Joint convergence along different subsequences of the signed cubic variation of fractional Brownian motion II

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Abstract

The purpose of this paper is to provide a complete description the convergence in distribution of two subsequences of the signed cubic variation of the fractional Brownian motion with Hurst parameter $H = 1/6$.

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1 Introduction

Suppose that $B = \{B(t), t \geq 0\}$ is a fractional Brownian motion with Hurst parameter $H = \frac{1}{6}$. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. In [6], Nualart and Ortiz-Latorre proved that the sequence of sums,

$$W_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} (B(j/n) - B((j-1)/n))^3,$$

converges in law to a Brownian motion $W = \{W(t), t \geq 0\}$, with variance $\kappa^2 t$ given by

$$\kappa^2 = \frac{3}{4} \sum_{m \in \mathbb{Z}} (|m+1|^{1/3} + |m-1|^{1/3} - 2|m|^{1/3})^3.$$

The process W is related to the signed cubic variation of B. A detailed analysis of this process has been recently developed by Swanson in [8], considering this variation as a class of sequences of processes.

In [1], Burdzy, Nualart and Swanson studied the convergence in distribution of the sequence of two-dimensional processes $\{(W_{a_n}(t), W_{b_n}(t))\}$, where $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are two strictly increasing sequences of natural numbers converging to infinity. A basic assumption for the results of [1] and also for the results of this paper is that $L_n \to L \in [0, \infty]$, where $L_n = b_n/a_n$. By [1, Corollary 3.6], if $L \in \{0, \infty\}$, then W_{a_n} and W_{b_n} converge to independent Brownian motions. We will therefore assume that $L \in (0, \infty)$.

The function $f_L(x) = \sum_{m\in\mathbb{Z}} f_{m,L}(x)$, where

$$f_{m,L}(x) = (|x-m+1|^{1/3} + |x-m-L|^{1/3} - |x-m|^{1/3} - |x-m+1-L|^{1/3})^3, \qquad (1.1)$$

plays a fundamental role in the analysis of the convergence in distribution of $\{(W_{a_n}(t), W_{b_n}(t))\}$. Under some conditions, the limit of this sequence is a two-dimensional Gaussian process X^ρ, independent of B, whose components are Brownian motions with variance $\kappa^2 t$, and with covariance $\int_0^t \rho(s)\,ds$ for some function ρ. In terms of the function $\rho \in C[0,\infty)$, the process X^ρ can be expressed as

$$X^\rho(t) = \int_0^t \sigma(s)\,d\mathbf{W}(s), \qquad (1.2)$$

where σ is given by

$$\sigma(t) = \kappa \begin{pmatrix} \sqrt{1-|\kappa^{-2}\rho(t)|^2} & \kappa^{-2}\rho(t) \\ 0 & 1 \end{pmatrix}, \qquad (1.3)$$

and $\mathbf{W} = (W^1, W^2)$ is a standard, 2-dimensional Brownian motion. More specifically, the main result of [1] is the following theorem, which is obtained using the central limit theorem for multiple stochastic integrals proved by Peccati and Tudor in [7] (see also [3]).

Theorem 1.1. *Let $I = \{n : L_n = L\}$ and $c_n = \gcd(a_n, b_n)$. Then $(B, W_{a_n}, W_{b_n}) \Rightarrow (B, X^\rho)$ in the Skorohod space $D_{\mathbb{R}^3}[0,\infty)$ as $n \to \infty$, in the following cases:*

(i) The set I^c is finite (which implies $L \in \mathbb{Q}$). In this case, if $L = p/q$, where $p, q \in \mathbb{N}$ are relatively prime, then for all $t \ge 0$,

$$\rho(t) = \frac{3}{4p} \sum_{j=1}^{q} f_L(j/q).$$

(ii) There exists $k \in \mathbb{N}$ such that $b_n = k \mod a_n$ for all n. In this case, for all $t \ge 0$,

$$\rho(t) = \frac{3}{4L} f_L(kt).$$

(iii) The set I is finite and $c_n \to \infty$. In this case, for all $t \ge 0$,

$$\rho(t) = \frac{3}{4L} \int_0^1 f_L(x)\,dx.$$

This type of result was motivated by the relationship between higher signed variations of fractional Brownian motions and the change of variable formulas in distribution for stochastic integrals with respect to these processes that have appeared recently in the literature (see [2, 4, 5]).

