Large deviations and a non-Markovian vanishing viscosity result. Part 2

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Outline

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Overview

- $X = X^{a(\cdot),n}$ solves $dX(t) = a(t)\,dt + \frac{1}{\sqrt{n}} dW(t)$.

$$\boxed{\inf_{a(\cdot)} \mathbb{E}\left[\int_0^T \ell(t,a(t))\,dt + h\left(X^{a(\cdot),n}\right)\right] \quad \underset{(n\to\infty)}{\longrightarrow} \quad \inf_{x(\cdot)}\left[\int_0^T \ell(t,x'(t))\,dt + h(x(\cdot))\right]}$$

should lead to convergence of solutions u_n to u of path-dependent PDEs

$$\boxed{\left(\partial_t + \frac{1}{2n}\partial_{xx}\right) u_n + \inf_a \left[\ell(t,a) + a\,\partial_x u_n\right] = 0 \qquad \text{resp.} \qquad \partial_t u + \inf_a \left[\ell(t,a) + a\,\partial_x u\right] = 0}$$

on $(0,T) \times C([0,T])$ with terminal condition $u_n(T,\cdot) = u(T,\cdot) = h$ on $C([0,T])$.

Outline

Setting

- $\Omega = C([0,T],\mathbb{R}^d)$ with $d = 1$ (for simplicity) equipped with $\|\cdot\|_\infty$.
- $X = (X(t))_{0\le t\le T}$ canonical process on Ω, i.e.,

$$X(t,\omega) = \omega(t).$$

- $\mathbb{F} = \mathbb{F}^X$.
- As *distance* on $[0,T]\times\Omega$, we use pseudo-metric $\mathbf{d}_\infty$:

$$\mathbf{d}_\infty((t_1,\omega_1),(t_2,\omega_2)) := |t_1 - t_2| + \sup_{s\in[0,T]} |\omega_1(s\wedge t_1) - \omega_2(s\wedge t_2)|.$$

Fact

$u : [0,T]\times\Omega \to \mathbb{R}$ semi-continuous (with respect to $\mathbf{d}_\infty$)

$\Longrightarrow$ u is non-anticipating: $u(t,x) = u(t, x(\cdot\wedge t))$.

Data and standing assumptions

Running cost

Let $\ell = \ell(t,a) : [0,T] \times \mathbb{R} \to [0,\infty]$ be Borel measurable. Assume:

- $\lim_{|a|\to\infty} \inf_t \frac{\ell(t,a)}{|a|} = \infty$ (Tonelli-Nagumo condition).
- For each t, $\ell(t,\cdot)$ is l.s.c., convex, and proper (i.e., $\ell(t,a) < \infty$ for some a).
- $\int_0^T \ell(t,0)\,dt < \infty$.

Example

(i) $\ell(t,a) = a^2$.
(ii) $\ell(t,a) = |a|^{3/2}$.

Terminal cost

Let $h = h(x) : C([0,T]) \to [0,\infty]$ be Borel measurable.

Example

$h(x) = 0$ if $x(t) = \sqrt{t}$ and $h(x) = \infty$ otherwise.

Outline

Stochastic control problems

Infima here are taken over all $a(\cdot) : [0, T] \times \Omega \to \mathbb{R}$ that are $\mathbb{F}$-progressively measurable and bounded.

Value function (given $n \in \mathbb{N}$)

Let $(t, x) \in [0, T] \times \Omega$. Define

$$v_n(t,x) := \inf_{a(\cdot)} \mathbb{E}\left[\int_t^T \ell\left(s, a\left(s, \frac{W^{t,x}}{\sqrt{n}}\right)\right) ds + h\left(\frac{W^{t,x}}{\sqrt{n}} + \int_t^{\cdot} a\left(s, \frac{W^{t,x}}{\sqrt{n}}\right) ds\right)\right],$$

where $W^{t,x}$ is a Wiener process starting at (t, x), i.e.,

$$W^{t,x}(s,\omega) = x(s), \quad s \in [0, t].$$

Or in a more compact form:

$$v_n(t,x) = \inf_{a(\cdot)} \mathbb{E}_{t,x,n}\left[\int_t^T \ell(s, a(s))\, ds + h(X + A^a - A^a(t))\right],$$

where $P_{t,x,n}$ is the probability on $(\Omega, \mathcal{F})$ such that $(s, \omega) \mapsto X(s, \omega) = \omega(s)$ satisfies

- $\sqrt{n}X$ is a Wiener process starting at (t, x) and
- $A^a(s) := \int_0^s a(r)\, dr$.

