Large deviations and a non-Markovian vanishing viscosity result

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Background and motivation

Setting:

- $\Omega = C([0,T],\mathbb{R}^d)$ with $d=1$ (for simplicity)
- P probability on $\mathcal{B}(\Omega)$
- $W=(W_t)_{0\le t\le T}$ standard P-Wiener process on Ω
- $\mathbb{F}=\mathbb{F}^W$

Schilder's theorem in Laplace form

(see, e.g., Dupuis–Ellis (97), Boué–Dupuis (AOP, 98)). For every $h\in C_b(\Omega)$,

$$\lim_{n\to\infty} -\frac{1}{n}\log E\left[e^{-nh(W/\sqrt{n})}\right] = \inf_{x\in AC([0,T])\text{ with } x(0)=0}\left\{\int_0^T \left|x'(t)\right|^2 dt + h(x)\right\}.$$

Non-exponential Schilder theorem in Laplace form

(Backhoff-Veraguas–Lacker–Tangpi (AAP, 2020)). For every $h\in C_b(\Omega)$,

$$\lim_{n\to\infty} -\rho^{-\ell_n}[-h(W/\sqrt{n})] = \inf_{x\in AC([0,T])\text{ with } x(0)=0}\left\{\int_0^T \ell(t,x'(t))\,dt + h(x)\right\}.$$

Here, $\ell_n(t,a)=\ell(t,a/\sqrt{n})$. (The operator $\rho^{-\ell_n}$ will be defined later.)

Applications of “large deviations" results and open problem

(Backhoff-Veraguas–Lacker–Tangpi (AAP, 2020))

Assume ℓ is coercive, i.e., $\boxed{\lim_{|a|\to\infty} \frac{\ell(t,a)}{|a|} = \infty.}$

- **General (non-Markovian) case:**
  “Solutions" of BSDEs converge to value function of a calculus-of-variations problem with path-dependent terminal cost $h : \Omega \to \mathbb{R}$.
- **Markovian case:** Say $h(\omega) = \tilde{h}(\omega_T)$.
  Vanishing viscosity result:
  “Solutions" u_n of (possibly super-quadratic) HJB equation

$$\partial_t u_n + \frac{1}{2n}\partial_{xx} u_n + \inf_{a\in\mathbb{R}} [\ell(t,a) + a \cdot \partial_x u_n] = 0 \quad \text{on } (0,T) \times \mathbb{R}$$

$$u_n(T,x) = \tilde{h}(x) \quad \text{on } \mathbb{R},$$

converge pointwise to value function u of a calculus-of-variations problem and “formally"

$$\partial_t u + \inf_{a\in\mathbb{R}} [\ell(t,a) + a \cdot \partial_x u] = 0 \quad \text{on } (0,T) \times \mathbb{R}$$

$$u(T,x) = \tilde{h}(x) \quad \text{on } \mathbb{R}.$$

Open problem: Is there a non-Markovian vanishing viscosity result in terms of (path-dependent) PDEs on $[0,T] \times \Omega$? (Recall that $\Omega = C([0,T])$.)

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A simple BSDE

Fix (bounded) $\mathcal{F}_T$-measurable random variable $h : \Omega \to \mathbb{R}$, i.e., $h = h(W)$.

- 1st try: Find a process $Y = (Y_t)_{0 \le t \le T}$ such that

$$dY_t = 0 \cdot dt \quad \text{on } [0, T], \qquad Y_T = h.$$

  Then $Y_t = h$, $t \in [0, T]$. Problem: Y is not $\mathbb{F}$-adapted.

