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# Weak convergence of the scaled median of independent Brownian motions

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**Abstract** We consider the median of $n$ independent Brownian motions, denoted by $M_n(t)$, and show that $\sqrt{n}\,M_n$ converges weakly to a centered Gaussian process. The chief difficulty is establishing tightness, which is proved through direct estimates on the increments of the median process. An explicit formula is given for the covariance function of the limit process. The limit process is also shown to be Hölder continuous with exponent $\gamma$ for all $\gamma < 1/4$.

**Keywords** Brownian motion · Median · Weak convergence · Fractional Brownian motion · Tightness

**Mathematics Subject Classification (2000)** 60F17 · 60G15 · 60J65 · 60K35

## 1 Introduction

Consider a dye diffusing in a homogeneous medium. When we view this phenomenon from a macroscopic perspective, what we see is a deterministic evolution of the density of the dye, governed by a partial differential equation. It is well understood that the solution of this equation can be represented probabilistically in terms of Brownian motion. The reason, of course, that Brownian motion enters into this situation is that, heuristically, we can imagine that each dye particle is performing such a random motion. In reality, however, a more accurate description of the particles is that they are following piece-wise linear trajectories and interacting through collisions.

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In 1968, Spitzer [5] provided a rigorous connection between a certain colliding particle model and the Brownian motion heuristics. In Spitzer's model, we begin with countably many particles distributed along the real line according to a Poisson distribution. At time $t = 0$, the particles begin moving with random velocities. These velocities are i.i.d., integrable, mean zero random variables. During their motion, the particles interact through elastic collisions. That is, whenever two particles meet, they exchange velocities (or, equivalently, they exchange trajectories). The particle which is closest to the origin at time $t = 0$ is called the "tagged" particle and we denote its position at time $t$ by $X(t)$. Spitzer showed that the law on $C[0,\infty)$ induced by the process $t \mapsto c^{-1/2}X(ct)$ converges weakly as $c \to \infty$ to the law of Brownian motion.

Spitzer's work was preceded by that of Harris [2] who showed that if the underlying motion of the particles is Brownian, instead of linear, then $c^{-1/4}X(ct)$ converges to fractional Brownian motion with Hurst parameter $H = 1/4$. These results were further generalized by Dürr et al. [1] in 1985. They showed, among other things, that if the individual particles perform fractional Brownian motion with Hurst parameter $H$, then $c^{-H/2}X(ct)$ converges to fractional Brownian motion with Hurst parameter $H/2$.

One thing to note in these more general models is the definition of an "elastic collision." When the particles perform Brownian motion, for example, the collisions are not isolated and it is not entirely clear how to exchange their trajectories at each point of intersection. In these situations, we generate the collision process by simply relabelling the particles at each time $t$ in a way that preserves their initial ordering. For instance, if there are only finitely particles, as there will be in our model, the location of the tagged particle is simply an order statistic of the locations of all of the particles. (In our model, it will be the median.)

In the work of Spitzer, Harris, and Dürr et al. the chief difficulty in proving convergence is establishing tightness. And in each of these models, the Poisson distribution of the initial particle configuration provides for tractable computations and is a central feature of the proofs. In this article, we will consider a model similar to Harris's, but without the initial Poisson distribution. Namely, we consider a sequence $\{B_j\}$ of independent Brownian motions starting at the origin. We let $M_n$ denote the median of the first $n$ of these, and study the scaled process $X_n = \sqrt{n}\,M_n$. As with the other models, our chief difficulty will be to prove tightness. We will prove this, however, by making direct estimates on the path of the "tagged" particle, without relying on any special features of the initial particle distribution. In the end, we will discover a limit process which behaves locally like fractional Brownian motion with Hurst parameter $H = 1/4$. This fact, formally stated in Theorem 2.1, lends support to the evident notion that Harris's initial Poisson distribution is, to a certain degree, just a technical convenience, and does not play a significant role in determining the local behavior of the limit.

Before proceeding with the formal analysis of our model, let us preview some of the techniques in the proof. The first key ingredient in the proof will be given by Theorem 5.1, which establishes a formula for the conditional law of the median in terms of probabilities associated with a certain random walk. The second ingredient will be Lemma 6.4, which gives estimates for this random walk in terms of its parameters. And the third ingredient will be Lemma 7.1 (and its modification, Lemma 8.1) which estimates those parameters in terms of the motion of the individual particles.

Since it would be natural to conjecture that the results of Spitzer and Dürr et al. would also hold in more general models, it is important to try to understand how these techniques might apply in a broader context. For example, we could try to generalize the results of Dürr et al. by replacing the Brownian motions in our model with fractional Brownian motions. Or we could replace them with reflected processes if we wanted to consider particles in a "box," reflecting off the walls of the box as well as each other. Such a model (in which the particles' paths are piece-wise linear) was studied by Tupper in [8], although in that paper, a seemingly ad hoc condition is imposed in order to prove tightness. (See the discussion after Theorem 2.3 in [8].) Other ways to generalize the model include giving our particles some nontrivial initial distribution, instead of starting them at the origin, or possibly considering a quantile (or even a family of quantiles) other than the median.

In any of these generalized models, the first and second ingredients outlined above would likely carry over with at most minor modifications. It is the third ingredient that would not transfer so easily. The estimates in Lemma 7.1 rely heavily on the fact that the individual particles are performing Brownian motion. Conceivably, analogous estimates could be worked out on a case-by-case basis for each different model under consideration. But the work of Harris [2], Spitzer [5], and Dürr et al. [1] suggests a deeper connection between the motion of the individual particles and the limit process. It is my belief that this connection would make itself known through these estimates.

But whether estimates can found in some general form or must be developed for each model individually, it is my hope that the techniques developed here can be used to extend the current family of results to a much broader range of colliding particle models.

## 2 The model and main result

In our model, we will consider a sequence of independent, standard, one-dimensional Brownian motions, $\{B_j(t)\}_{j=1}^{\infty}$. Let $M_n(t)$ denote the median of the first $n$ Brownian motions. To be precise, define the median function $\mathcal{M}_n : \mathbb{R}^n \to \mathbb{R}$ as follows: if $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\tau$ is a permutation of $\{1, \ldots, n\}$ such that $x_{\tau(1)} \leq x_{\tau(2)} \leq \cdots \leq x_{\tau(n)}$, then $\mathcal{M}_n(x_1, \ldots, x_n) = x_{\tau(k)}$, where $k = \lfloor (n+1)/2 \rfloor$ and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$. We then define the (continuous) median process $M_n(t) = \mathcal{M}_n(B_1(t), \ldots, B_n(t))$.

In terms of colliding particles, what we have here is a sequence of particle systems. In the $n$-th system there are $n$ particles performing Brownian motion. If these particles interact through elastic collisions, then their trajectories are given by the order statistics of $B_1(t), \ldots, B_n(t)$. We will investigate the behavior of the center particle's trajectory, $M_n(t)$.

In order to get a non-degenerate limit, we must consider the scaled median process $X_n(t) = \sqrt{n}\, M_n(t)$. The random variables $X_n = \{X_n(t) : 0 \le t < \infty\}$ take values in the space $C[0,\infty)$, which we endow with the topology of uniform convergence on compact sets. It will be shown that these processes converge weakly, by which we mean that they converge in law as $C[0,\infty)$-valued random variables.

**Theorem 2.1** *There exists a continuous process $X = \{X(t) : 0 \le t < \infty\}$ such that $X_{2n+1}$ converges weakly to $X$ as $n \to \infty$. Moreover, $X$ is a centered Gaussian process, which is locally Hölder continuous with exponent $\gamma$ for every $\gamma \in (0, 1/4)$, and has covariance function*

$$E[X(s)X(t)] = \sqrt{st}\, \sin^{-1}\left(\frac{s \wedge t}{\sqrt{st}}\right), \tag{2.1}$$

*where $\sin^{-1}(\cdot)$ takes values in $[-\pi/2, \pi/2]$.*

It can be shown by (2.1) that, for $t - s$ small, $E|X(t) - X(s)|^2 \approx \sqrt{t-s}$. In other words, the limit process has the same local fluctuations as fractional Brownian motion with Hurst parameter $H = 1/4$.

The chief difficulty in proving Theorem 2.1 will be to establish the tightness of the processes $X_{2n+1}$. Before dealing with this issue, let us first establish the convergence of the finite-dimensional distributions and the existence of the limit process. To begin, we will need the following result, which is a special case of Theorems 7.1.1 and 7.1.2 in [4].

**Theorem 2.2** *Let $\{\xi^{(n)}\}_{n=1}^\infty$ be an i.i.d. sequence of random vectors in $\mathbb{R}^d$ and define the component-wise median of $\xi^{(1)}, \ldots, \xi^{(n)}$ to be the vector $M^{(n)}$ with components $M_j^{(n)} = \mathcal{M}_n(\xi_j^{(1)}, \xi_j^{(2)}, \ldots, \xi_j^{(n)})$. Let $F_j(x) = P(\xi_j^{(1)} \le x)$, $G_{ij}(x,y) = P(\xi_i^{(1)} \le x, \xi_j^{(1)} \le y)$, and $\rho_{ij} = G_{ij}(0,0) - 1/4$. If*

(i) *$F_j(0) = 1/2$ and $F_j'(0) > 0$ for all $j$, and*
(ii) *$G_{ij}$ is continuous at $(0,0)$ for all $i$ and $j$,*

*then $\sqrt{n}\, M^{(n)}$ converges in law to a jointly Gaussian random vector $N$ satisfying*

$$EN_iN_j = \frac{\rho_{ij}}{F_i'(0)F_j'(0)}$$

*and $EN_i = 0$.*

For our purposes, we will need the following.

**Corollary 2.3** *If $\{\xi^{(n)}\}_{n=1}^\infty$ is an i.i.d. sequence of jointly Gaussian random vectors in $\mathbb{R}^d$ with mean zero and covariance matrix $\sigma$, then $\sqrt{n}\, M^{(n)}$ converges in law* *to a jointly Gaussian random vector Z with mean zero and covariance matrix* $\tau$*, where*

$$\tau_{ij} = EZ_iZ_j = \sqrt{\sigma_{ii}\sigma_{jj}}\,\sin^{-1}\left(\frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}\right)$$

*and* $\sin^{-1}(\cdot)$ *takes values in* $[-\pi/2, \pi/2]$.

*Proof* This follows easily from Theorem 2.2 and the well-known fact that if $X$ and $Y$ are jointly Gaussian with mean zero, then

$$P(X \le 0, Y \le 0) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\left(\frac{EXY}{\sqrt{EX^2 \cdot EY^2}}\right),$$

where $\sin^{-1}(\cdot)$ takes values in $[-\pi/2, \pi/2]$. □

**Theorem 2.4** *There exists a centered Gaussian process* $X = \{X(t) : 0 \le t < \infty\}$ *with covariance function* (2.1) *and which is locally Hölder continuous with exponent* $\gamma$ *for every* $\gamma \in (0, 1/4)$.

*Proof* Let $T$ be the set of finite sequences $\mathbf{t} = (t_1, \ldots, t_n)$ of distinct, nonnegative numbers, where the length $n$ of these sequences ranges over the set of positive integers. For each $\mathbf{t}$ of length $n$, let $\mathbf{Z_t} = (Z_1, \ldots, Z_n)$ be a jointly Gaussian random vector with mean zero and covariance

$$EZ_iZ_j = \sqrt{t_it_j}\,\sin^{-1}\left(\frac{t_i \wedge t_j}{\sqrt{t_it_j}}\right).$$

(By Corollary 2.3, with $\xi^{(j)} = (B_j(t_1), \ldots, B_j(t_n))$, such a $\mathbf{Z_t}$ exists.) Define the measure $Q_\mathbf{t}$ on $\mathbb{R}^n$ by $Q_\mathbf{t}(A) = P(\mathbf{Z_t} \in A)$. The family of finite-dimensional distributions, $\{Q_\mathbf{t}\}_{\mathbf{t}\in T}$, is clearly consistent, so there exists a real-valued process $X = \{X(t) : 0 \le t < \infty\}$ that has the desired finite-dimensional distributions. It remains only to show that this process has a continuous modification, which is locally Hölder-continuous with exponent $\gamma$ for every $\gamma \in (0, 1/4)$.

By the Kolmogorov–Čentsov Theorem (Theorem 2.2.8 in [3]), if, for each $T > 0$,

$$E|X(t) - X(s)|^\alpha \le C_T|t - s|^{1+\beta}$$

for some positive constants $\alpha, \beta$, and $C_T$ (depending on $T$) and all $0 \le s < t \le T$, then $X$ has a continuous modification which is locally Hölder-continuous with exponent $\gamma$ for every $\gamma \in (0, \beta/\alpha)$. Hence, it will suffice for us to show that for every $\alpha > 4$ and every $T > 0$,

$$E|X(t) - X(s)|^\alpha \le C|t - s|^{\alpha/4}$$

for some $C > 0$ (depending only on $T$ and $\alpha$) and all $0 \le s < t \le T$.