Deterministic control problem

Value function

Let $(t_0, x_0) \in [0, T] \times \Omega$. Define

$$v_0(t_0, x_0) := \inf_{x \in \mathcal{X}^{1,1}(t_0, x_0)} \left[\int_{t_0}^{T} \ell(t, x'(t))\, dt + h(x) \right].$$

Here, $\mathcal{X}^{1,1}(t_0, x_0)$ is the set of all $x \in \Omega$ such that

- $x = x_0$ on $[0, t_0]$ and
- $x|_{(t_0, T)} \in W^{1,1}(t_0, T)$.

Outline

Theorem (adaption of Thm. 2.2 in Backhoff–Veraguas et al.)

If $h \in C_b(\Omega)$, then $v_n \to v_0$ uniformly on compacta and v_0 is continuous.

Structure of proof: Use envelopes

$$v_*(t_0, x_0) := \sup_{\substack{\delta>0,\\ n\in\mathbb{N}}} \inf_{\substack{(t,x)\in O_\delta(t_0,x_0),\\ m\geq n}} v_m(t,x), \quad v^*(t_0, x_0) := \inf_{\substack{\delta>0,\\ n\in\mathbb{N}}} \sup_{\substack{(t,x)\in O_\delta(t_0,x_0),\\ m\geq n}} v_m(t,x).$$

Step 1: Show that $\boxed{v_0 \leq v_*}$. Weak (probabilistic) convergence argument. Tonelli-Nagumo crucial for tightness. (Similar to proving existence of optimal control.)

Step 2: Show that $\boxed{v^* \leq v_0}$. Truncation argument.

Step 3: Using $\boxed{v_* \leq v^*}$ concludes the proof.

No Lavrentiev phenomenon

$$v_0(t_0, x_0) = \inf_{x\in\mathcal{X}^{1,1}(t_0,x_0)} \left[\int_{t_0}^{T} \ell(t, x'(t))\, dt + h(x)\right] \qquad \text{(by definition)}$$

$$= \inf_{x\in\mathcal{X}^{1,\infty}(t_0,x_0)} \left[\int_{t_0}^{T} \ell(t, x'(t))\, dt + h(x)\right].$$

Non-markovian case (Thm. 2.7 in Backhoff–Veraguas et al.)

If $h \in C_b(\Omega)$ & $\ell \in C([0,T] \times \mathbb{R})$, then the minimal supersolution (Y_n, Z_n) of

$$dY_n(t) = -\sup_{a \in \mathbb{R}} \left[a\sqrt{n}\, Z_n(t) - \ell(t,a) \right] dt + Z_n(t)\, dW(t), \quad Y_n(T) = -h\left(\frac{W}{\sqrt{n}}\right), \tag{1}$$

satisfies $\boxed{v_n(t, W/\sqrt{n}) = -Y_n(t)} \to v_0(t,0)$ as $n \to \infty$.

Markovian case (Prop. 6.4 in Backhoff–Veraguas et al.)

If $\tilde{h} \in C_b(\mathbb{R})$ & $\ell \in C([0,T] \times \mathbb{R})$, then the maxim. visc. subsol. $u_n \in \mathrm{USC}([0,T] \times \mathbb{R})$ of

$$\partial_t u_n + \frac{1}{2n} \partial_{xx} u_n + \inf_{a \in \mathbb{R}} \left[\ell(t,a) + a \cdot \partial_x u_n \right] = 0 \quad \text{on } (0,T) \times \mathbb{R},$$
$$u_n(T,x) = \tilde{h}(x) \quad \text{on } \mathbb{R},$$

satisfies, for every $(t_0, x_0) \in [0,T] \times \mathbb{R}$,

$$\lim_{n \to \infty} u_n(t_0, x_0) = \inf_{x \in AC([t_0,T]) \text{ with } x(t_0) = x_0} \left[\int_{t_0}^{T} \ell\left(t, x'(t)\right) dt + \tilde{h}(x(T)) \right].$$

Open problem in Backhoff–Veraguas et al.