Theorem 1.1 covers many simple and interesting pairs of sequences, and helps to tell a surprising story about the asymptotic correlation between the sequences, $\{W_{a_n}\}$ and $\{W_{b_n}\}$, both of which are converging to a Brownian motion. For example, by Theorem 1.1(i), we may conclude that the asymptotic correlation of $W_n(t)$ and $W_{2n}(t)$ is a constant that does not depend on t, and whose numerical value is approximately 0.201. Likewise, Theorem 1.1(iii) shows that the asymptotic correlation of $W_{n^2}(t)$ and $W_{n(n+1)}(t)$ is not dependent on t and is approximately 0.102. Perhaps more surprisingly, Theorem 1.1(ii) shows that the asymptotic correlation of $W_n(t)$ and $W_{n+1}(t)$ *does* depend on t. Numerical calculations suggest that the correlation varies greatly with t, converging to 1 as $t \downarrow 0$, and being as low as about 0.075 for $t = 0.8$.

Nonetheless, there are many simple and interesting pairs of sequences that are *not* covered by Theorem 1.1. For example, the sequences $a_n = n^2$ and $b_n = (n+1)^2$ are not covered; nor are the sequences $a_n = 2n$ and $b_n = 3n + 1$. Additionally, many sequences whose ratios converge to an irrational number are not covered by this theorem.

The purpose of this paper is to provide a complete description of the asymptotic behavior of $W_{a_n}(t)$ and $W_{b_n}(t)$ for all sequences $\{a_n\}$ and $\{b_n\}$. We will show that the asymptotic correlation depends only on $L = \lim L_n$ when L is irrational; and when L is rational, it depends also on $\lim a_n|L_n - L|$. In the next section we state and prove this result and provide some remarks and examples.

2 Main result

Let X^ρ the two-dimensional process defined in (1.2). Recall that $f_L(x) = \sum_{m\in\mathbb{Z}} f_{m,L}(x)$, where $f_{m,L}$ is the function defined in (1.1). By [1, Lemma 2.6], the series defining f_L converges uniformly on $[0, 1]$. Also note that f_L is periodic with period 1. We first need the following technical result.

Lemma 2.1. *Let $L = p/q$, where $p, q \in \mathbb{N}$ are relatively prime numbers. Then, for any $x \in \mathbb{R}$ and $\eta = 1, \ldots, q$ we have $f_L(\eta L - x) = f_L(\widetilde{\eta} L + x)$, where $\widetilde{\eta} = q - \eta + 1$.*

Proof. For any $m \in \mathbb{Z}$ set $\widetilde{m} = -m + 1 + p$. Then

$$\begin{aligned} f_{m,L}(\eta L - x) &= \Bigg(\left|\frac{\eta p}{q} - x - m + 1\right|^{1/3} + \left|\frac{\eta p}{q} - x - m - \frac{p}{q}\right|^{1/3} \\ &\qquad - \left|\frac{\eta p}{q} - x - m\right|^{1/3} - \left|\frac{\eta p}{q} - x - m + 1 - \frac{p}{q}\right|^{1/3} \Bigg)^3 \\ &= \Bigg(\left| -\frac{\eta p}{q} + x - \widetilde{m} + p\right|^{1/3} + \left| -\frac{\eta p}{q} + x - \widetilde{m} + 1 + p + \frac{p}{q}\right|^{1/3} \\ &\qquad - \left| -\frac{\eta p}{q} + x - \widetilde{m} + 1 + p\right|^{1/3} - \left| -\frac{\eta p}{q} + x - \widetilde{m} + p + \frac{p}{q}\right|^{1/3} \Bigg)^3 . \end{aligned}$$

