delta t$

Properties of the limit process

Using the corollaries, we can prove F is a quartic variation process:

Theorem: $\Pi = \{0 = t_0 < t_1 < t_2 < \cdots\}$, $t_j \uparrow \infty$, $|\Pi| = \sup_j |t_j - t_{j-1}| < \infty$,

$$V_\Pi(t) = \sum_{0<t_j\le t} \left|F(t_j) - F(t_{j-1})\right|^4 .$$

For all $T > 0$,

$$\lim_{|\Pi|\to 0} E\left[\sup_{0\le t\le T}\left|V_\Pi(t) - \frac{6}{\pi}\int_0^t |u(q(s),s)|^{-2}\, ds\right|^2\right] = 0 .$$

All these local properties are shared with $B^{1/4}$. Also shared with the solution to the 1-d stochastic heat equation.

Related work: [S, 2007], [Burdzy, S, 2008 (preprint)]

Global properties may be different.

Properties of the limit process

Example of global properties:

$$B(0) \sim N(0,1),\ j(n) = \lfloor (n+1)/2 \rfloor,\ \alpha = 1/2,$$

Q_n is the median, $q(t) = 0$, $F_n(t) = n^{1/2} Q_n(t)$

$$F_n(\cdot) \Rightarrow F(\cdot) =_d X(\cdot + 1)$$

$$\rho_X(s,t) = \sqrt{st}\, \sin^{-1}\left(\frac{s \wedge t}{\sqrt{st}}\right)$$ [S, 2007]

$B^{1/4}(c\,\cdot) =_d c^{1/4} B^{1/4}(\cdot)$	$X(c\,\cdot) =_d c^{1/2} X(\cdot)$
$r_{1/4}(n) = E\left[B^{1/4}(1)\left(B^{1/4}(n+1) - B^{1/4}(n)\right)\right] \sim -\frac{1}{4} n^{-3/2}$	$r_X(n) = E\left[X(1)\left(X(n+1) - X(n)\right)\right] \sim -\frac{1}{6} n^{-2}$

Tightness

Theorem: If $\{F_n\}$ satisfies

(i) $\lim_{\lambda \to \infty} \sup_n P\left(|F_n(0)| > \lambda\right) = 0$, and

(ii) For all $T > 0$, $\exists$ constants $\alpha, \beta, C, n_0 > 0$ such that

$$P\left(|F_n(t) - F_n(s)| \geq \varepsilon\right) \leq C\varepsilon^{-\alpha} |t - s|^{1+\beta},$$

for all $n \geq n_0$, $s, t \in [0, T]$, and $\varepsilon \in (0,1)$,

then $\{F_n\}$ is tight.

- Verifying (i) is a relatively minor lemma.
- We verify (ii) by breaking into "regimes."

Tightness

Lemma: Fix $T>0$, $\Delta\in(0,1/2)$, and $p>2$. $\exists$ $C,n_0>0$ such that

$$P\left(|F_n(t)-F_n(s)|\geq\varepsilon\right)\leq C\left(\varepsilon^{-1}|t-s|^{1/4}\right)^p,$$

$\forall$ $n\geq n_0$, $s,t\in[0,T]$, $\varepsilon\in(0,1)$, and $n^{-1/2}\varepsilon\geq|t-s|^{1/2-\Delta}$.

Lemma: Fix $T>0$, $\Delta\in(0,1/2)$, and $p>2$. $\exists$ $C,n_0>0$ such that

$$P\left(|F_n(t)-F_n(s)|\geq\varepsilon\right)\leq C\left(\varepsilon^{-1}|t-s|^{1/4}\right)^p,$$

$\forall$ $n\geq n_0$, $s,t\in[0,T]$, $\varepsilon\in(0,1)$, and $n^{-1/2}\varepsilon\leq|t-s|^{1/2+\Delta}$.

Lemma: Fix $T>0$, $\Delta\in(0,1/8)$, and $p>2$. $\exists$ $C,n_0>0$ such that

$$P\left(|F_n(t)-F_n(s)|\geq\varepsilon\right)\leq C\left(\varepsilon^{-1}|t-s|^{1/4-2\Delta}\right)^p,$$

$\forall$ $n\geq n_0$, $s,t\in[0,T]$, $\varepsilon\in(0,1)$, and $|t-s|^{1/2+\Delta}\leq n^{-1/2}\varepsilon\leq|t-s|^{1/2-\Delta}$.