- 2nd try: Possibly more useful solution would be $Y_t := E[h|\mathcal{F}_t]$. In this case,

$$\boxed{dY_t = Z_t \, dW_t} \quad \text{on } [0, T], \qquad Y_T = h, \tag{1}$$

  for some process Z.
  Solution of BSDE (1) is a pair (Y, Z) of $\mathbb{F}$-progressive processes.
  Interpretation from financial math point of view:
  - W “price" process of underlying asset
  - h contingent claim/ derivative
  - Y price process of contingent claim
  - Z replicating/ hedging strategy

Feynman-Kac

Consider the BSDE

$$dY_t = Z_t\,dW_t \quad \text{on } [0,T], \qquad Y_T = \tilde{h}(W_T). \tag{2}$$

Assume that there is a $u \in C^{1,2}([0,T] \times \mathbb{R})$ such that

$$Y_t = u(t, W_t) \qquad \text{for all } t \in [0,T].$$

Then, from

$$\begin{aligned} du(t,W_t) &= \left[\partial_t + \frac{1}{2}\partial_{xx}\right] u(t,W_t)\,dt + \partial_x u(t,W_t)\,dW_t, \\ dY_t &= 0\,dt + Z_t\,dW_t, \end{aligned}$$

one can deduce that $Z_t = \partial_x u(t, W_t)$ and that u solves the heat equation

$$\begin{aligned} \left[\partial_t + \frac{1}{2}\partial_{xx}\right] u(t,x) = 0 \qquad &\text{on } (0,T) \times \mathbb{R}, \\ u(T,x) = \tilde{h}(x) \quad &\text{on } \mathbb{R}. \end{aligned} \tag{3}$$

Vice versa: If u solves (3), then $(Y,Z) := (u(t,W_t), \partial_x u(t,W_t))_{0\le t\le T}$ solves (2).

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Theorem (Pardoux–Peng (90), see also Zhang (2017))

Let $h \in L^2(\mathcal{F}_T, P)$ and let $f = f(t, y, z) : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be Borel with $f(\cdot, 0, 0) \in L^1$ and assume that there is an $L >$ such that, for all $t \in [0, T]$, y, $\tilde{y}$, z, $\tilde{z} \in \mathbb{R}$,

$$|f(t, y, z) - f(t, \tilde{y}, \tilde{z})| \le L\,(|y - \tilde{y}| + |z - \tilde{z}|)\,.$$

Then the BSDE

$$dY_t = -f(t, Y_t, Z_t)\,dt + Z_t\,dW_t \quad \text{on } [0, T], \qquad Y_T = h, \tag{4}$$

has a unique solution $(Y, Z) \in \mathbb{L}^2(\mathbb{F}) \times \mathbb{L}^2(\mathbb{F})$.

Theorem (Peng (91), see also Ma–Yong (99))

Let $u \in C^{1,2}([0, T] \times \mathbb{R})$ solve

$$\begin{aligned} \left[\partial_t + \frac{1}{2}\partial_{xx}\right] u(t, x) + f(t, u(t, x), \partial_x u(t, x)) = 0 \qquad &\text{on } (0, T) \times \mathbb{R}, \\ u(T, x) = \tilde{h}(x) \quad &\text{on } \mathbb{R}. \end{aligned} \tag{5}$$

Then $(Y, Z) := (u(t, W_t), \partial_x u(t, W_t)_{0 \le t \le T}$ solves (4) with $h := \tilde{h}(W_T)$.

Stochastic optimal control

Let $\ell : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\tilde{h} : \mathbb{R} \to \mathbb{R}$ be bounded and Borel.
Fix $m > 0$. Given $(t, x) \in [0, T) \times \mathbb{R}$, minimize

$$J(t, x, a(\cdot)) := E\left[\int_t^T \ell(s, a(s))\, ds + \tilde{h}\left(X_T^{t,x,a(\cdot)}\right)\right]$$

over all $a(\cdot) : [0, T] \times \Omega \to [-m, m]$ that are $\mathbb{F}$-progressive subject to

$$dX_s^{t,x,a(\cdot)} = a(s)\, ds + dW_s \quad \text{on } [t, T], \quad X_t^{t,x} = x.$$

The value function $V : [0, T] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$V(t, x) := \inf_{a(\cdot)} J(t, x, a(\cdot)).$$

BSDE connection: Assume that $V \in C^{1,2}$. Then V solves HJB equation

$$\left[\partial_t + \frac{1}{2}\partial_{xx}\right] V(t, x) + \inf_{|a|\leq m} [\ell(t, a) + a \cdot \partial_x V(t, x)] = 0, \qquad \text{on } (0, T) \times \mathbb{R},$$

$$V(T, x) = \tilde{h}(x) \quad \text{on } \mathbb{R}.$$

Moreover, by nonlinear Feynman-Kac, $(Y, Z) = (V(t, W_t), \partial_x V(t, W_t))_{0\leq t\leq T}$ solves

$$dY_t = -\inf_{|a|\leq m} [\ell(t, a) + a \cdot Z_t]\, dt + Z_t\, dW_t \text{ on } [0, T],\ Y_T = \tilde{h}(W_T).$$