Notice that $\widetilde{\eta} L = p + \frac{p}{q} - \frac{\eta p}{q}$. Therefore,

$$\begin{aligned} f_{m,L}(\eta L - x) &= \big(|\widetilde{\eta} L + x - \widetilde{m} - L|^{1/3} + |\widetilde{\eta} L + x - \widetilde{m} + 1|^{1/3} \\ &\qquad - |\widetilde{\eta} L + x - \widetilde{m} - L + 1|^{1/3} - |\widetilde{\eta} L + x - \widetilde{m}|^{1/3} \big)^3 \\ &= f_{\widetilde{m},L}(\widetilde{\eta} L + x). \end{aligned}$$

As a consequence,

$$f_L(\eta L - x) = \sum_{m\in\mathbb{Z}} f_{m,L}(\eta L - x) = \sum_{m\in\mathbb{Z}} f_{\widetilde{m},L}(\widetilde{\eta} L + x) = \sum_{\widetilde{m}\in\mathbb{Z}} f_{\widetilde{m},L}(\widetilde{\eta} L + x) = f_L(\widetilde{\eta} L + x),$$

which completes the proof. □

The next result is the main theorem of this paper. Together with the cases $L = 0$ and $L = \infty$, covered in [1, Corollary 3.6], this theorem gives a complete description of all subsequential limits of (W_{a_n}, W_{b_n}) for any pair of subsequences of $\{W_n\}$.

Theorem 2.2. *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be strictly increasing sequences in $\mathbb{N}$. Let $L_n = b_n/a_n$ and suppose $L_n \to L \in (0,\infty)$. Let $\delta_n = L_n - L$. Then, $(B, W_{a_n}, W_{b_n}) \Rightarrow (B, X^\rho)$ in $D_{\mathbb{R}^3}[0,\infty)$ as $n \to \infty$, in the following cases:*

(i) $L \in \mathbb{Q}$ and $a_n|\delta_n| \to k \in [0,\infty)$. In this case, if we write $L = p/q$, where $p, q \in \mathbb{N}$ are relatively prime, then, for all $t \geq 0$,

$$\rho(t) = \frac{3}{4p} \sum_{j=1}^{q} f_L\left(\frac{j}{q} + kt\right).$$

(ii) $L \in \mathbb{Q}$ and $a_n|\delta_n| \to \infty$, or $L \notin \mathbb{Q}$. In this case, for all $t \geq 0$,

$$\rho(t) = \frac{3}{4L} \int_0^1 f_L(x)\,dx.$$

Note that between the two parts of this theorem, there is, at least formally, a sort of continuity in k. For fixed q, since f_L is periodic with period 1, we have

$$\int_0^t \left[\frac{3}{4p} \sum_{j=1}^{q} f_L\left(\frac{j}{q} + ks\right)\right] ds \to \int_0^t \left[\frac{3}{4L} \int_0^1 f_L(x)\,dx\right] ds,$$

as $k \to \infty$.

To elaborate on the conditions in the two parts of this theorem and their connections to Theorem 1.1, first note that if L is rational and $L_n \neq L$, then

$$a_n|\delta_n| = \left|\frac{b_n q - a_n p}{q}\right| \geq \frac{1}{q}, \tag{2.1}$$

since the numerator is a nonzero integer. It follows that when $L \in \mathbb{Q}$, we have $a_n|\delta_n| \to 0$ if and only if $L_n = L$ for all but finitely many n. Therefore, Theorem 2.2(i) with $k = 0$ is equivalent to Theorem 1.1(i).

Next, if $L \in \mathbb{Q}$, $L_n \neq L$ for all but finitely many n, and $c_n = \gcd(a_n, b_n) \to \infty$, then (2.1) shows that for n sufficiently large, $a_n|\delta_n| \geq c_n/q \to \infty$. Hence, Theorem 1.1(iii) is a special case of Theorem 2.2(ii).

Lastly, to see that Theorem 1.1(ii) is a special case of Theorem 2.2(i), suppose there exists $k \in \mathbb{N}$ such that $b_n = k \mod a_n$ for all n. Then $b_n = \nu_n a_n + k$ for some integers ν_n.