Tightness

Lemma: Fix $T>0$, $\Delta\in(0,1/2)$, and $p>2$. $\exists\ C,n_0>0$ such that

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$\forall\ n\geq n_0$, $s,t\in[0,T]$, $\varepsilon\in(0,1)$, and $n^{-1/2}\varepsilon\geq\left|t-s\right|^{1/2-\Delta}$.

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$\forall\ n\geq n_0$, $s,t\in[0,T]$, $\varepsilon\in(0,1)$, and $\left|t-s\right|^{1/2+\Delta}\leq n^{-1/2}\varepsilon\leq\left|t-s\right|^{1/2-\Delta}$.

Tightness

$$P\left(\left|F_n(t)-F_n(s)\right|\geq\varepsilon\right)=P\left(\left|\left(Q_n(t)-q(t)\right)-\left(Q_n(s)-q(s)\right)\right|\geq n^{-1/2}\varepsilon\right)$$
$$=P\left(\left|\bar{B}_{j(n):n}(t)-\bar{B}_{j(n):n}(s)\right|\geq n^{-1/2}\varepsilon\right),$$

where $\bar{B}(t)=B(t)-q(t)$.

In general, if X and Y are dependent, what is

$$P\left(\left|Y_{j:n}-X_{j:n}\right|\geq y\right)?$$

The nonlinearity of the quantile function makes this a difficult question.

Connecting quantile to iid sums

(X,Y) – $\mathbb{R}^2$-valued random variable

Assume $(x,y) \mapsto P\left(X \le x, Y \le y\right)$ is continuous.

$$q_1(x,y) = P\left(Y > x + y \mid X < x\right)$$
$$q_2(x,y) = P\left(Y < x + y \mid X > x\right)$$

$$\varphi_{j:n}^{\le}(x,y) = P\left(\sum_{i=1}^{j-1} 1_{\{U_i \le q_1\}} \le \sum_{i=j+1}^{n} 1_{\{U_i \le q_2\}}\right),$$

where $\{U_i\}$ are iid Uniform$(0,1)$.

Similarly define $\varphi_{j:n}^{<}$, $\varphi_{j:n}^{\ge}$, and $\varphi_{j:n}^{>}$.

Connecting quantile to iid sums

$$q_1(x,y) = P\left(Y > x + y \mid X < x\right)$$

$$q_2(x,y) = P\left(Y < x + y \mid X > x\right)$$

$$\varphi_{j:n}^{\leq}(x,y) = P\left(\sum_{i=1}^{j-1} 1_{\{U_i \leq q_1\}} \leq \sum_{i=j+1}^{n} 1_{\{U_i \leq q_2\}}\right),$$

Theorem: If $\{(X_n, Y_n)\}$ are iid copies of (X, Y), then for all $y \in \mathbb{R}$,

$$\varphi_{j:n}^{<}(X_{j:n}, y) \leq P\left(Y_{j:n} - X_{j:n} < y \mid X_{j:n}\right) \leq \varphi_{j:n}^{\leq}(X_{j:n}, y) \text{ a.s.}$$

$$\varphi_{j:n}^{>}(X_{j:n}, y) \leq P\left(Y_{j:n} - X_{j:n} > y \mid X_{j:n}\right) \leq \varphi_{j:n}^{\geq}(X_{j:n}, y) \text{ a.s.}$$

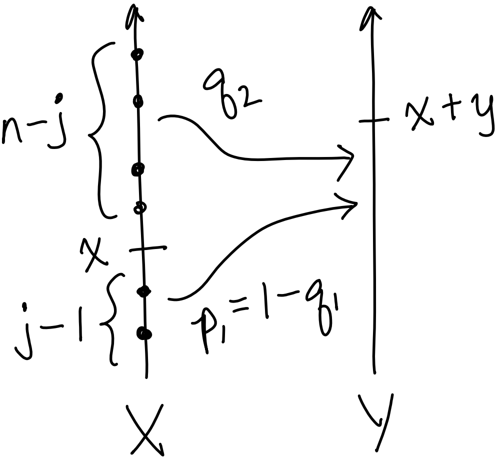
By taking complements, the two lines are equivalent.