First, observe that $X(t) - X(s)$ is normal with mean zero and variance

$$\begin{aligned} E|X(t) - X(s)|^2 &= EX(t)^2 + EX(s)^2 - 2EX(t)X(s) \\ &= \frac{\pi}{2}t + \frac{\pi}{2}s - 2\sqrt{st}\,\sin^{-1}\left(\sqrt{\frac{s}{t}}\right). \end{aligned}$$

An application of L'Hôpital's Rule shows that

$$\frac{\pi/2 - \sin^{-1} x}{\sqrt{1-x^2}} \to 1$$

as $x \to 1$. Hence, for some positive constant $C'$, we have $-\sin^{-1} x \le C'\sqrt{1-x^2} - \pi/2$ for all $0 \le x \le 1$. Now let $x = s/t$. Then

$$\begin{aligned} E|X(t) - X(s)|^2 &= t\left[\frac{\pi}{2} + \frac{\pi}{2}x - 2\sqrt{x}\,\sin^{-1}(\sqrt{x})\right] \\ &\le t\left[\frac{\pi}{2} + \frac{\pi}{2}x + 2\sqrt{x}\left(C'\sqrt{1-x} - \frac{\pi}{2}\right)\right] \\ &= t\left[\frac{\pi}{2}(1-\sqrt{x})^2 + 2C'\sqrt{x}\sqrt{1-x}\right]. \end{aligned}$$

Since $1 - \sqrt{x} \le \sqrt{1-x}$ for $0 \le x \le 1$,

$$\begin{aligned} E|X(t) - X(s)|^2 &\le t\left(\frac{\pi}{2}(1-x) + 2C'\sqrt{x}\sqrt{1-x}\right) \\ &\le t\left(\frac{\pi}{2}\sqrt{1-x} + 2C'\sqrt{1-x}\right) \\ &= \sqrt{t}\left(\frac{\pi}{2} + 2C'\right)\sqrt{t-s} \\ &\le C''|t-s|^{1/2}, \end{aligned}$$

where $C'' = \sqrt{T}(\pi/2 + 2C')$.

Now, for every $\alpha > 0$, there is a constant $K_\alpha$ such that if $N$ is normal with $EN = 0$, then $E|N|^\alpha = K_\alpha(EN^2)^{\alpha/2}$. Thus, for any $\alpha > 4$, $E|X(t) - X(s)|^\alpha \le C|t-s|^{\alpha/4}$, where $C = K_\alpha(C'')^{\alpha/2}$. □

**Theorem 2.5** *Let $X(t)$ be as in Theorem 2.4 and let $0 \le t_1 < \cdots < t_d$, $d \ge 1$, be arbitrary. Then $(X_n(t_1), \ldots, X_n(t_d))$ converges in law to $(X(t_1), \ldots, X(t_d))$ as $n \to \infty$.*

*Proof* This is an immediate consequence of Corollary 2.3. □

It now follows (see, for example, Theorem 2.4.15 in [3]) that Theorem 2.1 will be proved once we establish the following result.

**Theorem 2.6** *The sequence of processes $\{X_{2n+1}\}_{n=1}^{\infty}$ is tight.*

## 3 Conditions for tightness

A sufficient condition for tightness which will serve as the starting point for our analysis is the following.

**Theorem 3.1** *If $\{Z_n\}$ is a sequence of continuous stochastic processes such that*

(i) $\sup_n P(|Z_n(t)-Z_n(s)| \geq \varepsilon) \leq C_T \varepsilon^{-\alpha}|t-s|^{1+\beta}$ *whenever* $0 < \varepsilon < 1$, $T > 0$, *and* $0 \leq s, t \leq T$, *and*

(ii) $\sup_n E|Z_n(0)|^{\nu} < \infty$

*for some positive constants $\alpha$, $\beta$, $\nu$, and $C_T$ (depending on $T$), then $\{Z_n\}$ is tight.*

An alternative formulation of this theorem is one in which condition (i) is replaced by

$$\sup_{n\geq 1} E|Z_n(t) - Z_n(s)|^{\alpha} \leq C_T|t-s|^{1+\beta}. \tag{3.1}$$

For a proof of this alternative version, the reader is referred to Problem 2.4.11 in [3], which has a worked solution. An inspection of the proof shows that (3.1) is needed only to establish (via Chebyshev's inequality) condition (i).

Since the median process inherits the scaling property of Brownian motion, we will find it convenient to reformulate Theorem 3.1. Specifically, for any real number $c \geq 0$ and any $x \in \mathbb{R}^d$, we have $\mathcal{M}_n(cx) = c\mathcal{M}_n(x)$. Hence, the processes $X_n(c\,\cdot)$ and $\sqrt{c}\,X_n(\cdot)$ have the same law. For processes with this scaling property, we can modify Theorem 3.1 in the following way.

**Theorem 3.2** *Let $\{Z_n\}$ be a sequence of continuous stochastic processes. Suppose there exists $r > 0$ such that for every $c \geq 0$ and every $n$, the processes $Z_n(c\,\cdot)$ and $c^r Z_n(\cdot)$ have the same law. Suppose further that*

(i) $\sup_n P(|Z_n(1+\delta) - Z_n(1)| > \varepsilon) \leq C\varepsilon^{-\alpha}\delta^{1+\beta}$ *whenever* $0 < \varepsilon < 1$ *and* $0 < \delta < \delta_0$

*for some positive constants $\delta_0$, $C$, $\alpha$, and $\beta$. Define $\gamma = \min(\alpha r, \beta r, 1+\beta)$. If $\gamma > 1$ and*

(ii) $\sup_n E|Z_n(1)|^{\gamma/r} < \infty$,

*then the sequence $\{Z_n\}$ is tight.*

Theorem 3.2 follows directly from Theorem 3.1 (a complete proof can be found starting on p.36 of [7]). We will be applying it to the sequence $Z_n = X_{2n+1}$, in which case we have $r = 1/2$. We will find it quite straightforward to verify condition (ii). To verify condition (i), we will utilize the following lemma, which will be the central focus of the remainder of our analysis.

**Lemma 3.3** *There exists a constant $\delta_0 > 0$ and a family of constants $\{C_p\}_{p>2}$ such that for each $p > 2$,*

$$\sup_{n\geq 3} P\left(M_n(1+\delta) - M_n(1) > \frac{\varepsilon}{\sqrt{n}}\right) \leq C_p(\varepsilon^{-1}\delta^{1/6})^p \tag{3.2}$$

*whenever* $0 < \varepsilon < 1$ *and* $0 < \delta \leq \delta_0$.

It has already been remarked that the limit process $X$ behaves locally like a fractional Brownian motion with Hurst parameter $H = 1/4$. It seems reasonable, then, to conjecture that the right-hand side of (3.2) could be replaced by $C_p(\varepsilon^{-1}\delta^{1/4})^p$. Although this sharper bound was not obtained, the choice of 1/6 as the exponent in (3.2) appears to be arbitrary. Presumably, with minor modifications to the proofs presented here, the right-hand side of (3.2) could be replaced by $C_p(\varepsilon^{-1}\delta^{\nu})^p$ for any fixed $\nu < 1/4$.

*Proof of Theorem 2.6, given Lemma 3.3* We apply Theorem 3.2 to $Z_n = X_{2n+1}$ with $r = 1/2$. Choose any $p > 18$, let $\alpha = p$, and let $\beta = (p-6)/6$. Note that, in this case, $\gamma = \beta/2 > 1$.

To verify condition (i), let $\delta_0$ be as in Lemma 3.3. Since $X_{2n+1}(\cdot)$ and $-X_{2n+1}(\cdot)$ have the same law,

$$\begin{aligned}\sup_{n\geq 1} P(|X_{2n+1}(1+\delta) - X_{2n+1}(1)| > \varepsilon) &= 2\sup_{n\geq 1} P(X_{2n+1}(1+\delta) - X_{2n+1}(1) > \varepsilon)\\ &\leq 2C_p(\varepsilon^{-1}\delta^{1/6})^p\\ &= 2C_p\varepsilon^{-\alpha}\delta^{1+\beta}\end{aligned}$$

whenever $0 < \varepsilon < 1$ and $0 < \delta < \delta_0$.

To verify condition (ii), we will show that for any $q > 0$,

$$\sup_{n\geq 1} E|X_{2n+1}(1)|^q < \infty.$$

To see this, observe that for $n$ odd,

$$\begin{aligned}E|X_n(1)|^q &= \int_0^\infty qy^{q-1}P(|X_n(1)| > y)\,\mathrm{d}y\\ &= 2\int_0^\infty qy^{q-1}P(X_n(1) < -y)\,\mathrm{d}y.\end{aligned}$$

It will therefore suffice to show that for any $\kappa > 2$, there exists a finite constant $K$ such that

$$P(X_n(1) < -y) \leq Ky^{-\kappa} \tag{3.3}$$

for all $y > 0$ and all $n$.

To prove (3.3), we will consider two cases. First, assume $y \geq 2\sqrt{n}$. Note that by Theorem 1.3.2 in [4], $M_n(1)$ has density

$$f_n(x) = k\binom{n}{k}\frac{1}{2\pi}\,\Phi(x)^{k-1}\Phi(-x)^{n-k}\mathrm{e}^{-x^2/2} \tag{3.4}$$

where $k = \lfloor (n+1)/2 \rfloor$ and $\Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{x} \mathrm{e}^{-u^2/2}\,\mathrm{d}u$. Hence,

$$\begin{aligned}
P(X_n(1) < -y) &= P(M_n(1) < -y/\sqrt{n}) \\
&= \frac{n!}{(n-k)!(k-1)!} \int_{-\infty}^{-y/\sqrt{n}} \Phi(x)^{k-1}\Phi(-x)^{n-k}\Phi'(x)\,\mathrm{d}x \\
&\le \frac{n^k}{(k-1)!} \int_{-\infty}^{-y/\sqrt{n}} \Phi(x)^{k-1}\Phi'(x)\,\mathrm{d}x \\
&= \frac{n^k}{k!}\Phi(-y/\sqrt{n})^k.
\end{aligned}$$

By Stirling's formula, there exists a universal positive constant $C$ such that $k! \ge C^{-1}k^k\mathrm{e}^{-k}$. Also, writing $\int_x^\infty \mathrm{e}^{-u^2/2}\,\mathrm{d}u = \int_x^\infty u^{-1}\cdot u\mathrm{e}^{-u^2/2}\,\mathrm{d}u$ and integrating by parts, it follows that

$$\sqrt{2\pi}\,\Phi(-x) \le x^{-1}\mathrm{e}^{-x^2/2} \tag{3.5}$$

for all $x > 0$. Thus,

$$P(X_n(1) < -y) \le C\frac{n^k}{k^k\mathrm{e}^{-k}}\left(\frac{\sqrt{n}}{y}\,\mathrm{e}^{-y^2/2n}\right)^k.$$

Since $y \ge 2\sqrt{n}$ and $n/k \le 2$, we have

$$P(X_n(1) < -y) \le C\mathrm{e}^{k(1-y^2/2n)}.$$

Since $1 - y^2/2n < 0$ and $k \ge n/2$, we have

$$P(X_n(1) < -y) \le C\mathrm{e}^{n/2-y^2/4} \le C\mathrm{e}^{-y^2/8}.$$

Finally, given $\kappa > 2$, there exists $K$ such that $C\mathrm{e}^{-y^2/8} \le Ky^{-\kappa}$ for all $y > 0$, which verifies (3.3) in the case $y \ge 2\sqrt{n}$.

Now assume $y < 2\sqrt{n}$. In this case,

$$\begin{aligned}
P(X_n(1) < -y) &= P(M_n(1) < -y/\sqrt{n}) \\
&= P\Bigg(\sum_{j=1}^n 1_{\{B_j(1) < -y/\sqrt{n}\}} \ge \frac{n}{2}\Bigg) \\
&= P\Bigg(\sum_{j=1}^n \xi_j \ge n\left(\frac{1}{2} - \mu\right)\Bigg),
\end{aligned}$$

where $\mu = \Phi(-y/\sqrt{n})$ and $\xi_j = 1_{\{B_j(1) < -y/\sqrt{n}\}} - \mu$. By Burkholder's inequality (see, for example, Theorem 6.3.10 in [6]), there exists a constant $K'$, depending only on $\kappa$, such that

$$E\left|\sum_{j=1}^{n}\xi_j\right|^{\kappa} \leq K'E\left|\sum_{j=1}^{n}|\xi_j|^2\right|^{\kappa/2}.$$

Hence, since $\kappa > 2$, Jensen's inequality and the fact that $|\xi_j| \leq 1$ a.s. imply

$$E\left|\sum_{j=1}^{n}\xi_j\right|^{\kappa} \leq K'n^{\kappa/2}E\sum_{j=1}^{n}\frac{1}{n}|\xi_j|^k \leq K'n^{\kappa/2}.$$

Chebyshev's inequality now gives

$$P(X_n(1) < -y) \leq \frac{K'n^{\kappa/2}}{\left|n\left(\frac{1}{2}-\mu\right)\right|^{\kappa}} = K'\left|\sqrt{n}\left(\tfrac{1}{2}-\mu\right)\right|^{-\kappa}.$$

Since

$$\sqrt{n}\left(\tfrac{1}{2}-\mu\right) = \frac{\sqrt{n}}{\sqrt{2\pi}}\int_0^{y/\sqrt{n}} \mathrm{e}^{-u^2/2}\,\mathrm{d}u \geq \frac{y}{\sqrt{2\pi}}\,\mathrm{e}^{-y^2/2n} \geq \frac{y}{\sqrt{2\pi}}\,\mathrm{e}^{-2},$$

we have that $P(X_n(1) < -y) \leq Ky^{-\kappa}$, where $K = K'(\mathrm{e}^{-2}/\sqrt{2\pi})^{-\kappa}$. This verifies (3.3) when $y < 2\sqrt{n}$ and completes the proof. □

Our goal for the remainder of this article is to establish (3.2). Since each individual Brownian particle can be expected to move a distance of $\sqrt{\delta}$ between time $t=1$ and $t=1+\delta$, we will accomplish our goal by considering three different "jump regimes." They are: the large jump regime in which $\varepsilon/\sqrt{n}$ is much larger than $\sqrt{\delta}$, the small jump regime in which $\varepsilon/\sqrt{n}$ is much smaller than $\sqrt{\delta}$, and the medium jump regime in which these two quantities are comparable. In the first two regimes, we will establish the sharp bound mentioned in the remark following Lemma 3.3. The bound in the medium jump regime will be established by modifying the techniques used in the small jump regime. This modification will result in the weaker bound given in (3.2).