Thus, $L_n = \nu_n + k/a_n$. Letting $n \to \infty$ shows that $L \in \mathbb{N}$ and $\nu_n = L$ for all but finitely many n. We therefore have $a_n|\delta_n| = |b_n - a_n L| = k$, for large enough n. In this case, using $p = L$ and $q = 1$ and the fact that f_L is periodic with period 1, we find that the function ρ in Theorem 2.2(i) agrees with the function ρ in Theorem 1.1(ii).

Before giving the formal proof of Theorem 2.2 we would like to explain the main ideas in comparison with the proof of Theorem 1.1. Let $\{x\} = x - \lfloor x \rfloor$. In [1], it is shown that the covariance between the components of the limit process X^ρ is given by

$$\int_0^t \rho(s)\, ds = \frac{3}{4L} \sum_{m \in \mathbb{Z}} \lim_{n\to\infty} \frac{1}{a_n} \sum_{j=1}^{\lfloor a_n t \rfloor} f_{m,L}(\{jL_n\}),$$

provided the above limits exist for each $m \in \mathbb{Z}$. The principal challenge in analyzing these limits has been that the above summands, $f_{m,L}(\{jL_n\})$, could not be replaced by $f_{m,L}(\{jL\})$. This is because, although L_n is close to L for large n, $\{jL_n\}$ is not uniformly close to $\{jL\}$ as j ranges from 1 to $\lfloor a_n t \rfloor$. In [1], we studied these limits via the decomposition

$$\sum_{j=1}^{\lfloor a_n t \rfloor} f_{m,L}(\{jL_n\}) = \alpha_n \sum_{j=0}^{q_n - 1} f_{m,L}(\{jL_n\}) + \sum_{j=1}^{r_n} f_{m,L}(\{jL_n\}).$$

Here, $b_n/a_n = p_n/q_n$, where p_n and q_n are relatively prime, and $\lfloor a_n t \rfloor = \alpha_n q_n + r_n$ with $\alpha_n \in \mathbb{Z}$ and $0 \le r_n < q_n$.

To prove Theorem 2.2 in the case that $L \in \mathbb{Q}$, we use a different decomposition. Let $L = p/q$, where $p, q \in \mathbb{N}$ are relatively prime. We then write

$$\sum_{j=1}^{\lfloor a_n t \rfloor} f_{m,L}(\{jL_n\}) \approx \sum_{\eta=1}^{q} \sum_{i=0}^{\alpha_n - 1} f_{m,L}(\{(iq + \eta)L_n\}).$$

In this case, since q is fixed and finite, we are able to use the approximation

$$\sum_{j=1}^{\lfloor a_n t \rfloor} f_{m,L}(\{jL_n\}) \approx \sum_{\eta=1}^{q} \sum_{i=0}^{\alpha_n - 1} f_{m,L}(\{iqL_n + \eta L\}).$$

Since $qL = p$, we have $iqL_n = ip + iq\delta_n$. Thus, we have

$$\sum_{j=1}^{\lfloor a_n t \rfloor} f_{m,L}(\{jL_n\}) \approx \sum_{\eta=1}^{q} \sum_{i=0}^{\alpha_n - 1} f_{m,L}(\{iq\delta_n + \eta L\}).$$

Using a Riemann-sum argument, we will show that for each fixed η,

$$\sum_{i=0}^{\alpha_n - 1} f_{m,L}(\{iq\delta_n + \eta L\}) \approx \frac{1}{q\delta_n} \int_0^{a_n \delta_n t} f_{m,L}(\{x + \eta L\})\, dx,$$

giving

$$\int_0^t \rho(s)\, ds = \frac{3}{4L} \sum_{m \in \mathbb{Z}} \frac{1}{q} \sum_{\eta=1}^{q} \lim_{n\to\infty} \frac{1}{a_n \delta_n} \int_0^{a_n \delta_n t} f_{m,L}(\{x + \eta L\})\, dx.$$

We then prove the theorem case-by-case, depending on the asymptotic behavior of the sequence $a_n\delta_n$. Note that the actual analysis in the proof is made somewhat more delicate by the fact that δ_n may be negative.