Connecting quantile to iid sums

$$q_1(x,y) = P\left(Y > x+y \mid X < x\right), q_2(x,y) = P\left(Y < x+y \mid X > x\right)$$

$$P\left(Y_{j:n} - X_{j:n} < y \mid X_{j:n} = x\right) = P\left(Y_{j:n} < x+y \mid X_{j:n} = x\right)$$

Heuristic:



We need

$$\sum_{i=1}^{j-1} 1_{\{U_i \le p_1\}} + \sum_{i=j+1}^{n} 1_{\{U_i \le q_2\}} \ge j$$

$$\sum_{i=1}^{j-1} \left(1 - 1_{\{U_i \le q_1\}}\right) + \sum_{i=j+1}^{n} 1_{\{U_i \le q_2\}} \ge j$$

$$-\sum_{i=1}^{j-1} 1_{\{U_i \le q_1\}} + \sum_{i=j+1}^{n} 1_{\{U_i \le q_2\}} > 0$$

Probability is $\varphi_{j:n}^{<}(x,y)$.

(Have not accounted for particle at x.)

Estimating the iid sums

Theorem: Fix $\alpha \in (0,1)$. Let $\sigma = \alpha q_1 + (1-\alpha) q_2, \mu = \alpha q_1 - (1-\alpha) q_2$. Suppose $j(n)/n = \alpha + o(n^{-1/2})$. Then $\forall r > 1$, $\exists\ C, n_0 > 0$ such that $\forall n \geq n_0$,

$$\varphi^{\leq}_{j(n):n}(x,y) = P\left(\sum_{i=1}^{j-1} 1_{\{U_i \leq q_1\}} \leq \sum_{i=j+1}^{n} 1_{\{U_i \leq q_2\}} \right) \leq C \frac{\sigma^r}{n^r \mu^{2r}}$$

whenever $\mu > 0$. Note that C does not depend on q_1 or q_2.

Follows from…

Lemma: For all $r \geq 1$, $\exists\ C, n_0 > 0$ such that

$$E\left| \sum_{i=1}^{n} \left(1_{\{U_i \leq p\}} - p \right) \right|^{2r} \leq C\Big((np)^r \vee (np) \Big),$$

for all $n \geq n_0$ and all $p \in [0,1]$.

Applying the estimates

Recall: $F_n(t) = n^{1/2}\left(Q_n(t) - q(t)\right) = n^{1/2}\left(B_{j(n):n}(t) - q(t)\right)$

Fix $s < t$. Define $\delta = t - s$.

$$X = B(s) - q(s) \qquad X_{j(n):n} = Q_n(s) - q(s)$$
$$Y = B(t) - q(t) \qquad Y_{j(n):n} = Q_n(t) - q(t)$$

$$q_1(x,y) = P\left(B(t) > q(t) + x + y \mid B(s) < q(s) + x\right)$$
$$q_2(x,y) = P\left(B(t) < q(t) + x + y \mid B(s) > q(s) + x\right)$$

$$\begin{aligned} P\left(\left|F_n(t) - F_n(s)\right| > \varepsilon\right) &= P\left(F_n(t) - F_n(s) < -\varepsilon\right) + P\left(F_n(t) - F_n(s) > \varepsilon\right) \\ &= P\left(Y_{j:n} - X_{j:n} < -n^{-1/2}\varepsilon\right) + P\left(Y_{j:n} - X_{j:n} > n^{-1/2}\varepsilon\right) \end{aligned}$$

(The two probabilities are estimates similarly.)