## 4 The large jump regime

The large jump regime is the easiest of the three to deal with. The probability that the median makes a large jump can be bounded above by the probability that at least one Brownian particle makes a large jump. Since the latter probability is exponentially small, the derivation of (3.2) is immediate.

**Lemma 4.1** *Fix $p > 0$ and $0 < \Delta < 1/2$. Suppose that $\varepsilon, \delta \in (0,1)$ and $n \in \mathbb{N}$ satisfy $\varepsilon/\sqrt{n} \geq \delta^{1/2-\Delta}$. Then*

$$P\left(M_n(1+\delta) - M_n(1) > \frac{\varepsilon}{\sqrt{n}}\right) \leq C(\varepsilon^{-1}\delta^{1/4})^p,$$

*where $C$ depends only on $p$ and $\Delta$.*

*Proof* Suppose that $B_j(1+\delta,\omega) - B_j(1,\omega) \leq \varepsilon/\sqrt{n}$ for all $j$. Then, for each $j$ such that $B_j(1,\omega) \leq M_n(1,\omega)$, we have $B_j(1+\delta,\omega) \leq M_n(1,\omega) + \varepsilon/\sqrt{n}$. Note that there are at least $k = \lfloor (n+1)/2 \rfloor$ such values of $j$. It follows that $M_n(1+\delta,\omega) \leq M_n(1,\omega) + \varepsilon/\sqrt{n}$. Therefore,

$$\bigcap_{j=1}^{n} \left\{B_j(1+\delta) - B_j(1) \leq \varepsilon/\sqrt{n}\right\} \subset \left\{M_n(1+\delta) - M_n(1) \leq \varepsilon/\sqrt{n}\right\},$$

which gives

$$\begin{aligned} P\left(M_n(1+\delta) - M_n(1) > \frac{\varepsilon}{\sqrt{n}}\right) &\leq P\left(\bigcup_{j=1}^{n} \{B_j(1+\delta) - B_j(1) > \varepsilon/\sqrt{n}\}\right) \\ &\leq n\Phi(-\varepsilon/\sqrt{n\delta}). \end{aligned}$$

For each $r > 0$, there exists $C_r$ such that $\Phi(-x) \leq C_r x^{-r}$ for all $x > 0$. Taking $r = (p/4+1)/\Delta$ gives

$$P\left(M_n(1+\delta) - M_n(1) > \frac{\varepsilon}{\sqrt{n}}\right) \leq nC_r\left(\frac{\varepsilon}{\sqrt{n\delta}}\right)^{-r} \leq nC_r(\delta^{-\Delta})^{-r} = C_r n\delta^{p/4+1}.$$

The proof is completed by observing that $n \leq \varepsilon^2\delta^{-1} \leq \varepsilon^{-p}\delta^{-1}$. □

This establishes the necessary bound for the large jump regime. The other regimes, as we will see, are considerably more difficult to deal with.

## 5 Conditioning the median

To establish (3.2) for the small and medium jump regimes, we will use conditioning. It may seem natural, at first, to condition on the locations of all the Brownian particles at time $t = 1$. It turns out, however, that this is, in some sense, too much information. Rather, we shall condition only on the location of the median particle at time $t = 1$.

Let us first give a heuristic description of this conditioning. Suppose that $M_n(1) = x$. This tells us three things. First, we have a single Brownian particle whose location is $x$. Second, we have roughly $n/2$ Brownian particles whose locations are less than $x$. Other than this condition on their locations, these particles are independent and identically distributed. We will refer to these particles as the "lower" particles. Third, we have roughly $n/2$ i.i.d. Brownian particles whose locations are greater than $x$. These will naturally be referred to as the "upper" particles.

Let us now fix $y > 0$ and consider the event $D = \{M_n(1+\delta) - M_n(1) > y\}$. This event will occur if and only if there are at least $n/2$ particles whose location at time $t = 1+\delta$ is greater than $x+y$. Particles that satisfy this condition will be said to have "jumped." Let $U(j)$ be the event that the $j$-th upper particle jumps, and let $L(j)$ be the event that the $j$-th lower particle does *not* jump. Then the total number of particles that jump is

$$\sum 1_{U(j)} + \left(\frac{n}{2} - \sum 1_{L(j)}\right).$$

The event $D$ will occur if and only if this sum is at least $n/2$, which occurs if and only if $\sum Y_j \geq 0$, where $Y_j = 1_{U(j)} - 1_{L(j)}$ are i.i.d. $\{-1,0,1\}$-valued random variables. Through conditioning, then, we are able to transform the event of interest into one involving an i.i.d. sum.

With these heuristics in place, let us establish the rigorous result. Define

$$\begin{aligned} p_1 = p_1(x,y,\delta) &= P(B(1+\delta) < x+y \mid B(1) < x) && (5.1)\\ p_2 = p_2(x,y,\delta) &= P(B(1+\delta) > x+y \mid B(1) > x) && (5.2)\\ &= p_1(-x,-y,\delta) \end{aligned}$$

and

$$q_j = 1 - p_j. \tag{5.3}$$

In the language of our heuristics, $p_1$ is the probability that a lower particle does not jump and $p_2$ is the probability that an upper particle does jump.

Now, for each fixed triple $(x,y,\delta)$, let $\{\xi_j^L\}_{j=1}^\infty$ and $\{\xi_j^U\}_{j=1}^\infty$ be sequences of i.i.d $\{0,1\}$-valued random variables with $P(\xi_j^L = 1) = p_1$ and $P(\xi_j^U = 1) = p_2$. Define $Y_j = \xi_j^U - \xi_j^L$. Observe that $\{Y_j\}_{j=1}^\infty$ is an i.i.d. sequence of $\{-1,0,1\}$-valued random variables and, for future reference, define

$$\begin{aligned} \tilde{p}_1 &= P(Y_j = -1) = p_1 q_2 && (5.4)\\ \tilde{p}_2 &= P(Y_j = 1) = p_2 q_1 && (5.5)\\ \tilde{\varepsilon} &= P(Y_j \neq 0) = \tilde{p}_1 + \tilde{p}_2 && (5.6)\\ \tilde{\mu} &= -EY_j = \tilde{p}_1 - \tilde{p}_2. && (5.7) \end{aligned}$$

Finally, let $S_k = \sum_{j=1}^k Y_j$ and $\varphi_k(x,y,\delta) = P(S_k \geq 0)$.

Our heuristics suggest that

$$P(M_n(1+\delta) - M_n(1) > y | M_n(1) = x) \approx \varphi_{n/2}(x, y, \delta).$$

For a rigorous statement, the following inequality will serve our purposes.

**Theorem 5.1** *Let $n \geq 3$ and $k = \lfloor (n+1)/2 \rfloor$. Then for all $y > 0$ and all $\delta > 0$,*

$$P(M_n(1+\delta) - M_n(1) > y) \leq \int_{-\infty}^{\infty} \varphi_{k-1}(x, y, \delta) f_n(x)\, dx,$$

*where $f_n(x)$ is the density of $M_n(1)$, given by* (3.4).

*Proof* First, let us observe that

$$\begin{aligned}
\varphi_k(x, y, \delta) = P(S_k \geq 0) &= P\left(\sum_{j=1}^{k} \xi_j^U \geq \sum_{j=1}^{k} \xi_j^L\right) \\
&= \sum_{\ell=0}^{k} \sum_{m=\ell}^{k} P\left(\sum_{j=1}^{k} \xi_j^L = \ell,\ \sum_{j=1}^{k} \xi_j^U = m\right) \\
&= \sum_{\ell=0}^{k} \sum_{m=\ell}^{k} \binom{k}{\ell}\binom{k}{m} p_1^{\ell} q_1^{k-\ell} p_2^m q_2^{k-m}.
\end{aligned}$$

Let us also adopt the following notation: for $h > 0$, let $p_{1,h} = p_1(x+h, y-h, \delta)$ and

$$\varphi_k^h(x, y, \delta) = \sum_{\ell=0}^{k} \sum_{m=\ell}^{k} \binom{k}{\ell}\binom{k}{m} p_{1,h}^{\ell} q_{1,h}^{k-\ell} p_2^m q_2^{k-m},$$

where $q_{1,h} = 1 - p_{1,h}$. Finally, let $\Delta M_n = M_n(1+\delta) - M_n(1)$.

Now, fix $\delta > 0$ and $y > 0$. Let $K \in \mathbb{N}$ and let $h > 0$ with $K/h \in \mathbb{N}$. Then

$$P(\Delta M_n > y,\ |M_n(1)| \leq K) \leq \sum_{\substack{x \in h\mathbb{Z} \\ |x| \leq K}} P\left(M_n(1+\delta) > x + y,\ M_n(1) \in [x, x+h)\right).$$

Let $\mathcal{S}_n = \{1, \ldots, n\}$ and let $S = S_n$ denote the collection of all ordered pairs $(I, j)$ where $I \subset \mathcal{S}_n$ and $j \in \mathcal{S}_n$ satisfy $|I| = k-1$ and $j \notin I$. For $(I, j) \in S$, $x \in \mathbb{R}$, and $h > 0$, define $I(j)^c = \mathcal{S}_n \setminus (I \cup \{j\})$ and

$$\begin{aligned} A(I,j,x,h) &= \{B_j(1) \in [x,x+h)\} \\ &\qquad \cap \{B_i(1) < B_j(1),\ \forall i \in I\} \cap \{B_i(1) > B_j(1),\ \forall i \in I(j)^c\}, \\ \tilde{A}(I,j,x,h) &= \{B_j(1) \in [x,x+h)\} \\ &\qquad \cap \{B_i(1) < x+h,\ \forall i \in I\} \cap \{B_i(1) > x,\ \forall i \in I(j)^c\}. \end{aligned}$$

Note that $\{M_n(1) \in [x,x+h)\} = \bigcup\{A(I,j,x,h) : (I,j) \in S\}$ up to a set of measure zero, and that this is a disjoint union. Therefore,

$$\begin{aligned} P(M_n(1+\delta) > x+y,\ M_n(1) \in [x,x+h)) &= \sum_{(I,j)\in S} P(M_n(1+\delta) \\ &\qquad > x+y,\ A(I,j,x,h)) \\ &\le \sum_{(I,j)\in S} P(M_n(1+\delta) \\ &\qquad > x+y,\ \tilde{A}(I,j,x,h)), \end{aligned}$$

since $A(I,j,x,h) \subset \tilde{A}(I,j,x,h)$.

Now fix $(I,j) \in S$ and $x \in \mathbb{R}$. Define

$$\begin{aligned} N_1 &= \sum_{i\in I} 1_{\{B_i(1+\delta)<x+y\}} \\ N_2 &= \sum_{i\in I(j)^c} 1_{\{B_i(1+\delta)>x+y\}} \\ N &= \sum_{i=1}^{n} 1_{\{B_i(1+\delta)>x+y\}} \end{aligned}$$

and note that $\{M_n(1+\delta) > x+y\} = \{N \geq n-k+1\}$. Also note that, up to a set of measure zero,

$$\begin{aligned} N &= N_2 + (k-1) - N_1 + 1_{\{B_j(1+\delta)>x+y\}} \\ &\le N_2 - N_1 + k. \end{aligned}$$

Thus, if $d(n) = n-2k+1$, then $\{M_n(1+\delta) > x+y\} \subset \{N_2 - N_1 \geq d(n)\}$. This gives

$$\begin{aligned} P(M_n(1+\delta) > x+y,\ \tilde{A}(I,j,x,h)) &\le P(N_2 - N_1 \geq d(n), \tilde{A}(I,j,x,h)) \\ &= \sum_{\ell=0}^{k-1} \sum_{m=d(n)+\ell}^{n-k} P(N_1 = \ell, N_2 = m,\ \tilde{A}(I,j,x,h)). \end{aligned}$$

Hence, if we define

$$P_1(\ell) = P(\{N_1 = \ell\} \cap \{B_i(1) < x + h, \forall i \in I\}),$$
$$P_2(m) = P(\{N_2 = m\} \cap \{B_i(1) > x, \forall i \in I(j)^c\}),$$

then we can write

$$P(M_n(1+\delta) > x + y, \tilde{A}(I,j,x,h)) \leq \sum_{\ell=0}^{k-1} \sum_{m=d(n)+\ell}^{n-k} P(B_j(1) \in [x, x+h)) P_1(\ell) P_2(m).$$

Since

$$P(\tilde{A}(I,j,x,h)) = P(B_j(1) \in [x, x+h)) \Phi(x+h)^{k-1} \Phi(-x)^{n-k},$$

this gives

$$P(M_n(1+\delta) > x + y | \tilde{A}(I,j,x,h)) \leq \sum_{\ell=0}^{k-1} \sum_{m=d(n)+\ell}^{n-k} \frac{P_1(\ell)}{\Phi(x+h)^{k-1}} \cdot \frac{P_2(m)}{\Phi(-x)^{n-k}}$$

for each fixed $I$, $j$, and $x$.