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Quadratic and super-quadratic BSDEs

Consider BSDE

$$dY_t = -f(t, Y_t, Z_t)\,dt + Z_t\,dW_t \quad \text{on } [0, T], \qquad Y_T = h. \tag{6}$$

Quadratic BSDE: $|f(t, y, z)| \leq C(1 + |z|^2)$

Well-posedness of solutions (Kobylanski, AOP, 2000)

Super-quadratic BSDE: $\lim_{|z|\to\infty} \dfrac{|f(t, y, z)|}{|z|^2} = \infty$

- Ill-posedness of solutions (Delbaen–Hu–Bao, PTRF, 2011)
- Well-posedness of minimal supersolutions (Drapeau–Heyne–Kupper, AOP, 2013)

$(Y, Z) \in \mathbb{L}^2(\mathbb{F}) \times \mathbb{L}^2(\mathbb{F})$ supersolution of (6) if Y is càdlàg, $\int Z\,dW$ a supermartingale,

- $Y_s \geq Y_t + \int_t^s [-f(r, Y_r, Z_r)]\,dr + \int_t^s Z_r\,dW_r$, and
- $Y_T \geq h$.

(Y, Z) minimal supersolution of (6) if

- $(\tilde{Y}, \tilde{Z})$ supersolution of (6) $\quad \Longrightarrow \quad Y \leq \tilde{Y}$.

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Subject of study

PPDE

$$\boxed{\begin{aligned} -\partial_t u - F(t,\omega,u,\partial_\omega u,\partial^2_{\omega\omega} u) &= 0 \qquad \text{on } [0,T)\times\Omega, \\ u(T,\omega) &= h(\omega) \quad \text{on } \Omega. \end{aligned}}$$

- Ω path space:

$$C([0,T],\mathbb{R}^d),\ D([0,T],\mathbb{R}^d),\ C([0,T],H),\ldots$$

- $F = F(t,\omega,y,z,\gamma)$ Hamiltonian
- $u = u(t,\omega) : [0,T]\times\Omega \to \mathbb{R}$ *non-anticipating*, i.e.,

$$u(t,\omega) = u(t,\omega_{\cdot\wedge t}) \qquad \text{“}=\text{"} \qquad u(t,\{\omega_s\}_{s\le t})$$

PPDE

$$\boxed{\begin{aligned} -\partial_t u - F(t,\omega,u,\partial_\omega u,\partial^2_{\omega\omega} u) = 0 \qquad & \text{on } [0,T) \times \Omega, \\ u(T,\omega) = h(\omega) \quad & \text{on } \Omega. \end{aligned}}$$

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WARNING: Don't confuse PPDE with stochastic (or rough) PDE

$$u(t,x,\omega) = u_0 + \int_0^t F[\cdots]\,ds + \int_0^t G[\cdots]\,d\omega_s$$

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Motivation and applications I

Intrinsic mathematical:

- Very few results available for PDEs on those (infinite-dimensional) path spaces
- In contrast: Large literature for PDEs on Hilbert space

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Optimal control of delay equations and/or with path-dependent terminal cost:

$$\text{Minimize } J(t,\omega,a(\cdot)) := \sup_{s\leq t} \left| x^{t,\omega,a(\cdot)}(s) \right|$$

subject to $x = x^{t,\omega,a(\cdot)}$ solving

$$\begin{aligned} x'(s) &= f(x(s-1),a(s)) \text{ on } (t,T), \\ x &= \omega \text{ on } [0,t]. \end{aligned}$$

Formally, value function $v(t,\omega) := \inf_{a(\cdot)} J(t,\omega,a(\cdot))$ solves

$$\begin{aligned} -\partial_t v - \inf_a \left[f(\omega(s-1))\, \partial_\omega v \right] &= 0, \\ v(T,\omega) &= \sup_{s\leq T} |\omega(s)|\,. \end{aligned}$$