Applying the estimates

$$P\left(Y_{j:n} - X_{j:n} < -n^{-1/2}\varepsilon\right) \le E\left[\varphi_{j:n}^{\le}(X_{j:n}, -n^{-1/2}\varepsilon)\right]$$
$$\le \sup_{|x|\le K} \varphi_{j:n}^{\le}(x, -n^{-1/2}\varepsilon) + P\left(\left|X_{j:n}\right| \ge K\right),$$

where $K = n^{-1/2}\varepsilon\delta^{-1/4}$. Straightforward to prove that:

$$P\left(\left|X_{j:n}\right| \ge n^{-1/2}\varepsilon\delta^{-1/4}\right) = P\left(\left|F_n(s)\right| \ge \varepsilon\delta^{-1/4}\right)$$
$$\le C_p\left(\varepsilon\delta^{-1/4}\right)^{-p}$$
$$= C_p\left(\varepsilon^{-1}\left|t-s\right|^{1/4}\right)^{p}.$$

Need to estimate $\varphi_{j:n}^{\le}(x, -n^{-1/2}\varepsilon)$ when $\left|x\right| \le n^{-1/2}\varepsilon\delta^{-1/4}$.

Applying the estimates

$$q_1(x,y) = P\left(B(t) > q(t) + x + y \mid B(s) < q(s) + x\right)$$
$$q_2(x,y) = P\left(B(t) < q(t) + x + y \mid B(s) > q(s) + x\right)$$
$$\sigma(x,y) = \alpha q_1(x,y) + (1-\alpha) q_2(x,y)$$
$$\mu(x,y) = \alpha q_1(x,y) - (1-\alpha) q_2(x,y)$$

$$\varphi^{\leq}_{j(n):n}(x, -n^{-1/2}\varepsilon) \leq C \frac{\sigma(x, -n^{-1/2}\varepsilon)^{p/2}}{n^{p/2} \mu(x, -n^{-1/2}\varepsilon)^{p}} .$$

Need to estimate $\sigma(x, -n^{-1/2}\varepsilon)$ (upper bound) and $\mu(x, -n^{-1/2}\varepsilon)$ (lower bound).

Applying the estimates

$$q_1(x,y) = P\left(B(t) > q(t) + x + y \mid B(s) < q(s) + x\right)$$
$$q_2(x,y) = P\left(B(t) < q(t) + x + y \mid B(s) > q(s) + x\right)$$
$$\sigma(x,y) = \alpha q_1(x,y) + (1-\alpha) q_2(x,y)$$
$$\mu(x,y) = \alpha q_1(x,y) - (1-\alpha) q_2(x,y)$$

$$\Psi(x,y) = P\left(B(t) > q(t) + x + y, B(s) < q(s) + x\right), \ \Psi(0,0) \approx C\delta^{1/2}$$

Taylor expansions give:

$$\alpha q_1(x,y) = \Psi(0,0) - \frac{1}{2} u(q(s),s) y + \frac{1}{2\sqrt{2\pi\delta}} u(q(s),s) y^2 + R_1$$
$$(1-\alpha) q_2(x,y) = \Psi(0,0) + \frac{1}{2} u(q(s),s) y + \frac{1}{2\sqrt{2\pi\delta}} u(q(s),s) y^2 + R_2$$
$$\left|R_j\right| \le C\left(\left(|x| + |y|\right)\left(\delta^{1/2} + |y| + \delta^{-1/2} |y|^2\right) + \delta^{-3/2} |y|^4\right)$$

The rest of the proof is a technical piecing together of these estimates.

Applying the estimates

$$q_1(x,y) = P\left(B(t) > q(t) + x + y \mid B(s) < q(s) + x\right)$$

$$q_2(x,y) = P\left(B(t) < q(t) + x + y \mid B(s) > q(s) + x\right)$$

$$\alpha q_1(x,y) = \Psi(0,0) - \frac{1}{2} u(q(s),s) y + \frac{1}{2\sqrt{2\pi\delta}} u(q(s),s) y^2 + R_1$$

$$(1-\alpha) q_2(x,y) = \Psi(0,0) + \frac{1}{2} u(q(s),s) y + \frac{1}{2\sqrt{2\pi\delta}} u(q(s),s) y^2 + R_2$$

This final step (applying the estimates with Taylor expansions) will need to be done separately for each future project. (The previous steps will carry over unchanged.)

E.g.: diffusion quantiles – analyze transition densities
quantiles of Gaussian processes – analyze the covariance function.

On the other hand, the final form of this Taylor expansion looks quite general. It seems plausible that under appropriate conditions, it will still be valid in other models.