For the case $L \notin \mathbb{Q}$, the proof will be done by adapting the method of proof of the equidistribution theorem based on Fourier series expansions. This theorem says that for any interval $I \subset [0,1)$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k : \{kL\} \in I, 1 \le k \le n\}| = |I|,$$

and a simple proof can be found in [9, Theorem 1.8].

Proof of Theorem 2.2. Let

$$S_n(t) = E[W_{a_n}(t)W_{b_n}(t)].$$

By [1, Theorem 3.1 and Lemma 3.5], it will suffice to show that

$$S_n(t) \to \int_0^t \rho(s)\,ds, \tag{2.2}$$

for each $t \ge 0$.

Fix $t \ge 0$. Since $W_n(t) = 0$ if $\lfloor nt \rfloor = 0$, we may assume $t > 0$ and n is sufficiently large so that $\lfloor a_n t \rfloor > 0$ and $\lfloor b_n t \rfloor > 0$. Recall that $\{x\} = x - \lfloor x \rfloor$, and let $\widehat{f}_{m,L}(x) = f_{m,L}(\{x\})$, where $f_{m,L}$ is the function introduced in (1.1).

In the reference [1] it is proved (see [1, (3.18), (3.20), and Remark 3.3]) that

$$\lim_{n\to\infty} S_n(t) = \frac{3}{4L} \sum_{m\in\mathbb{Z}} \lim_{n\to\infty} \widetilde{\beta}(m,n), \tag{2.3}$$

where

$$\widetilde{\beta}(m,n) = \frac{1}{a_n} \sum_{j=1}^{\lfloor a_n t \rfloor} \widehat{f}_{m,L}(jL_n),$$

provided that, for each fixed $m \in \mathbb{Z}$, the limit $\lim_{n\to\infty} \widetilde{\beta}(m,n)$ exists. The proof will now be done in several steps.

Step 1. Assume $L \in \mathbb{Q}$ and $a_n|\delta_n| \to k \in (0,\infty]$. Let us write $L = p/q$, where p and q are relatively prime. Choose n_0 such that for all $n \ge n_0$, we have $\lfloor a_n t \rfloor > q$. For each $n \ge n_0$, write $\lfloor a_n t \rfloor = \alpha_n q + r_n$, where $\alpha_n \in \mathbb{N}$ and $0 \le r_n < q$. Since $a_n \to \infty$ and $\widehat{f}_{m,L}$ is bounded, it follows that

$$\begin{aligned}
\lim_{n\to\infty} \widetilde{\beta}(m,n) &= \lim_{n\to\infty} \frac{1}{a_n} \sum_{j=1}^{\alpha_n q} \widehat{f}_{m,L}(jL_n) \\
&= \lim_{n\to\infty} \frac{1}{a_n} \sum_{\eta=1}^{q} \sum_{i=0}^{\alpha_n - 1} \widehat{f}_{m,L}((iq+\eta)L_n) \\
&= \lim_{n\to\infty} \frac{1}{a_n} \sum_{\eta=1}^{q} \sum_{i=0}^{\alpha_n - 1} \widehat{f}_{m,L}(ip + \eta L + (iq+\eta)\delta_n) \\
&= \sum_{\eta=1}^{q} \left(\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=0}^{\alpha_n - 1} \widehat{f}_{m,L}(\eta L + \operatorname{sgn}(\delta_n) x_i) \right),
\end{aligned}$$

where $x_i = (iq + \eta)|\delta_n|$. Our assumption that $a_n|\delta_n| \to k \in (0, \infty]$ implies that there exists $n_1 \geq n_0$ such that $\delta_n \neq 0$ for all $n \geq n_1$. Set $\Delta x = x_{i+1} - x_i = q|\delta_n|$. Then

$$\lim_{n\to\infty} \widetilde{\beta}(m,n) = \frac{1}{q}\sum_{\eta=1}^{q}\left(\lim_{n\to\infty}\frac{1}{a_n|\delta_n|}\sum_{i=0}^{\alpha_n-1}\widehat{f}_{m,L}(\eta L + \operatorname{sgn}(\delta_n)x_i)\Delta x\right).$$