Pricing of path-dependent options in mathematical finance:

$$u(t,\omega) := \mathbb{E}_{t,\omega}[\xi],$$

where ξ is a functional of Brownian motion B, e.g.,

$$\xi(\tilde{\omega}) = \max\left\{\sup_{s\le t} |\omega(s)|, \sup_{t<s\le T} |B(s,\tilde{\omega})|\right\} \text{ for } \mathbb{P}_{t,\omega}\text{-a.e. } \tilde{\omega}.$$

Formally, u solves

$$-\left[\partial_t u + \frac{1}{2}\partial^2_{\omega\omega} u\right] = 0,$$
$$u(T,\cdot) = \xi.$$

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   - Motivation
   - Path derivatives
   - Viscosity solutions

Path derivatives: 1st order case

$$\boxed{\Omega = C([0,T],\mathbb{R}) \text{ from now on.}}$$

Kim (since 1980s), Lukoyanov (since late 1990s).

Definition

$u : [0,T]\times\Omega \to \mathbb{R}$ is in $\mathcal{C}^{1,1}$ if u is non-anticipating, continuous, and

$\exists$ functions $\partial_t u$, $\partial_\omega u : [0,T]\times\Omega \to \mathbb{R}$ non-anticipating and continuous:
$\forall \omega \in AC([0,T]) : \forall s \leq t :$

$$u(t,\omega) - u(s,\omega) = \int_s^t \left[\partial_t u(r,\omega) + \partial_\omega u(r,\omega)\,\omega'(r)\right]\, dr.$$

Example: If f and g are smooth and

$$u(t,\omega) = f(t,\omega(t)) + \int_0^t g(s,\omega(s))\, ds$$

then

$$\partial_t u(t,\omega) = \frac{\partial}{\partial_t} f(t,\omega(t)) + g(t,\omega(t)), \qquad \partial_\omega u(t,\omega) = \frac{\partial}{\partial x} f(t,\omega(t)).$$

Path derivatives: 2nd order case

Dupire (2009) and, in 2010s, Cont, Fournié, Ekren, K., Touzi, Zhang, etc.

Definition

$u : [0, T] \times \Omega \to \mathbb{R}$ is in $\mathcal{C}^{1,2}$ if u is non-anticipating, continuous, and

$\exists$ functions $\partial_t u$, $\partial_\omega u$, $\partial^2_{\omega\omega} u : [0, T] \times \Omega \to \mathbb{R}$ non-anticipating and continuous: $\forall$ Itô semimartingale X, i.e., $dX_t = b_t\, dt + \sigma_t\, dW_t$: $\forall s \le t$:

$$u(t, X) - u(s, X) = \int_s^t \left[\partial_t u(r, X) + \partial_\omega u(r, X)\, b_r + \frac{1}{2}\partial^2_{\omega\omega} u(r, X)\, \sigma_r^2\right] dr + \int_s^t [\partial_\omega (r, X)\, \sigma_r]\, dW_r$$

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Crandall–Lions (TAMS, 1983).

Definition

$u : [0, T] \times \mathbb{R} \to \mathbb{R}$ continuous is a viscosity subsolution of

$$-\partial_t u(t,x) - \tilde{F}(t, \partial_x u(t,x)) = 0 \text{ at } (t,x),$$

if, for each test function $\varphi \in \underline{\mathcal{A}}u(t,x)$,

$$-\partial_t \varphi(t,x) - \tilde{F}(t, \partial_x \varphi(t,x)) \leq 0.$$

The test function space is defined by

$$\underline{\mathcal{A}}u(t,x) := \Big\{ \varphi \in C^{1,2}([0,T] \times \mathbb{R}) : \\ 0 = (\varphi - u)(t,x) = \inf_{(s,y)} (\varphi - u)(s,y) \Big\}$$

Lukoyanov (2007) and Bayraktar–K. (JFA, 2018).
Fix $L \geq 0$. Assume $|F(t,z) - F(t,\tilde{z})| \leq L\,|z - \tilde{z}|$.