Is there a non-Markovian vanishing viscosity result in terms of (path-dependent) PDEs on $[0, T] \times \Omega$?

In other words: Are there "solutions" $u_n : [0, T] \times \Omega \to \mathbb{R}$ of

$$\partial_t u_n + \frac{1}{2n} \partial_{xx} u_n + \inf_{a \in \mathbb{R}} [\ell(t, a) + a \cdot \partial_x u_n] = 0 \quad \text{on } (0, T) \times \Omega,$$

$$u_n(T, x) = h(x) \quad \text{on } \Omega,$$

that converge to **the** "solution" $u : [0, T] \times \Omega \to \mathbb{R}$ of

$$\partial_t u + \inf_{a \in \mathbb{R}} [\ell(t, a) + a \cdot \partial_x u] = 0 \quad \text{on } (0, T) \times \Omega,$$

$$u(T, x) = h(x) \quad \text{on } \Omega?$$

Outline

Relevant literature on path-dependent PDEs

- In nearly the whole literature on (generalized solutions of) path-dependent PDEs, **linear growth** in $\partial_x u$ is required.
- Only exception: Kaise, Kato, and Takahasi (2018)
  - Object of study: Calculus-of-variations problem with path-dependent terminal cost (as we do here but they require h to be **Lipschitz**).
  - Main result: Existence and uniqueness of and among **Lipschitz** viscosity solutions of corresponding path-dependent (1st order) HJB equation.

- Given $a \in \mathbb{R}$, consider path $A^a \in \Omega$, $t \mapsto at$.
- Given $u : [0,T] \times \Omega \to \mathbb{R} \cup \{+\infty\}$, $(t_0, x_0) \in [0,T) \times \Omega$, and $a \in \mathbb{R}$, put

$$d_- u(t_0,x_0)(1,a) := \varliminf_{\delta \downarrow 0} \frac{u(t_0+\delta, x_0(\cdot \wedge t_0) + A^a - A^a(t_0)) - u(t_0,x_0)}{\delta} \qquad \text{and}$$

$$d_+ u(t_0,x_0)(1,a) := \varlimsup_{\delta \downarrow 0} \frac{u(t_0+\delta, x_0(\cdot \wedge t_0) + A^a - A^a(t_0)) - u(t_0,x_0)}{\delta}.$$

Recall the path-dependent PDE

$$\partial_t u + \inf_{a\in\mathbb{R}} [\ell(t,a) + a \cdot \partial_x u] = 0 \quad \text{on } (0,T) \times \Omega, \qquad u(T,x) = h(x) \quad \text{on } \Omega. \tag{2}$$

Definition

(i) $u \in \mathrm{LSC}([0,T] \times \Omega)$ Dini supersolution of (2) if $u(T,\cdot) \geq h$ and $\forall\, (t_0,x_0) \in [0,T) \times \Omega$,

$$\inf_{a\in\mathbb{R}} [d_- u(t_0,x_0)(1,a) + \ell(t_0,a)] \leq 0.$$

(ii) $u \in \mathrm{USC}([0,T] \times \Omega)$ Dini subsolution of (2) if $u(T,\cdot) \leq h$ and $\forall\, (t_0,x_0) \in [0,T) \times \Omega$,

$$\inf_{a\in\mathbb{R}} [d_+ u(t_0,x_0)(1,a) + \ell(t_0,a)] \geq 0.$$

- Given $a \in \mathbb{R}$, consider path $A^a \in \Omega$, $t \mapsto at$.
- Given $(t_0, x_0) \in [0, T) \times \Omega$, W^{t_0,x_0} is a Wiener process starting at (t_0, x_0).
- Given $u : [0, T] \times \Omega \to \mathbb{R}$, $(t_0, x_0) \in [0, T) \times \Omega$, $n \in \mathbb{N}$, $a \in \mathbb{R}$, put

$$d_+^{1,2} u(t_0, x_0)(1, a, n^{-1}) := \overline{\lim_{\delta \downarrow 0}} \frac{\mathbb{E}\left[u\left(t_0 + \delta, \frac{W^{t_0,x_0}}{\sqrt{n}} + A^a - A^a(t_0)\right) - u(t_0, x_0)\right]}{\delta}.$$