To simplify this double sum, let

$$\begin{aligned} \psi(x,y,\delta) &= P(B(1+\delta) < x + y,\ B(1) < x) \\ &= \int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right) \Phi'(t)\, \mathrm{d}t. \end{aligned} \tag{5.8}$$

Then by symmetry and independence,

$$P_1(\ell) = \binom{k-1}{\ell} [\psi(x+h, y-h)]^{\ell} [\Phi(x+h) - \psi(x+h, y-h)]^{k-1-\ell},$$
$$P_2(m) = \binom{n-k}{m} [\psi(-x,-y)]^{m} [\Phi(-x) - \psi(-x,-y)]^{n-k-m}.$$

Also note that

$$\frac{\psi(x+h, y-h)}{\Phi(x+h)} = P(B(1+\delta) < x + y | B(1) < x + h) = p_{1,h}$$

and

$$\frac{\psi(-x,-y)}{\Phi(-x)} = P(B(1+\delta) > x + y | B(1) > x) = p_2,$$

which yields

$$P(M_n(1+\delta) > x+y|\tilde{A}(I,j,x,h)) \leq \sum_{\ell=0}^{k-1} \sum_{m=d(n)+\ell}^{n-k} \binom{k-1}{\ell}\binom{n-k}{m} p_{1,h}^{\ell} q_{1,h}^{k-1-\ell} p_2^m q_2^{n-k-m}$$

for each fixed $I$, $j$, and $x$.

Now suppose $n$ is odd. In this case, $d(n)=0$ and $n-k=k-1$, so

$$P(M_n(1+\delta) > x+y|\tilde{A}(I,j,x,h)) \leq \varphi_{k-1}^h(x,y,\delta). \tag{5.9}$$

On the other hand, if $n$ is even, then $d(n)=1$ and $n-k=k$, so

$$\begin{aligned} P(M_n(1+\delta) > x+y|\tilde{A}(I,j,x,h)) &\leq \sum_{\ell=0}^{k-1} \sum_{m=\ell+1}^{k} \binom{k-1}{\ell}\binom{k}{m} p_{1,h}^{\ell} q_{1,h}^{k-1-\ell} p_2^m q_2^{k-m} \\ &= \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} p_{1,h}^{\ell} q_{1,h}^{k-1-\ell} \sum_{m=\ell+1}^{k} \binom{k}{m} p_2^m q_2^{k-m}. \end{aligned}$$

But

$$\begin{aligned} \sum_{m=\ell+1}^{k} \binom{k}{m} p_2^m q_2^{k-m} &= P\left(\sum_{j=1}^{k} \xi_j^U > \ell\right) \leq P\left(\sum_{j=1}^{k-1} \xi_j^U \geq \ell\right) \\ &= \sum_{m=\ell}^{k-1} \binom{k-1}{m} p_2^m q_2^{k-1-m}, \end{aligned}$$

so (5.9) holds in this case as well.

Putting it all together, we have

$$\begin{aligned} P(\Delta M_n > y,\, |M_n(1)| \leq K) &\leq \sum_{\substack{x\in h\mathbb{Z} \\ |x|\leq K}} \sum_{(I,j)\in S} P(M_n(1+\delta) > x+y,\, \tilde{A}(I,j,x,h)) \\ &\leq \sum_{\substack{x\in h\mathbb{Z} \\ |x|\leq K}} \sum_{(I,j)\in S} \varphi_{k-1}^h(x,y,\delta) P(\tilde{A}(I,j,x,h)) \\ &= \sum_{\substack{x\in h\mathbb{Z} \\ |x|\leq K}} \sum_{(I,j)\in S} \varphi_{k-1}^h(x,y,\delta) \frac{P(\tilde{A})}{P(A)} P(A(I,j,x,h)). \end{aligned}$$

Note that $P(A(I,j,x,h)) \geq P(B_j(1) \in [x, x+h))\Phi(x)^{k-1}\Phi(-x-h)^{n-k}$, so that

$$\frac{P(\tilde{A})}{P(A)} \leq \left[\frac{\Phi(x+h)}{\Phi(x)}\right]^{k-1}\left[\frac{\Phi(-x)}{\Phi(-x-h)}\right]^{n-k}.$$

If we denote the right-hand side of this inequality by $g_h(x)$, then by dominated convergence,

$$\begin{aligned} P(\Delta M_n > y, |M_n(1)| \leq K) &\leq \sum_{\substack{x \in h\mathbb{Z} \\ |x| \leq K}} \varphi_{k-1}^h(x,y,\delta) g_h(x) \sum_{(I,j)\in S} P(A(I,j,x,h)) \\ &= \sum_{\substack{x \in h\mathbb{Z} \\ |x| \leq K}} \varphi_{k-1}^h(x,y,\delta) g_h(x) P(M_n(1) \in [x, x+h)) \\ &\to \int_{-K}^{K} \varphi_{k-1}(x,y,\delta) f_n(x)\,\mathrm{d}x. \end{aligned}$$

Letting $K \to \infty$ finishes the proof. □

The estimate in Theorem 5.1 can be simplified even further and we will find it convenient to use the following.

**Corollary 5.2** *Let $n \geq 3$, $k = \lfloor (n+1)/2 \rfloor$, $y > 0$, and $\delta > 0$. Then*

$$P(M_n(1+\delta) - M_n(1) > y) \leq \varphi_{k-1}(x_0, y, \delta) + P(M_n(1) \leq x_0) \tag{5.10}$$

*for all $x_0 \in \mathbb{R}$.*

*Proof* We will first show that $x \mapsto \varphi_{k-1}(x,y,\delta)$ is decreasing, for which it will suffice to show that $x \mapsto p_1(x,y,\delta)$ is increasing. To see this, recall that $\varphi_{k-1}(x,y,\delta) = P(\sum_{j=1}^{k-1} Y_j \geq 0)$. If $x \mapsto p_1(x,y,\delta)$ is increasing, then $x \mapsto p_2(x,y,\delta) = p_1(-x,-y,\delta)$ is decreasing. Hence, by (5.4) and (5.5), $P(Y_j = -1) = p_1(1-p_2)$ increases with $x$ and $P(Y_j = 1) = p_2(1-p_1)$ decreases with $x$, which shows that $x \mapsto \varphi_{k-1}(x,y,\delta)$ is decreasing.

With $\psi$ as in (5.8), we have $p_1 = \psi/\Phi(x)$ and

$$\partial_x p_1 = -\frac{\Phi'(x)}{[\Phi(x)]^2}\psi + \frac{1}{\Phi(x)}\left[\Phi\left(\frac{y}{\sqrt{\delta}}\right)\Phi'(x) + \frac{1}{\sqrt{\delta}}\int_{-\infty}^{x}\Phi'\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi'(t)\,\mathrm{d}t\right]. \tag{5.11}$$

Integrating by parts gives

$$\psi(x,y,\delta) = \Phi\left(\frac{y}{\sqrt{\delta}}\right)\Phi(x) + \frac{1}{\sqrt{\delta}}\int_{-\infty}^{x}\Phi'\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi(t)\,\mathrm{d}t.$$

Substituting this into (5.11) gives

$$\begin{aligned}\partial_x p_1 &= -\frac{\Phi'(x)}{[\Phi(x)]^2\sqrt{\delta}}\int_{-\infty}^{x}\Phi'\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi(t)\,\mathrm{d}t\\&\quad+\frac{1}{\Phi(x)\sqrt{\delta}}\int_{-\infty}^{x}\Phi'\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi'(t)\,\mathrm{d}t\\&=\frac{1}{\Phi(x)\sqrt{\delta}}\int_{-\infty}^{x}\Phi'\left(\frac{x+y-t}{\sqrt{\delta}}\right)\left[\frac{\Phi'(t)}{\Phi(t)}-\frac{\Phi'(x)}{\Phi(x)}\right]\Phi(t)\,\mathrm{d}t.\end{aligned}\tag{5.12}$$

Note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\Phi'(x)}{\Phi(x)}\right]&=\frac{\Phi''(x)\Phi(x)-[\Phi'(x)]^2}{[\Phi(x)]^2}\\&=\frac{1}{[\Phi(x)]^2}\left(-\frac{1}{\sqrt{2\pi}}x\mathrm{e}^{-x^2/2}\Phi(x)-\frac{1}{2\pi}\mathrm{e}^{-x^2}\right)\\&=-\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}[\Phi(x)]^2}\left(x\Phi(x)+\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\right).\end{aligned}$$

Clearly, $x\,\Phi(x)+\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}\geq 0$ for $x\geq 0$. If $x<0$, then by (3.5),

$$\begin{aligned}x\Phi(x)+\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}&=x\Phi(-|x|)+\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\\&\geq x\frac{1}{\sqrt{2\pi}}|x|^{-1}\mathrm{e}^{-x^2/2}+\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}=0.\end{aligned}$$

Thus, $x\mapsto\Phi'(x)/\Phi(x)$ is decreasing, so by (5.12), $\partial_x p_1\geq 0$.

Hence, $x\mapsto\varphi_{k-1}(x,y,\delta)$ is decreasing, and using Theorem 5.1,

$$\begin{aligned}P(M_n(1+\delta)-M_n(1)>y)&\leq\int_{-\infty}^{\infty}\varphi_{k-1}(x,y,\delta)f_n(x)\,\mathrm{d}x\leq\int_{-\infty}^{x_0}\varphi_{k-1}(x,y,\delta)f_n(x)\,\mathrm{d}x\\&\quad+\varphi_{k-1}(x_0,y,\delta)\int_{x_0}^{\infty}f_n(x)\,\mathrm{d}x\\&\leq\int_{-\infty}^{x_0}f_n(x)\,\mathrm{d}x+\varphi_{k-1}(x_0,y,\delta)\int_{-\infty}^{\infty}f_n(x)\,\mathrm{d}x\\&=P(M_n(1)\leq x_0)+\varphi_{k-1}(x_0,y,\delta),\end{aligned}$$

where $x_0\in\mathbb{R}$ is arbitrary. □

Recall that our only remaining goal is to establish the inequality (3.2) for the small and medium jump regimes. In applying Corollary 5.2 to this task, we must set $y = \varepsilon/\sqrt{n}$. Our choice for $x_0$, however, is less clear. On the one hand, we want $x_0$ to be large so that the first term on the right-hand of (5.10) is small. On the other hand, we need $x_0$ to be sufficiently far into the negative real line so that the second term is small. The value of $x_0$ that will strike a balance for us is given in the following lemma.

**Lemma 5.3** *Let $\varepsilon > 0$, $\delta > 0$, and $n \in \mathbb{N}$. Define $x_0 = -\varepsilon/(\delta^{1/4}\sqrt{n})$. Then for all $p > 2$,*

$$P(M_n(1) \leq x_0) \leq C_p(\varepsilon^{-1}\delta^{1/4})^p,$$

*where $C_p$ is a finite constant depending only p.*

*Proof* This follows immediately from (3.3). □

In light of this lemma and Corollary 5.2, we will establish inequality (3.2) once we verify that

$$\varphi_{k-1}\left(-\frac{\varepsilon}{\delta^{1/4}\sqrt{n}}, \frac{\varepsilon}{\sqrt{n}}, \delta\right) \leq C_p(\varepsilon^{-1}\delta^{1/6})^p \tag{5.13}$$

for all values of $\varepsilon$, $\delta$, and $n$ in the small and medium jump regimes.

## 6 Estimates for a random walk

In this section, we wish to find useful estimates for $\varphi_k(x, y, \delta) = P(S_k \geq 0)$. The process $\{S_n\}_{n=1}^{\infty}$ is, of course, a biased random walk which, in the cases we are interested in, has a negative drift. Let us recall the definition of $S_n$. In this section, we will temporarily abandon the tilde notation for the sake of simplicity.

We take as given a sequence of $\{-1, 0, 1\}$-valued random variables with $p_1 = P(Y_j = -1)$ and $p_2 = P(Y_j = 1)$. We define $\varepsilon = p_1 + p_2$ and $\mu = p_1 - p_2$, so that $P(Y_j = 0) = 1 - \varepsilon$. We then define $S_n = \sum_{j=1}^n Y_j$.

As mentioned, we will be interested in the case where $\mu > 0$, so that the walk has a negative drift. Besides this, however, we will also be interested in the case where $\varepsilon$ is small. That is, besides the negative drift, our walk will have the property that, for most time steps, it does not move. Our first estimate is a straightforward application of Chebyshev's inequality. It is a fairly simple result and serves as our starting point, but it will not be sufficient by itself. Note, in particular, that it does not make any noteworthy use of the fact that $\varepsilon$ is small.