Let $\varepsilon > 0$ be arbitrary. Since $f_{m,L}$ is continuous, we may find $n_2 \geq n_1$ such that for all $n \geq n_2$,

$$\sup_{\substack{|x-y|\leq \Delta x \\ x,y\in[0,1]}} |f_{m,L}(x) - f_{m,L}(y)| < \varepsilon.$$

Note that if $\lfloor x \rfloor = \lfloor y \rfloor$, then $\{x\} - \{y\} = x - y$. Thus,

$$\sup_{\substack{|x-y|\leq \Delta x \\ \lfloor x \rfloor = \lfloor y \rfloor}} |\widehat{f}_{m,L}(x) - \widehat{f}_{m,L}(y)| < \varepsilon, \tag{2.4}$$

for all $n \geq n_2$. Let

$$J_n = \{0 \leq i < \alpha_n : \lfloor \eta L + \operatorname{sgn}(\delta_n)x_i \rfloor = \lfloor \eta L + \operatorname{sgn}(\delta_n)x_{i+1} \rfloor\}.$$

Note that if $i \in J_n$ and $x \in [x_i, x_{i+1}]$, then

$$\lfloor \eta L + \operatorname{sgn}(\delta_n)x \rfloor = \lfloor \eta L + \operatorname{sgn}(\delta_n)x_i \rfloor.$$

Thus, using (2.4), we obtain

$$\left|\widehat{f}_{m,L}(\eta L + \operatorname{sgn}(\delta_n)x_i)\Delta x - \int_{x_i}^{x_{i+1}} \widehat{f}_{m,L}(\eta L + \operatorname{sgn}(\delta_n)x)\,dx\right| \leq \varepsilon \Delta x = \varepsilon q|\delta_n|,$$

for all $i \in J_n$ and $n \geq n_2$. Also, since $\widehat{f}_{m,L}$ is bounded, there is a constant M such that

$$\left|\widehat{f}_{m,L}(\eta L + \operatorname{sgn}(\delta_n)x_i)\Delta x - \int_{x_i}^{x_{i+1}} \widehat{f}_{m,L}(\eta L + \operatorname{sgn}(\delta_n)x)\,dx\right| \leq M\Delta x = Mq|\delta_n|,$$

for all $i \notin J_n$ and $n \geq n_2$. Therefore,

$$\sum_{i=0}^{\alpha_n-1}\widehat{f}_{m,L}(\eta L + \operatorname{sgn}(\delta_n)x_i)\Delta x = \int_{x_0}^{x_{\alpha_n}} \widehat{f}_{m,L}(\eta L + \operatorname{sgn}(\delta_n)x)\,dx + R_n,$$

where

$$|R_n| \leq (\varepsilon|J| + M(\alpha_n - |J|))q|\delta_n|.$$

Note that $\alpha_n - |J|$ is the number of times that the monotonic sequence $\{\eta L + \operatorname{sgn}(\delta_n)x_i\}_{i=0}^{\alpha_n}$ crosses an integer. Thus, $\alpha_n - |J| \leq |x_{\alpha_n} - x_0| + 1 = \alpha_n q|\delta_n| + 1$. Combined with $|J| \leq \alpha_n$ and $\alpha_n \leq a_n t/q$, we have

$$|R_n| \leq \varepsilon a_n|\delta_n|t + Mqa_n|\delta_n|^2 t + Mq|\delta_n|.$$

Hence, since $a_n \to \infty$, we have

$$\limsup_{n\to\infty} \frac{|R_n|}{a_n|\delta_n|} \le \varepsilon t.$$

Since ε was arbitrary, it follows that

$$\lim_{n\to\infty} \widetilde{\beta}(m,n) = \frac{1}{q}\sum_{\eta=1}^{q}\left(\lim_{n\to\infty}\frac{1}{a_n|\delta_n|}\int_{x_0}^{x_{\alpha_n}} \widehat{f}_{m,L}(\eta L + \operatorname{sgn}(\delta_n)x)\,dx\right).$$