Definition

$u : [0,T] \times \Omega \to \mathbb{R}$ non-anticipating, continuous is a viscosity L-subsolution of

$$-\partial_t u(t,\omega) - F(t, \partial_\omega u(t,\omega)) = 0 \text{ at } (t,\omega),$$

if, for each test function $\varphi \in \underline{\mathcal{A}}^L u(t,\omega)$,

$$-\partial_t \varphi(t,\omega) - F(t, \partial_\omega \varphi(t,\omega)) \leq 0.$$

The test function space is defined by

$$\underline{\mathcal{A}}^L u(t,\omega) := \Big\{ \varphi \in \mathcal{C}^{1,2} : 0 = (\varphi - u)(t,\omega) = \inf_{(s,X) \in [t,T] \times \mathcal{X}^L(t,\omega)} (\varphi - u)(s,X) \Big\},$$

where the (compact) set $\mathcal{X}^L(t,\omega)$ is defined by

$$\mathcal{X}^L(t,\omega) := \{ X \in \Omega : X|_{[0,t]} = \omega|_{[0,t]} \text{ and } X|_{[t,T]} \in AC([t,T]) \text{ with } \sup_{t \leq s \leq t} \left| X'(s) \right| \leq L \}.$$

Ekren–K.–Touzi–Zhang (AOP, 2014), Ekren–Touzi–Zhang (AOP, 2016ab), Ren–Rosestolato (SIMA, 2020), etc.
Fix $L \geq 0$. Assume $|F(t, z, \gamma) - F(t, \tilde{z}, \tilde{\gamma})| \leq L\,(|z - \tilde{z}| + |\gamma - \tilde{\gamma}|)$.

Definition

$u : [0, T] \times \Omega \to \mathbb{R}$ non-anticipating, continuous is a viscosity $\mathcal{P}^L$-subsolution of

$$-\partial_t u(t, \omega) - F(t, \partial_\omega u(t\omega), \partial^2_{\omega\omega} u(t, \omega)) = 0 \text{ at } (t, \omega),$$

if, for each test function $\varphi \in \underline{\mathcal{A}}^{\mathcal{P}^L} u(t, \omega)$,

$$-\partial_t \varphi(t, \omega) - F(t, \partial_\omega u(t\omega), \partial^2_{\omega\omega} \varphi(t, \omega)) \leq 0.$$

The test function space is defined by

$$\underline{\mathcal{A}}^{\mathcal{P}^L} u(t, \omega) := \Big\{ \varphi \in \mathcal{C}^{1,2} : \\ 0 = (\varphi - u)(t, \omega) = \inf_{\tau \in \mathcal{T}^t} \inf_{\mathbb{P} \in \mathcal{P}^L(t,\omega)} \mathbb{E}^{\mathbb{P}} [(\varphi - u)(\tau, X)] \Big\}.$$

Viscosity solutions: 2nd order PPDEs: Part 2

The test function space is defined by

$$\underline{\mathcal{A}}^{\mathcal{P}^L} u(t,\omega) := \Big\{ \varphi \in \mathcal{C}^{1,2} : \\ 0 = (\varphi - u)(t,\omega) = \inf_{\tau \in \mathcal{T}^t} \inf_{\mathbb{P} \in \mathcal{P}^L(t,\omega)} \mathbb{E}^{\mathbb{P}} \left[(\varphi - u)(\tau, X) \right] \Big\},$$

where

- $\mathcal{T}^t = \{$all $[t, T]$-valued stopping times$\}$,
- X canonical process on Ω, i.e., $X_t(\tilde{\omega}) = \tilde{\omega}(t)$,
- and the $\mathcal{P}^L(t,\omega)$ is defined by

$$\mathcal{P}^L(t,\omega) := \Big\{ \text{all probability measures } \mathbb{P} \text{ on } \Omega \text{ such that, } \mathbb{P}\text{-a.s.,} \\ X|_{[0,t]} = \omega|_{[0,t]} \text{ and} \\ X|_{[t,T]} \text{ is an Itô-semimartingale of the form} \\ dX_s = b_s\, ds + \sigma_s\, dW_s \\ \text{with } \|b\|_\infty \le L \text{ and } \|\sigma\|_\infty \le \sqrt{2L}. \Big\}$$