**Lemma 6.1** *If $\varepsilon > 0$ and $\mu > 0$, then for all $p > 1$, there exists $C_p$, depending only on p, such that*

$$P(S_n \geq 0) \leq C_p \frac{\varepsilon}{n^p \mu^{2p}} \tag{6.1}$$

*for all n.*

*Proof* Since $EY_j = -\mu$, Chebyshev's inequality gives

$$P(S_n \geq 0) = P(S_n + n\mu \geq n\mu) \leq \frac{E|S_n + n\mu|^{2p}}{n^{2p}\mu^{2p}}.$$

By Burkholder's and Jensen's inequalities,

$$E|S_n + n\mu|^{2p} = E\left|\sum_{j=1}^{n}(Y_j + \mu)\right|^{2p} \leq \tilde{C}_p E\left|\sum_{j=1}^{n}|Y_j + \mu|^2\right|^{p} \leq \tilde{C}_p n^p E|Y_1 + \mu|^{2p}.$$

Also,

$$\begin{aligned} E|Y_1 + \mu|^{2p} &= p_1(1-\mu)^{2p} + (1-\varepsilon)\mu^{2p} + p_2(1+\mu)^{2p} \\ &\leq 2^{2p}(p_1 + p_2) + \mu^{2p} \\ &\leq (2^{2p} + 1)\varepsilon \end{aligned}$$

since $\mu \leq \varepsilon$. Thus, (6.1) holds with $C_p = \tilde{C}_p(2^{2p} + 1)$. □

As it stands, (6.1) will not suit our needs. We will find it necessary for the numerator on the right-hand side of (6.1) to contain $\varepsilon^p$ rather than $\varepsilon$. To accomplish this, we must appeal to the fact that, for the most part, this random walk does not move. To this end, we begin with two lemmas.

**Lemma 6.2** *For $n \in \mathbb{N}$, $k \in \{0, \ldots, n\}$, $p \in (0,1)$, and $x \in \mathbb{R}$, let $f(n,k,p) = \binom{n}{k}p^k q^{n-k}$, where $q = 1-p$, and let $g(n,x,p) = (2\pi npq)^{-1/2}\exp\{-(x–np)^2/2npq\}$. Then*

$$\sup_{n\in\mathbb{N}}\left(\sup_{k\in\{0,\ldots,n\}}\frac{f(n,k,p)}{g(n,k,p)}\right) < \infty$$

*if and only if $p = 1/2$. However, there exists a universal constant $C$, independent of $p$, such that $f(n,k,p)/g(n,k,p) \leq C$ for all $n \in \mathbb{N}$ and all $k \in \{0, \ldots, \lfloor np \rfloor\}$, provided $p \leq 1/2$.*

*Proof* It will first be shown that there exists a universal constant $C$ such that

(i) if $p \leq 1/2$, then $f(n,0,p)/g(n,0,p) \leq C$, and
(ii) if $p \leq 1/2$ and $\lfloor np \rfloor \geq 1$, then $f(n,1,p)/g(n,1,p) \leq C$.

We will start by showing that if $\alpha > 0$, then there exists a constant $C_\alpha$, depending only on $\alpha$, such that for all $p \leq 1/2$,

$$(np)^\alpha (q\mathrm{e}^{p/2q})^n \leq C_\alpha. \quad (6.2)$$

To prove this, first consider $2/5 \le p \le 1/2$. In this case, $q\mathrm{e}^{p/2q} \le \frac{3}{5}\mathrm{e}^{1/2} < 1$. Thus,

$$(np)^\alpha (q\mathrm{e}^{p/2q})^n \le \sup_n \left[ n^\alpha \left( \frac{3}{5}\mathrm{e}^{1/2} \right)^n \right] < \infty.$$

Next, consider $0 < p < 2/5$. Since $\frac{\mathrm{d}}{\mathrm{d}q}[\log(q^{5/6}\mathrm{e}^{p/2q})] = (5q-3)/6q^2 > 0$ for $q > 3/5$, it follows that in this case, $q^{5/6}\mathrm{e}^{p/2q} \le 1$. Hence,

$$(np)^\alpha (q\mathrm{e}^{p/2q})^n \le (np)^\alpha q^{n/6} = (n^\alpha q^{n/6}) p^\alpha.$$

Elementary calculus shows that $x \mapsto x^\alpha q^{x/6}$ attains its maximum on $[0,\infty)$ at $x = -6\alpha/\log q$. Thus,

$$n^\alpha q^{n/6} p^\alpha \le \left( \frac{6\alpha}{e} \right)^\alpha \left( \frac{1-q}{|\log q|} \right)^\alpha.$$

Since $(q-1)/\log q \to 1$ as $q \to 1$, this proves (6.2). Thus, if $p \le 1/2$, then

$$\frac{f(n,0,p)}{g(n,0,p)} = \sqrt{2\pi npq}\, q^n \mathrm{e}^{np/2q} = \sqrt{2\pi q}(np)^{1/2}(q\mathrm{e}^{p/2q})^n \le \sqrt{2\pi}\, C_{1/2},$$

and if $p \le 1/2$ and $np \ge 1$, then

$$\begin{aligned}
\frac{f(n,1,p)}{g(n,1,p)} &= \sqrt{2\pi npq}\, npq^{n-1} \exp\left\{ \frac{np}{2q} - \frac{1}{q} + \frac{1}{2npq} \right\} \\
&\le \sqrt{2\pi q}\, q^{n-1} (np)^{3/2} \mathrm{e}^{np/2q} \\
&= \sqrt{\frac{2\pi}{q}} (np)^{3/2} (q\mathrm{e}^{p/2q})^n \\
&\le \sqrt{4\pi}\, C_{3/2},
\end{aligned}$$

which verifies (i) and (ii).

Now, for $k \in \{1,\dots,n-1\}$, Stirling's formula implies that $f(n,k,p)$ is bounded above and below by universal, positive constant multiples of

$$\frac{n^{n+\frac{1}{2}}}{(n-k)^{n-k+\frac{1}{2}} k^{k+\frac{1}{2}}}\, p^k q^{n-k}.$$

Let us define

$$\begin{aligned} F(k) = F(n,k,p) &= \log\left(\frac{n^{n+\frac{1}{2}}}{(n-k)^{n-k+\frac{1}{2}}k^{k+\frac{1}{2}}}\,p^k q^{n-k}\right) - \log(\sqrt{2\pi}\,g(n,k,p)) \\ &= (n+1)\log n - (n-k+\tfrac{1}{2})\log(n-k) - (k+\tfrac{1}{2})\log k \\ &\quad +(k+\tfrac{1}{2})\log p + (n-k+\tfrac{1}{2})\log q + (k-np)^2/2npq, \end{aligned}$$

so that there are universal, positive constants $C_1$ and $C_2$ such that

$$\log C_1 + F(k) \le \log\left[\frac{f(n,k,p)}{g(n,k,p)}\right] \le \log C_2 + F(k) \tag{6.3}$$

for all $k \in \{1,\dots,n-1\}$. Note that $F(k)$ is well-defined for all real $k \in (0,n)$.

We can directly compute that

$$F(n/2) = \frac{1}{2}\log(4pq) + \frac{n}{2}(G(p) + G(1-p)),$$

where $G(p) = \log 2 + \log p + 1/(4p) - 1/2$. Now, $G'(p) = 1/p - 1/(4p^2)$, which gives

$$G'(p) - G'(1-p) = \left(\frac{q-p}{pq}\right)\left(1 - \frac{1}{4pq}\right).$$

Since $1-1/(4pq) < 0$ for all $p \neq 1/2$, the function $p \mapsto G(p)+G(1-p)$ is strictly decreasing on $(0,1/2)$ and strictly increasing on $(1/2,0)$. Since $G(1/2) = 0$, we have that $G(p)+G(1-p) > 0$ for all $p \neq 1/2$. Thus, if $p \neq 1/2$, then $F(n/2) \to \infty$ as $n \to \infty$. It now follows from (6.3) that

$$\sup_{n\in\mathbb{N}}\left(\sup_{k\in\{0,\dots,n\}} \frac{f(n,k,p)}{g(n,k,p)}\right) = \infty$$

whenever $p \neq 1/2$.

Now suppose $p \le 1/2$ and let $k \in [2, np]$. We can compute that for all $x \in (0,n)$,

$$\begin{aligned} F'(x) &= \log(n-x) + \frac{1}{2(n-x)} - \log x - \frac{1}{2x}\log\frac{p}{q} + \frac{x}{npq} - \frac{1}{q} \\ F''(x) &= -\frac{1}{n-x} + \frac{1}{2(n-x)^2} - \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{npq} \\ F'''(x) &= -\frac{1}{(n-x)^2} + \frac{1}{(n-x)^3} + \frac{1}{x^2} - \frac{1}{x^3} \\ F^{(4)}(x) &= \frac{3-2(n-x)}{(n-x)^4} + \frac{3-2x}{x^4}. \end{aligned}$$

It is easily verified that $F(np) = 0$ and $F'(np) = (p-q)/2npq$, so that we may write

$$F(k) = -\int_k^{np} F'(t)\,\mathrm{d}t = -\int_k^{np} \left( \frac{p-q}{2npq} - \int_t^{np} F''(s)\,\mathrm{d}s \right) \mathrm{d}t$$
$$\leq \frac{q-p}{2q} + \int_k^{np} \int_k^{s} F''(s)\,\mathrm{d}t\,\mathrm{d}s.$$

Since $F^{(4)} \leq 0$ on $[2, n-2]$ and $F'''(n/2) = 0$, it follows that $F''' \geq 0$ on $[2, n/2]$, which implies $F''$ is increasing on $[2, n/2]$. Since $F''(np) = (p^2+q^2)/2n^2p^2q^2$, we have

$$F(k) \leq \frac{1}{2} + \frac{p^2+q^2}{2n^2p^2q^2} \int_k^{np} (s-k)\,\mathrm{d}s \leq \frac{1}{2} + \frac{p^2+q^2}{2n^2p^2q^2} n^2p^2 \leq \frac{3}{2}$$

for all $p \leq 1/2$.

It now follows from (6.3) and (i), (ii) that there is a universal constant $C$, independent of $p$, such that $f(n,k,p)/g(n,k,p) \leq C$ for all $n \in \mathbb{N}$ and all $k \in \{0, \ldots, \lfloor np \rfloor\}$, provided $p \leq 1/2$. Also, if $p = 1/2$, symmetry gives the same bound for $k \in \{\lfloor n/2 \rfloor + 1, \ldots, n\}$, and it follows that

$$\sup_{n \in \mathbb{N}} \left( \sup_{k \in \{0,\ldots,n\}} \frac{f(n,k,p)}{g(n,k,p)} \right) < \infty,$$

which completes the proof. □

**Lemma 6.3** *Let $0 < \varepsilon < 1/2$ and suppose that $\{\xi_j\}_{j=1}^{\infty}$ are i.i.d. $\{0,1\}$-valued random variables with $P(\xi_1 = 1) = \varepsilon$. Let $T_n = \sum_{j=1}^{n} \xi_j$. Then for each $p > 1$, there exists a finite constant $C_p$, depending only on p, such that*

$$E[T_n^{-p} 1_{\{T_n > 0\}}] \leq C_p \frac{1}{(\varepsilon n)^p}$$

*for all $n \in \mathbb{N}$.*

*Proof* Observe that

$$E[T_n^{-p} 1_{\{T_n > 0\}}] = E[T_n^{-p} 1_{\{1 \leq T_n \leq \varepsilon n/2\}}] + E[T_n^{-p} 1_{\{T_n > \varepsilon n/2\}}]$$
$$\leq P\left(T_n \leq \frac{\varepsilon n}{2}\right) + \left(\frac{\varepsilon n}{2}\right)^{-p}.$$

Hence, it will suffice to show that

$$P\left(T_n \leq \frac{\varepsilon n}{2}\right) \leq C_p \frac{1}{(\varepsilon n)^p}.$$

To see this, let $f$ and $g$ be as in Lemma 6.2 with $p = \varepsilon$, so that there exists a universal, finite constant $C$, independent of $\varepsilon$, such that $f(n,k,\varepsilon) \leq Cg(n,k,\varepsilon)$ for all $n \in \mathbb{N}$ and all $k \in \{0,\dots,\lfloor \varepsilon n \rfloor\}$. Let $m = \lfloor \varepsilon n/2 \rfloor$, so that

$$P\left(T_n \leq \frac{\varepsilon n}{2}\right) = P(T_n \leq m) = \sum_{k=0}^{m} P(T_n = k) \leq C \sum_{k=0}^{m} g(n,k,\varepsilon).$$

If $\varepsilon n \leq 4$, then $P(T_n \leq m) \leq 1 \leq 4^p/(\varepsilon n)^p$, so that we may assume without loss of generality that $\varepsilon n > 4$. Note that $x \mapsto g(n,x,\varepsilon)$ is increasing on $[0,\varepsilon n]$ and $\varepsilon n > 4$ implies $m + 1 \leq (\varepsilon n/2) + 1 < 3\varepsilon n/4$. Thus,

$$\begin{aligned} P(T_n \leq m) &\leq C \int_0^{m+1} g(n,x,\varepsilon)\,\mathrm{d}x \leq C \int_{-\infty}^{3\varepsilon n/4} g(n,x,\varepsilon)\,\mathrm{d}x \\ &= \frac{C}{\sqrt{2\pi t}} \int_{-\infty}^{3\varepsilon n/4} \mathrm{e}^{-(x-\varepsilon n)^2/2t}\,\mathrm{d}x, \end{aligned}$$

where $t = n\varepsilon(1-\varepsilon)$. By a change of variables,

$$P(T_n \leq m) \leq C\Phi\left(-\frac{\varepsilon n}{4\sqrt{t}}\right) \leq C\Phi\left(-\frac{\sqrt{\varepsilon n}}{4}\right).$$

By (3.5),

$$P(T_n \leq m) \leq \frac{C}{\sqrt{2\pi}} \cdot \frac{4}{\sqrt{\varepsilon n}}\,\mathrm{e}^{-\varepsilon n/32} \leq C\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{-\varepsilon n/32}.$$

Since there exists $K_p < \infty$ such that $x^p \mathrm{e}^{-x/32} \leq K_p$ for all $x \in [0,\infty)$, we have

$$P(T_n \leq m) \leq C\sqrt{\frac{2}{\pi}}\,K_p \frac{1}{(\varepsilon n)^p},$$

which finishes the proof. □

With these lemmas in place, we may now make the needed improvement to Lemma 6.1.