Recall the path-dependent PDE

$$\partial_t u + \frac{1}{2n} \partial_{xx} u + \inf_{a \in \mathbb{R}} [\ell(t, a) + a \cdot \partial_x u] = 0 \quad \text{on } (0, T) \times \Omega, \quad u(T, \cdot) = h. \tag{3}$$

Definition

$u \in \mathrm{USC}([0, T] \times \Omega)$ Dini subsolution of (3) if $u(T, \cdot) \leq h$ and $\forall\, (t_0, x_0) \in [0, T) \times \Omega$,

$$\inf_{a \in \mathbb{R}} \left[d_+^{1,2} u(t_0, x_0)(1, a, n^{-1}) + \ell(t_0, a)\right] \geq 0.$$

A minimax-type notion of solutions in the 1st order case

- Advantages: Allows state constraints (example follows) and discontinuity in time.
- Disadvantages: Very weak notion.

Recall the path-dependent PDE

$$\partial_t u + \inf_{a \in \mathbb{R}} [\ell(t,a) + a \cdot \partial_x u] = 0 \quad \text{on } (0,T) \times \Omega, \qquad u(T,x) = h(x) \quad \text{on } \Omega. \tag{4}$$

Definition

Let $u : [0,T] \times \Omega \to [0,\infty]$.
(i) $u \in \mathrm{LSC}$ is minimax supersol. of (4) if $u(T,\cdot) \geq h$ and $\forall\,(t_0,x_0) \in \mathrm{dom}(u)$ with $t_0 < T$,

$$\inf_{x \in \mathcal{X}^{1,1}(t_0,x_0)} \underline{\lim}_{\delta \downarrow 0} \left[u(t_0+\delta, x) - u(t_0,x_0) + \int_{t_0}^{t_0+\delta} \ell(s, x'(s))\, ds \right] \delta^{-1} \leq 0.$$

(ii) $u \in \mathrm{LSC}$ is **l.s.c.**minimax subs. of (4) if $u(T,\cdot) \leq h$ & $\forall (t_0,x_0) \in [0,T) \times \Omega \forall t \in (t_0,T)$

$$\sup_{\substack{x \in \mathcal{X}^{1,1}(t_0,x_0) \\ \text{with } (t,x) \in \mathrm{dom}(u) \\ \text{and } \int_{t_0}^t \ell(s,x'(s))\,ds < \infty}} \underline{\lim}_{\delta \downarrow 0} \left[u(t-\delta, x) - u(t,x) - \int_{t-\delta}^{t} \ell(s, x'(s))\, ds \right] \delta^{-1} \leq 0.$$

An example with state constraints

- $K := \{t \mapsto \sqrt{t}\}$ (considered as a one-element subset of Ω)
- $h := \infty \cdot \mathbf{1}_{K^c}$
- $\ell(t,a) = |a|^{3/2}$

Then

$$v_0(t_0,x_0) = \inf_{x \in \mathcal{X}^{1,1}(t_0,x_0)} \left[\int_{t_0}^{T} |a|^{3/2}\, dt + \infty \cdot \mathbf{1}_{K^c}(x) \right]$$

$$= \begin{cases} \int_{t_0}^{T} 2^{-3/2}\, t^{-3/4}\, dt = 2^{1/2} \left(T^{1/4} - t_0^{1/4} \right) & \text{if } x_0|_{[0,t_0]}(t) = t^{1/2}, \\ \infty & \text{otherwise.} \end{cases}$$

Note: If $v_0(t_0,x_0) < \infty$ and $t_0 < T$, then, for every constant perturbation $a \in \mathbb{R}$,

$$d_- v_0(t_0,x_0)(1,a) = \varliminf_{\delta \downarrow 0} \frac{v_0(t_0+\delta, x_0(\cdot \wedge t_0) + A^a - A^a(t_0)) - v_0(t_0,x_0)}{\delta} = \infty.$$

Thus the HJB inequality for Dini supersolutions

$$\inf_{a \in \mathbb{R}} [d_- v_0(t_0,x_0)(1,a) + |a|^{3/2}] \le 0 \qquad \text{never holds.}$$

Possible resolution: Allow non-constant perturbations in semi-derivatives (cf. last slide).