**Lemma 6.4** *If $0<\varepsilon<1/2$ and $\mu>0$, then for all $p>1$, there exists $C_p$, depending only on p, such that*

$$P(S_n \geq 0) \leq C_p \frac{\varepsilon^p}{n^p \mu^{2p}} \tag{6.4}$$

*for all n.*

*Proof* Let $\{\tilde{Y}_j\}_{j=1}^{\infty}$ be a sequence of i.i.d. $\{-1,1\}$-valued random variables with $P(\tilde{Y}_1 = -1) = p_1/\varepsilon$. Let $\{\xi_j\}_{j=1}^{\infty}$ be a sequence of i.i.d. $\{0,1\}$-valued random variables, independent of $\{\tilde{Y}_j\}_{j=1}^{\infty}$, with $P(\xi_1 = 1) = \varepsilon$. Then $\{\tilde{Y}_j\xi_j\}_{j=1}^{\infty}$ is an i.i.d. sequence of random variables which has the same law as $\{Y_j\}_{j=1}^{\infty}$.

Let $\tilde{S}_n = \sum_{j=1}^n \tilde{Y}_j$ and note that by Lemma 6.1,

$$P(\tilde{S}_n \geq 0) \leq \tilde{C}_p \frac{1}{n^p(\mu/\varepsilon)^{2p}} = \tilde{C}_p \frac{\varepsilon^{2p}}{n^p \mu^{2p}}. \tag{6.5}$$

Define $\xi^{(n)} = (\xi_1, \ldots, \xi_n)$, so that

$$\begin{aligned} P(S_n \geq 0) &= P\Bigg(\sum_{j=1}^n \tilde{Y}_j\xi_j \geq 0\Bigg) = \sum_{k=0}^n \sum_{\substack{\alpha\in\{0,1\}^n \\ |\alpha|=k}} P\Bigg(\sum_{j=1}^n \tilde{Y}_j\xi_j \geq 0,\, \xi^{(n)} = \alpha\Bigg) \\ &= \sum_{k=0}^n \sum_{\substack{\alpha\in\{0,1\}^n \\ |\alpha|=k}} P\Bigg(\sum_{\{j:\alpha_j=1\}} \tilde{Y}_j \geq 0,\, \xi^{(n)} = \alpha\Bigg), \end{aligned}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $T_n = \sum_{j=1}^n \xi_j$, then by symmetry and independence,

$$P(S_n \geq 0) = \sum_{k=0}^n \sum_{\substack{\alpha\in\{0,1\}^n \\ |\alpha|=k}} P\Bigg(\sum_{j=1}^k \tilde{Y}_j \geq 0\Bigg) P(\xi^{(n)} = \alpha) = \sum_{k=0}^n P(\tilde{S}_k \geq 0) P(T_n = k).$$

Using (6.5) and Lemma 6.3,

$$\begin{aligned} P(S_n \geq 0) &\leq P(T_n = 0) + \tilde{C}_p \frac{\varepsilon^{2p}}{\mu^{2p}} \sum_{k=1}^n k^{-p} P(T_n = k) \\ &= (1-\varepsilon)^n + \tilde{C}_p \frac{\varepsilon^{2p}}{\mu^{2p}} E[T_n^{-p} 1_{\{T_n>0\}}] \\ &\leq (1-\varepsilon)^n + \tilde{C}'_p \frac{\varepsilon^{2p}}{\mu^{2p}} \frac{1}{(\varepsilon n)^p}, \end{aligned}$$

Note that $1-\varepsilon \le \mathrm{e}^{-\varepsilon}$, so that

$$(1-\varepsilon)^n \le \mathrm{e}^{-\varepsilon n} \le \tilde{C}''_p \frac{1}{(\varepsilon n)^p} = \tilde{C}''_p \frac{\varepsilon^p}{n^p \varepsilon^{2p}} \le \tilde{C}''_p \frac{\varepsilon^p}{n^p \mu^{2p}},$$

which gives (6.4) with $C_p = \tilde{C}''_p + \tilde{C}'_p$. □

## 7 The small jump regime

Let us now put the pieces together and establish (3.2) for the small jump regime. Recall from Sect. 5 that it will suffice to establish (5.13). Using the notation of (5.1)–(5.7), Lemma 6.4 will give us that, for $p > 1$,

$$\varphi_{k-1}(x,y,\delta) \le C_p \frac{\tilde{\varepsilon}^p}{(k-1)^p \tilde{\mu}^{2p}}, \tag{7.1}$$

provided $\tilde{\varepsilon} = \tilde{\varepsilon}(x,y,\delta) < 1/2$ and $\tilde{\mu} = \tilde{\mu}(x,y,\delta) > 0$. We will be applying this with $x = -\varepsilon/(\delta^{1/4}\sqrt{n})$ and $y = \varepsilon/\sqrt{n}$, but recall that in the small jump regime, we can write $\varepsilon/\sqrt{n} = \delta^{1/2+\alpha}$ for some $\alpha > 0$. As such, the following lemma will help us check the provisions of (7.1).

**Lemma 7.1** *For each* $\Delta > 0$*, there exists* $\delta_0 > 0$ *such that*

(i) $\tilde{\mu}(-\delta^{1/4+\alpha}, \delta^{1/2+\alpha}, \delta) \ge \frac{1}{\sqrt{2\pi}} \delta^{1/2+\alpha}$, *and*

(ii) $\tilde{\varepsilon}(-\delta^{1/4+\alpha}, \delta^{1/2+\alpha}, \delta) \le 1000\,\delta^{1/2} < \frac{1}{2}$

*for all* $\alpha \ge \Delta$ *and all* $0 < \delta \le \delta_0$.

*Proof* For fixed $\delta > 0$, let $\psi(x,y) = \psi(x,y,\delta)$ be given by (5.8). We wish to show that

$$\psi(x,y) = \frac{1}{2} - \frac{1}{2\pi}\tan^{-1}\sqrt{\delta} + \frac{x}{\sqrt{2\pi}} + \frac{y}{2\sqrt{2\pi}} + \frac{\sqrt{\delta}}{4\pi}(x+y)^2 - \frac{y^2}{4\pi\sqrt{\delta}} + \tilde{R}(x,y), \tag{7.2}$$

where

$$|\tilde{R}(x,y)| \le (|x|+|y|)^3 + \frac{|x||y|^2}{\sqrt{\delta}}(|x|+|y|) + \frac{|y|^4}{\delta^{3/2}} + \delta^{3/2}(x+y)^2 + \delta(|x|+|y|) \tag{7.3}$$

for all $x, y \in \mathbb{R}$.

We will first show that for $i \ge 0$ and $j \ge 1$,

$$\partial_x^i \psi = \int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right) \Phi^{(i+1)}(t)\,\mathrm{d}t, \tag{7.4}$$

$$\partial_x^i \partial_y^j \psi = -\left(\frac{1}{\sqrt{\delta}}\right)^{j-1} \Phi^{(j-1)}\left(\frac{y}{\sqrt{\delta}}\right) \Phi^{(i+1)}(x) + \partial_x^{i+1}\partial_y^{j-1}\psi. \tag{7.5}$$

For $i = 0$, (7.4) is just the definition of $\psi$. If (7.4) is true for some $i \geq 0$, then using integration by parts gives

$$\begin{aligned}\partial_x^{i+1}\psi &= \partial_x\left[\int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi^{(i+1)}(t)\,\mathrm{d}t\right]\\ &= \Phi\left(\frac{y}{\sqrt{\delta}}\right)\Phi^{(i+1)}(x) + \frac{1}{\sqrt{\delta}}\int_{-\infty}^{x} \Phi'\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi^{(i+1)}(t)\,\mathrm{d}t\\ &= \int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi^{(i+2)}(t)\,\mathrm{d}t,\end{aligned}$$

so by induction, (7.4) holds for all $i \geq 0$. For (7.5), first consider $j = 1$. Then

$$\begin{aligned}\partial_x^{i}\partial_y\psi &= \partial_y\left[\int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi^{(i+1)}(t)\,\mathrm{d}t\right]\\ &= \int_{-\infty}^{x} \partial_y\left[\Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right)\right]\Phi^{(i+1)}(t)\,\mathrm{d}t\\ &= \int_{-\infty}^{x} \partial_x\left[\Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right)\right]\Phi^{(i+1)}(t)\,\mathrm{d}t\\ &= \partial_x\left[\int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi^{(i+1)}(t)\,\mathrm{d}t\right] - \Phi\left(\frac{y}{\sqrt{\delta}}\right)\Phi^{(i+1)}(x)\\ &= -\Phi\left(\frac{y}{\sqrt{\delta}}\right)\Phi^{(i+1)}(x) + \partial_x^{i+1}\psi,\end{aligned}$$

and (7.5) holds for all $i \geq 0$ when $j = 1$. Now suppose (7.5) holds for some $j \geq 1$ and all $i \geq 0$. Then

$$\begin{aligned}\partial_x^{i}\partial_y^{j+1}\psi &= \partial_y\left[-\left(\frac{1}{\sqrt{\delta}}\right)^{j-1}\Phi^{(j-1)}\left(\frac{y}{\sqrt{\delta}}\right)\Phi^{(i+1)}(x) + \partial_x^{i+1}\partial_y^{j-1}\psi\right]\\ &= -\left(\frac{1}{\sqrt{\delta}}\right)^{j}\Phi^{(j)}\left(\frac{y}{\sqrt{\delta}}\right)\Phi^{(i+1)}(x) + \partial_x^{i+1}\partial_y^{j}\psi.\end{aligned}$$

By induction, (7.5) holds for all $i \geq 0$ and $j \geq 1$.

By Taylor's Theorem we have that

$$\psi(x,y) = \psi(0,0) + x\psi_x(0,0) + y\psi_y(0,0) + \frac{1}{2!}[x^2\psi_{xx}(0,0) + 2xy\psi_{xy}(0,0) + y^2\psi_{yy}(0,0)] + R^{(1)}(x,y), \quad (7.6)$$

where

$$R^{(1)}(x,y) = \frac{1}{3!}[x^3\psi_{xxx}(\bar{x},\bar{y}) + 3x^2y\psi_{xxy}(\bar{x},\bar{y}) + 3xy^2\psi_{xyy}(\bar{x},\bar{y}) + y^3\psi_{yyy}(\bar{x},\bar{y})]$$

and $(\bar{x},\bar{y}) = (\theta x, \theta y)$ for some $\theta \in (0,1)$. Using (7.4), (7.5), and direct integration, we can verify that (7.6) becomes

$$\psi(x,y) = \frac{1}{2} - \frac{1}{2\pi}\tan^{-1}\sqrt{\delta} + \frac{x}{2\sqrt{2\pi}}\left(1 + \frac{1}{\sqrt{1+\delta}}\right) + \frac{y}{2\sqrt{2\pi}\sqrt{1+\delta}} + \frac{(x+y)^2\sqrt{\delta}}{4\pi(1+\delta)} - \frac{y^2}{4\pi\sqrt{\delta}} + R^{(1)}(x,y).$$

Now,

$$\frac{x}{2\sqrt{2\pi}}\left(1 + \frac{1}{\sqrt{1+\delta}}\right) = \frac{x}{\sqrt{2\pi}} + \frac{x}{2\sqrt{2\pi}}\left(\frac{1}{\sqrt{1+\delta}} - 1\right)$$

$$\frac{y}{2\sqrt{2\pi}\sqrt{1+\delta}} = \frac{y}{2\sqrt{2\pi}} + \frac{y}{2\sqrt{2\pi}}\left(\frac{1}{\sqrt{1+\delta}} - 1\right)$$

$$\frac{(x+y)^2\sqrt{\delta}}{4\pi(1+\delta)} = \frac{\sqrt{\delta}}{4\pi}(x+y)^2 + \frac{\sqrt{\delta}}{4\pi}(x+y)^2\left(\frac{1}{1+\delta} - 1\right).$$

Thus, if

$$R^{(2)}(x,y) = \frac{x+y}{2\sqrt{2\pi}}\left(\frac{1}{\sqrt{1+\delta}} - 1\right) - \frac{\delta^{3/2}(x+y)^2}{4\pi(1+\delta)},$$

then (7.2) holds with $\tilde{R} = R^{(1)} + R^{(2)}$.

Since $|(1+\delta)^{-1/2} - 1| < \delta$, we have $|R^{(2)}(x,y)| \le \delta(|x|+|y|) + \delta^{3/2}(x+y)^2$. To estimate $R^{(1)}$, we must estimate the third partial derivatives of $\psi$. Using (7.4), we have

$$|\psi_{xxx}(x,y)| = \left|\int_{-\infty}^{x} \Phi\left(\frac{x+y-t}{\sqrt{\delta}}\right)\Phi^{(4)}(t)\,\mathrm{d}t\right| \le \int_{-\infty}^{\infty} |\Phi^{(4)}(t)|\,\mathrm{d}t.$$

Since $\Phi^{(4)}(t) = (3t - t^3)\Phi'(t)$, we have

$$|\psi_{xxx}(x,y)| \leq 2\int_0^\infty (3t + t^3)\Phi'(t)\,\mathrm{d}t = \frac{10}{\sqrt{2\pi}}.$$

Similarly, by (7.5),

$$|\psi_{xxy}(x,y)| = \left| -\Phi\left(\frac{y}{\sqrt{\delta}}\right)\Phi'''(x) + \psi_{xxx}(x,y) \right| \leq |\Phi'''(x)| + \frac{10}{\sqrt{2\pi}}.$$

Since $|\Phi'''(x)| \leq 2(2\pi)^{-1/2}$ for all $x \in \mathbb{R}$, we have that

$$|\psi_{xxy}(x,y)| \leq \frac{12}{\sqrt{2\pi}}.$$

Likewise, the formulas

$$\psi_{xyy} = -\frac{1}{\sqrt{\delta}}\Phi'\left(\frac{y}{\sqrt{\delta}}\right)\Phi''(x) + \psi_{xxy} = \frac{x}{\sqrt{\delta}}\Phi'\left(\frac{y}{\sqrt{\delta}}\right)\Phi'(x) + \psi_{xxy}$$

and

$$\psi_{yyy} = -\frac{1}{\delta}\Phi''\left(\frac{y}{\sqrt{\delta}}\right)\Phi'(x) + \psi_{xyy} = \frac{y}{\delta^{3/2}}\Phi'\left(\frac{y}{\sqrt{\delta}}\right)\Phi'(x) + \psi_{xyy}$$

can be used to verify that

$$\begin{aligned}
|\psi_{xyy}(x,y)| &\leq (|x|\delta^{-1/2} + 12\sqrt{2\pi})/(2\pi) \\
|\psi_{yyy}(x,y)| &\leq (|y|\delta^{-3/2} + |x|\delta^{-1/2} + 12\sqrt{2\pi})/(2\pi).
\end{aligned}$$

Piecing this together, we have

$$\begin{aligned}
|R^{(1)}(x,y)| &\leq \frac{1}{3!}\left[\frac{10|x|^3}{\sqrt{2\pi}} + \frac{36|x|^2|y|}{\sqrt{2\pi}} + 3|x||y|^2\left(\frac{|x|}{2\pi\sqrt{\delta}} + \frac{12}{\sqrt{2\pi}}\right)\right. \\
&\quad \left. + |y|^3\left(\frac{|y|}{2\pi\delta^{3/2}} + \frac{|x|}{2\pi\sqrt{\delta}} + \frac{12}{\sqrt{2\pi}}\right)\right] \\
&\leq \frac{1}{3!}\left[\frac{12}{\sqrt{2\pi}}(|x| + |y|)^3 + \frac{3|x||y|^2}{2\pi\sqrt{\delta}}(|x| + |y|) + \frac{|y|^4}{2\pi\delta^{3/2}}\right] \\
&\leq (|x| + |y|)^3 + \frac{|x||y|^2}{\sqrt{\delta}}(|x| + |y|) + \frac{|y|^4}{\delta^{3/2}}.
\end{aligned}$$

Combined with the estimate for $R^{(2)}$, this verifies (7.3).

Now, observe that $p_1(x,y,\delta) = \psi(x,y)/\Phi(x)$. Write $\Phi(x) = \frac{1}{2} + \frac{x}{\sqrt{2\pi}} + r_1(x)$, where $r_1(x) = \frac{1}{2}x^2\Phi''(\bar{x})$ and $\bar{x} = \theta x$ for some $\theta \in (0,1)$. Note that $|r_1(x)| \le \frac{1}{2\sqrt{2\pi}}|x|^3$. For $x \neq -\sqrt{\pi/2}$, write $\Phi(x)^{-1} = (\frac{1}{2} + \frac{x}{\sqrt{2\pi}})^{-1} + r_2(x)$, where $r_2(x) = -r_1(x)\Phi(x)^{-1}(\frac{1}{2} + \frac{x}{\sqrt{2\pi}})^{-1}$. Similarly, we may write $\Phi(x)^{-1} = 2 + r_3(x)$, where

$$r_3(x) = r_2(x) + \frac{1}{\frac{1}{2} + \frac{x}{\sqrt{2\pi}}} - 2 = r_2(x) - \frac{4x}{\sqrt{2\pi} + 2x}.$$

Let us now assume $|x| \le 1$. Then $x \neq -\sqrt{\pi/2}$ and the above applies. Note that

$$|r_2(x)| \le \frac{|r_1(x)|}{\Phi(-1)\left(\frac{1}{2} - \frac{1}{\sqrt{2\pi}}\right)}$$

Since $\Phi(-1) \ge \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \ge \frac{1}{10}$, we have $|r_2(x)| \le 100|r_1(x)| \le \frac{50}{\sqrt{2\pi}}|x|^3$. Also,

$$|r_3(x)| \le |r_2(x)| + \left(\frac{4}{\sqrt{2\pi} - 2}\right)|x| \le \frac{50}{\sqrt{2\pi}}|x|^3 + \frac{20}{\sqrt{2\pi}}|x|.$$

Since $|x| \le 1$, this gives $|r_3(x)| \le \frac{70}{\sqrt{2\pi}}|x|$. Applying (7.2) yields

$$\begin{aligned} p_1(x,y,\delta) &= \psi(x,y)\Phi(x)^{-1} \\ &= \left(\frac{1}{2} + \frac{x}{\sqrt{2\pi}}\right)\left(\left(\frac{1}{2} + \frac{x}{\sqrt{2\pi}}\right)^{-1} + r_2(x)\right) \\ &\quad + \left(-\frac{1}{2\pi}\tan^{-1}\sqrt{\delta} + \frac{y}{2\sqrt{2\pi}} + \frac{\sqrt{\delta}}{4\pi}(x+y)^2 - \frac{y^2}{4\pi\sqrt{\delta}}\right)(2 + r_3(x)) \\ &\quad + \tilde{R}(x,y)\Phi(x)^{-1} \\ &= 1 - \frac{1}{\pi}\tan^{-1}\sqrt{\delta} + \frac{y}{\sqrt{2\pi}} + \frac{\sqrt{\delta}}{2\pi}(x+y)^2 - \frac{y^2}{2\pi\sqrt{\delta}} + R_\delta(x,y), \end{aligned} \tag{7.7}$$

where

$$\begin{aligned} |R_\delta(x,y)| &\le |r_2(x)| + \left(\frac{\tan^{-1}\sqrt{\delta}}{2\pi} + \frac{|y|}{2\sqrt{2\pi}} + \frac{\sqrt{\delta}}{4\pi}(x+y)^2 + \frac{y^2}{4\pi\sqrt{\delta}}\right)|r_3(x)| \\ &\quad + \frac{|\tilde{R}(x,y)|}{\Phi(-1)} \\ &\le \frac{50}{\sqrt{2\pi}}|x|^3 + \left(\frac{\sqrt{\delta}}{2\pi} + \frac{|y|}{2\sqrt{2\pi}} + \frac{\sqrt{\delta}}{4\pi}(x+y)^2 + \frac{y^2}{4\pi\sqrt{\delta}}\right)\frac{70}{\sqrt{2\pi}}|x| \\ &\quad + 10|\tilde{R}(x,y)|. \end{aligned}$$

Hence,

$$
\begin{aligned}
|R_\delta(x,y)| \le{}& \frac{50}{\sqrt{2\pi}}|x|^3 + \left(\frac{\sqrt{\delta}}{2\pi} + \frac{|y|}{2\sqrt{2\pi}} + \frac{\sqrt{\delta}}{4\pi}(x+y)^2 + \frac{y^2}{4\pi\sqrt{\delta}}\right)\frac{70}{\sqrt{2\pi}}|x| \\
&+10\left[(|x|+|y|)^3 + \frac{|x||y|^2}{\sqrt{\delta}}(|x|+|y|) + \frac{|y|^4}{\delta^{3/2}} + \delta^{3/2}(x+y)^2\right. \\
&\left. +\delta(|x|+|y|)\right]
\end{aligned}
$$

by (7.3).

Now suppose that $\delta \le 1$ and $\alpha, \beta \in \mathbb{R}$. Let $y = \delta^{1/2+\alpha}$, $x = -\delta^{1/4+\beta}$, and assume that $y \le -x \le 1$. Using the fact that $|x| + |y| \le 2|x| \le 2$, we have

$$
\begin{aligned}
|R_\delta(x,y)| \le{}& \frac{50}{\sqrt{2\pi}}|x|^3 + \frac{70}{(2\pi)^{3/2}}|x|\left(\sqrt{\delta} + 2|y| + 2\sqrt{\delta} + \frac{y^2}{\sqrt{\delta}}\right) \\
&+10\left(8|x|^3 + 2\frac{|x||y|^2}{\sqrt{\delta}} + \frac{|y|^4}{\delta^{3/2}} + 4\delta^{3/2}x^2 + 2\delta|x|\right) \\
\le{}& 25|x|^3 + 5|x|\left(3\sqrt{\delta} + 2|y| + \frac{y^2}{\sqrt{\delta}}\right) \\
&+80|x|^3 + 20\frac{|x||y|^2}{\sqrt{\delta}} + 10\frac{|y|^4}{\delta^{3/2}} + 40\delta^{3/2}x^2 + 20\delta|x|
\end{aligned}
$$

which reduces to

$$
\begin{aligned}
|R_\delta(x,y)| \le{}& 105\delta^{3/4+3\beta} + 15\delta^{3/4+\beta} + 10\delta^{3/4+\alpha+\beta} + 25\delta^{3/4+2\alpha+\beta} \\
&+10\delta^{1/2+4\alpha} + 40\delta^{2+2\beta} + 20\delta^{5/4+\beta}.
\end{aligned}
$$

To simplify further, suppose $\alpha > 0$. Then

$$
\begin{aligned}
|R_\delta(x,y)| \le{}& 105\delta^{3/4+3\beta} + 15\delta^{3/4+\beta} + 10\delta^{3/4+\beta} + 25\delta^{3/4+\beta} \\
&+10\delta^{1/2+4\alpha} + 40\delta^{2+2\beta} + 20\delta^{3/4+\beta} \\
={}& 105\delta^{3/4+3\beta} + 70\delta^{3/4+\beta} + 10\delta^{1/2+4\alpha} + 40\delta^{2+2\beta}.
\end{aligned}
$$

Now, if $\beta \ge 0$, then $2+2\beta > 3/4+\beta$, and $|R_\delta(x,y)| \le 115\delta^{3/4+3\beta} + 110\delta^{3/4+\beta} + 10\delta^{1/2+4\alpha}$. Otherwise, if $\beta < 0$, then $2+2\beta > 3/4+3\beta$, and $|R_\delta(x,y)| \le 145\delta^{3/4+3\beta} + 70\delta^{3/4+\beta} + 10\delta^{1/2+4\alpha}$. In either case,

$$
|R_\delta(x,y)| \le 150(\delta^{3/4+3\beta} + \delta^{3/4+\beta} + \delta^{1/2+4\alpha})
$$

whenever $\alpha > 0$. On the other hand, suppose $\alpha < 0$. Then

$$
\begin{aligned}
|R_\delta(x,y)| \le{}& 105\delta^{3/4+3\beta} + 15\delta^{3/4+2\alpha+\beta} + 10\delta^{3/4+2\alpha+\beta} + 25\delta^{3/4+2\alpha+\beta} \\
&+10\delta^{1/2+4\alpha} + 40\delta^{2+2\beta} + 20\delta^{3/4+2\alpha+\beta} \\
={}& 105\delta^{3/4+3\beta} + 70\delta^{3/4+2\alpha+\beta} + 10\delta^{1/2+4\alpha} + 40\delta^{2+2\beta}.
\end{aligned}
$$

If $\beta \geq 0$, then $2+2\beta > 3/4+\beta \geq 3/4+2\alpha+\beta$; if $\beta < 0$, then $2+2\beta > 3/4+3\beta$. We therefore have

$$|R_\delta(x,y)| \leq 150(\delta^{3/4+3\beta} + \delta^{3/4+2\alpha+\beta} + \delta^{1/2+4\alpha})$$

whenever $\alpha < 0$.