Main wellposedness results

Recall the path-dependent PDEs

$$\partial_t u + \inf_{a\in\mathbb{R}} [\ell(t,a) + a\cdot\partial_x u] = 0 \quad \text{on } (0,T)\times\Omega, \qquad u(T,\cdot) = h \text{ on } \Omega, \tag{5}$$

$$\partial_t u + \frac{1}{2n}\partial_{xx} u + \inf_{a\in\mathbb{R}} [\ell(t,a) + a\cdot\partial_x u] = 0 \qquad \text{on } (0,T)\times\Omega, \quad u(T,\cdot) = h \text{ on } \Omega. \tag{6}$$

Theorem (A)

(i) Let $h:\Omega\to[0,\infty]$ be l.s.c. and proper (i.e., $h(x)<\infty$ for some x). Then v_0 is the unique l.s.c. minimax solution of (5) that is bounded from below.
(ii) Let h be u.s.c. and bounded. Let $\ell\in C([0,T]\times\mathbb{R},\mathbb{R})$. Then v_0 is the unique maximal bounded Dini subsolution of (5).

Theorem (B)

Let h be u.s.c. and bounded. Let $\ell\in C([0,T]\times\mathbb{R},\mathbb{R})$. Then v_n is the unique maximal bounded Dini subsolution of (6).

Step 1: Minimax supersolution property "$\Longleftrightarrow$" viability property

$$\forall (t_0, x_0) : \forall t > t_0 \ : \exists x \in \mathcal{X}^{1,1}(t_0, x_0) : \boxed{u(t_0, x_0) \geq \int_{t_0}^{t} \ell(s, x'(s))\, ds + u(t, x).}$$

Step 2: Minimax subsolution property "$\Longleftrightarrow$" monotonicity property

$$\forall (t_0, x_0) : \forall t > t_0 \ : \forall x \in \mathcal{X}^{1,1}(t_0, x_0) : \boxed{u(t_0, x_0) \leq \int_{t_0}^{t} \ell(s, x'(s))\, ds + u(t, x).}$$

Step 3 (existence): v_0 satisfies DPP "$\Longleftrightarrow$" viability and monotonicity.
Step 4 (comparison): Follows from Steps 1, 2, and the terminal condition.

Structure of proof of Theorem (B)

Step 1: Dini subsolution property "$\Longleftrightarrow$" monotonicity property

$$\forall (t_0, x_0) : \ \forall t > t_0 \ : \forall a \in \mathbb{R} : \boxed{u(t_0, x_0) \leq \mathbb{E}_{t_0, x_0, n} \left[\int_{t_0}^{t} \ell(s, a)\, ds + u(t, X + A^a - A^a(t_0)) \right]}.$$

Step 2 (existence): Use BSDEs and Feynman-Kac to show that v_n satisfies one half of DPP "$\Longleftrightarrow$" monotonicity.
Step 3 (uniqueness, i.e., v_n is maximal Dini subsolution): Follows from Step 1, monotonicity, and approximation of bounded controls through elementary controls.

Outline

Conclusion: A non-Markovian vanishing viscosity result

Let $h \in C_b(\Omega)$ and $\ell \in C([0,T] \times \mathbb{R}, \mathbb{R})$.
Then the maximal Dini subsolutions $u_n : [0,T] \times \Omega \to \mathbb{R}$ of

$$\partial_t u_n + \frac{1}{2n} \partial_{xx} u_n + \inf_{a \in \mathbb{R}} [\ell(t,a) + a \cdot \partial_x u_n] = 0 \quad \text{on } (0,T) \times \Omega,$$
$$u_n(T,x) = h(x) \quad \text{on } \Omega,$$

converge uniformly on compacta to the maximal Dini subsolution $u : [0,T] \times \Omega \to \mathbb{R}$ of

$$\partial_t u + \inf_{a \in \mathbb{R}} [\ell(t,a) + a \cdot \partial_x u] = 0 \quad \text{on } (0,T) \times \Omega,$$
$$u(T,x) = h(x) \quad \text{on } \Omega.$$