In summary, we have an expansion for $p_1(x,y,\delta)$ given by (7.7), together with a remainder estimate of the form

$$|R_\delta(x,y)| \leq 150\Big(\delta^{3/4+3\beta} + \delta^{3/4+2(\alpha\wedge 0)+\beta} + \delta^{1/2+4\alpha}\Big), \tag{7.8}$$

valid for $0 < \delta \leq 1$ whenever $y = \delta^{1/2+\alpha}$ and $x = -\delta^{1/4+\beta}$ satisfy $y \leq -x \leq 1$. Moreover, by symmetry, the same bound holds for $|R_\delta(-x,-y)|$.

Now fix $\Delta > 0$. Choose $\delta_0 \leq 1$ such that

$$900\left(\delta_0^{1/4} \vee \delta_0^{3\Delta}\right) < (2\pi)^{-1/2}. \tag{7.9}$$

Let $\alpha \geq \Delta$ and $0 < \delta \leq \delta_0$. Set $\beta = \alpha$, $y = \delta^{1/2+\alpha}$, and $x = -\delta^{1/4+\beta}$. Note that by (5.1)–(5.7)

$$\tilde{\mu}(x,y,\delta) = p_1(x,y,\delta) - p_1(-x,-y,\delta),$$

so by (7.7)

$$\tilde{\mu} = \frac{2y}{\sqrt{2\pi}} + R_\delta(x,y) - R_\delta(-x,-y).$$

Since $\delta \leq 1$, we have $y \leq -x \leq 1$. Hence, by (7.8) and (7.9),

$$\begin{aligned}
|R_\delta(x,y) - R_\delta(-x,-y)| &\leq 300(2\delta^{3/4+\alpha} + \delta^{1/2+4\alpha})\\
&= 300(2\delta^{1/4} + \delta^{3\alpha})y\\
&\leq 300(2\delta_0^{1/4} + \delta_0^{3\Delta})y\\
&\leq 900\left(\delta_0^{1/4} \vee \delta_0^{3\Delta}\right) y < (2\pi)^{-1/2}y.
\end{aligned}$$

Therefore, $\tilde{\mu} \geq (2\pi)^{-1/2}y$, which proves (i).

For (ii), observe that $\tilde{\mu} > 0$ implies $q_1 < q_2$. Hence $\tilde{\varepsilon} = p_1q_2 + p_2q_1 \leq 2q_2$. Moreover,

$$\begin{aligned}
q_2 &= 1 - p_1(-x,-y,\delta)\\
&\leq \frac{1}{\pi}\tan^{-1}\sqrt{\delta} + \frac{|y|}{\sqrt{2\pi}} + \frac{\sqrt{\delta}}{2\pi}(x+y)^2 + \frac{y^2}{2\pi\sqrt{\delta}} + |R_\delta(-x,-y)|\\
&\leq \delta^{1/2} + \delta^{1/2+\alpha} + \delta^{1+2\alpha} + \delta^{1/2+2\alpha} + 150(\delta^{3/4+3\alpha} + \delta^{3/4+\alpha} + \delta^{1/2+4\alpha})\\
&\leq 500\,\delta^{1/2},
\end{aligned} \tag{7.10}$$

so $\tilde{\varepsilon} \leq 1{,}000\,\delta^{1/2}$. By making $\delta_0$ smaller if necessary we can ensure that $1{,}000\,\delta^{1/2} < 1/2$. □

**Lemma 7.2** *Let $p > 2$. Fix $0 < \Delta < 1/2$ and let $\delta_0$ be as in Lemma 7.1. Suppose $\varepsilon > 0$, $0 < \delta \leq \delta_0$, and $n \geq 3$ satisfy $\varepsilon/\sqrt{n} \leq \delta^{1/2+\Delta}$. Then*

$$P\left(M_n(1+\delta) - M_n(1) > \frac{\varepsilon}{\sqrt{n}}\right) \leq C(\varepsilon^{-1}\delta^{1/4})^p,$$

*where C depends only on p and $\Delta$.*

*Proof* Let $y = \varepsilon/\sqrt{n}$ and choose $\alpha \geq \Delta$ such that $y = \delta^{1/2+\alpha}$. Set $x_0 = -\delta^{1/4+\alpha}$. By Corollary 5.2, Lemma 5.3, Lemma 6.4, and Lemma 7.1,

$$P\left(M_n(1+\delta) - M_n(1) > \frac{\varepsilon}{\sqrt{n}}\right) \leq C_{p/2}\frac{\tilde{\varepsilon}^{p/2}}{(k-1)^{p/2}\tilde{\mu}^p} + C_p(\varepsilon^{-1}\delta^{1/4})^p,$$

where $\tilde{\varepsilon} = \tilde{\varepsilon}(x_0, y, \delta) \leq 1{,}000\,\delta^{1/2} < 1/2$ and

$$\tilde{\mu} = \tilde{\mu}(x_0, y, \delta) \geq \frac{1}{\sqrt{2\pi}}\,\delta^{1/2+\alpha} = \frac{1}{\sqrt{2\pi}} \cdot \frac{\varepsilon}{\sqrt{n}} > 0.$$

Hence,

$$\frac{\tilde{\varepsilon}^{p/2}}{(k-1)^{p/2}\tilde{\mu}^p} \leq C\frac{\delta^{p/4}}{n^{p/2}(\varepsilon/\sqrt{n})^p} = C(\varepsilon^{-1}\delta^{1/4})^p,$$

which completes the proof. □

## 8 The medium jump regime and final proof

Our analysis of the medium jump regime will require only minor modifications to the methods of Sect. 7.

**Lemma 8.1** *Fix $0 < \Delta < 1/16$ and set $\Delta' = (1-16\Delta)/12 > 0$. Then there exists $\delta_0 > 0$ such that*

(i) $\tilde{\mu}(-\delta^{1/4+\alpha}, \delta^{1/2+\alpha}, \delta) \geq \frac{1}{\sqrt{2\pi}}\,\delta^{1/2+\Delta}$, *and*

(ii) $\tilde{\varepsilon}(-\delta^{1/4+\alpha}, \delta^{1/2+\alpha}, \delta) \leq 1{,}000\,\delta^{1/2-4\Delta'} < \frac{1}{2}$

*for all $-\Delta' \leq \alpha \leq \Delta$ and all $0 < \delta \leq \delta_0$.*

*Proof* For fixed $0 < \Delta < 1/16$, choose $\delta_0 > 0$ as in Lemma 7.1. By (5.1), $p_1$ is increasing in $y$. Hence, if $x = -\delta^{1/4+\alpha}$ and $y = \delta^{1/2+\Delta}$, then by (7.7) and (7.8),

$$\begin{aligned}
\tilde{\mu}(x, \delta^{1/2+\alpha}, \delta) &= p_1(x, \delta^{1/2+\alpha}, \delta) - p_1(-x, -\delta^{1/2+\alpha}, \delta) \\
&\geq p_1(x, y, \delta) - p_1(-x, -y, \delta) \\
&= \frac{2y}{\sqrt{2\pi}} + R_\delta(x, y) - R_\delta(-x, -y),
\end{aligned}$$

where

$$|R_\delta(x,y) - R_\delta(-x,-y)| \le 300\Big(\delta^{3/4+3\alpha} + \delta^{3/4+\alpha} + \delta^{1/2+4\Delta}\Big)$$
$$\le 300(2\delta^{3/4-3\Delta'} + \delta^{1/2+4\Delta}).$$

However, note that $3/4 - 3\Delta' = 1/2 + 4\Delta$. Hence, by (7.9),

$$|R_\delta(x,y) - R_\delta(-x,-y)| \le 900\delta^{1/2+4\Delta} = 900\delta^{3\Delta} y < (2\pi)^{-1/2} y.$$

Therefore, $\tilde{\mu} \ge (2\pi)^{-1/2} y$, which proves (i).
For (ii), observe that $\tilde{\varepsilon} \le 2q_2$ and, as in (7.10),

$$\begin{aligned} q_2 &= 1 - p_1(-x, -\delta^{1/2+\alpha}, \delta) \\ &\le \delta^{1/2} + \delta^{1/2+\alpha} + \delta^{1+2\alpha} + \delta^{1/2+2\alpha} + 150(\delta^{3/4+3\alpha} + \delta^{3/4+\alpha} + \delta^{1/2+4\alpha}) \\ &\le 4\delta^{1/2-2\Delta'} + 150(2\delta^{3/4-3\Delta'} + \delta^{1/2-4\Delta'}) \\ &\le 500\,\delta^{1/2-4\Delta'}. \end{aligned}$$

Note that $1/2 - 4\Delta' > 1/6$, so that by making $\delta_0$ smaller if necessary, we can ensure that $1,000\,\delta^{1/2-4\Delta'} < 1/2$. □

**Lemma 8.2** *Fix $p > 2$. Let $\Delta = 1/18$ and choose $\delta_0 > 0$ as in Lemma 8.1. Suppose $\varepsilon > 0$, $0 < \delta \le \delta_0$, and $n \ge 3$ satisfy $\delta^{5/9} \le \varepsilon/\sqrt{n} \le \delta^{53/108}$. Then*

$$P\left(M_n(1+\delta) - M_n(1) > \frac{\varepsilon}{\sqrt{n}}\right) \le C(\varepsilon^{-1}\delta^{1/6})^p,$$

*where $C$ depends only on $p$.*

*Proof* Let $\Delta = 1/18$ and $\Delta' = (1-16\Delta)/12 = 1/108$ and observe that $y = \varepsilon/\sqrt{n} = \delta^{1/2+\alpha}$ for some $\alpha \in [-\Delta', \Delta]$. Set $x_0 = -\delta^{1/4+\alpha}$. By Corollary 5.2, Lemma 5.3, Lemma 6.4, and Lemma 8.1,

$$P\left(M_n(1+\delta) - M_n(1) > \frac{\varepsilon}{\sqrt{n}}\right) \le C_{p/2}\frac{\tilde{\varepsilon}^{p/2}}{(k-1)^{p/2}\tilde{\mu}^p} + C_p(\varepsilon^{-1}\delta^{1/4})^p,$$

where $\tilde{\varepsilon} = \tilde{\varepsilon}(x_0, y, \delta) \le 1,000\,\delta^{1/2-4\Delta'} < 1/2$ and

$$\tilde{\mu} = \tilde{\mu}(x_0, y, \delta) \ge \frac{1}{\sqrt{2\pi}}\,\delta^{1/2+\Delta} > 0.$$

Note that $n = \varepsilon^2 y^{-2} = \varepsilon^2\delta^{-1-2\alpha}$. Hence,

$$\begin{aligned} \frac{\tilde{\varepsilon}^{p/2}}{(k-1)^{p/2}\tilde{\mu}^p} &\le C(\tilde{\varepsilon}\varepsilon^{-2}\delta^{1+2\alpha}\tilde{\mu}^{-2})^{p/2} \le C(\delta^{1/2-4\Delta'}\varepsilon^{-2}\delta^{1-2\Delta'}\delta^{-1-2\Delta})^{p/2} \\ &= C(\varepsilon^{-2}\delta^{1/2-6\Delta'-2\Delta})^{p/2}. \end{aligned}$$

Since $1/2 - 6\Delta' - 2\Delta = 1/3$, this completes the proof. □

With the completion of our lemmas, we have made short work of the only proof that remains.

*Proof of Lemma 3.3* Take $\Delta = 1/108$ in Lemma 4.1 and, for each $p > 2$, let $C_{p,1}$ be the constant that appears in that lemma. Then take $\Delta = 1/18$ in Lemma 8.1. Let $\delta_0 > 0$ be as in that lemma and note that the conclusions of Lemmas 7.2 and 8.2 hold for this choice of $\delta_0$. For each $p > 2$, let $C_{p,2}$ be the larger of the constants appearing in those two lemmas and let $C_p = C_{p,1} \vee C_{p,2}$.

Now let $0 < \varepsilon < 1$, $0 < \delta \leq \delta_0$, and $n \geq 3$. Choose $\alpha > -1/2$ such that $\varepsilon/\sqrt{n} = \delta^{1/2+\alpha}$. If $\alpha \leq -1/108$, then by Lemma 4.1,

$$P\left(M_n(1+\delta) - M_n(1) > \frac{\varepsilon}{\sqrt{n}}\right) \leq C_{p,1}(\varepsilon^{-1}\delta^{1/4})^p \leq C_p(\varepsilon^{-1}\delta^{1/6})^p.$$

If $\alpha \geq 1/18$, then by Lemma 7.2,

$$P\left(M_n(1+\delta) - M_n(1) > \frac{\varepsilon}{\sqrt{n}}\right) \leq C_{p,2}(\varepsilon^{-1}\delta^{1/4})^p \leq C_p(\varepsilon^{-1}\delta^{1/6})^p.$$

If $-1/108 \leq \alpha \leq 1/18$, then by Lemma 8.2,

$$P\left(M_n(1+\delta) - M_n(1) > \frac{\varepsilon}{\sqrt{n}}\right) \leq C_{p,2}(\varepsilon^{-1}\delta^{1/6})^p \leq C_p(\varepsilon^{-1}\delta^{1/6})^p,$$

and we are done. □

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