Now, note that $x_0 = \eta|\delta_n|$ and

$$x_{\alpha_n} = (\alpha_n q + \eta)|\delta_n| = (\lfloor a_n t\rfloor - r_n + \eta)|\delta_n|.$$

Since $|\eta - r_n| \le q$, we have $|x_{\alpha_n} - a_n|\delta_n|t| \le (q+1)|\delta_n|$. Thus, since $a_n \to \infty$ and $\widehat{f}_{m,L}$ is bounded, we have

$$\lim_{n\to\infty} \widetilde{\beta}(m,n) = \frac{1}{q}\sum_{\eta=1}^{q}\left(\lim_{n\to\infty}\frac{1}{a_n|\delta_n|}\int_{0}^{a_n|\delta_n|t} \widehat{f}_{m,L}(\eta L + \operatorname{sgn}(\delta_n)x)\,dx\right). \tag{2.5}$$

Step 2. Assume $L \in \mathbb{Q}$ and $a_n|\delta_n| \to \infty$. Then, taking into account that the function $\widehat{f}_{m,L}$ has period one, we can write

$$\int_0^{a_n|\delta_n|t} \widehat{f}_{m,L}(\eta L + \operatorname{sgn}(\delta_n)x)\,dx$$
$$= \lfloor a_n|\delta_n|t\rfloor \int_0^1 \widehat{f}_{m,L}(x)\,dx + \int_0^{a_n|\delta_n|t - \lfloor a_n|\delta_n|t\rfloor} \widehat{f}_{m,L}(\eta L + \operatorname{sgn}(\delta_n)x)\,dx.$$

From (2.5) and the fact that $\widehat{f}_{m,L}$ is bounded, we then obtain

$$\lim_{n\to\infty} \widetilde{\beta}(m,n) = t\int_0^1 \widehat{f}_{m,L}(x)\,dx.$$

By (2.3) and the fact that $f_L = \sum_{m\in\mathbb{Z}} f_{m,L}$ is periodic with period 1, this gives

$$\lim_{n\to\infty} S_n(t) = \frac{3t}{4L}\int_0^1 f_L(x)\,dx.$$

In light of (2.2), this completes half the proof of Theorem 2.2(ii). To complete the proof of Theorem 2.2(ii), it remains only to consider the case $L \notin \mathbb{Q}$, and this will be done in the final step of this proof.

Step 3. Assume $L \in \mathbb{Q}$, $a_n|\delta_n| \to k \in (0,\infty)$, and $\delta_n > 0$ for all n. From (2.5), we have

$$\lim_{n\to\infty} \widetilde{\beta}(m,n) = \frac{1}{q}\sum_{\eta=1}^{q}\left(\frac{1}{k}\int_0^{kt} \widehat{f}_{m,L}(\eta L + x)\,dx\right).$$

From (2.3), the fact that f_L has period 1, the identity $L = p/q$, and the substitution $x = ks$, this gives

$$\begin{aligned}\lim_{n\to\infty} S_n(t) &= \frac{3}{4Lk}\int_0^{kt} \frac{1}{q}\sum_{\eta=1}^{q} f_L(\eta L + x)\,dx \\ &= \int_0^t \frac{3}{4p}\sum_{\eta=1}^{q} f_L(\eta L + ks)\,ds.\end{aligned}$$

Step 4. Assume $L \in \mathbb{Q}$, $a_n|\delta_n| \to k \in (0,\infty)$, and $\delta_n < 0$ for all n. As in Step 3, we have

$$\lim_{n\to\infty} S_n(t) = \int_0^t \frac{3}{4p}\sum_{\eta=1}^{q} f_L(\eta L - ks)\,ds.$$

By Lemma 2.1,

$$\begin{aligned}\lim_{n\to\infty} S_n(t) &= \int_0^t \frac{3}{4p}\sum_{\eta=1}^{q} f_L((q-\eta+1)L + ks)\,ds \\ &= \int_0^t \frac{3}{4p}\sum_{\eta=1}^{q} f_L(\eta L + ks)\,ds.\end{aligned}$$

Step 5. We now prove Theorem 2.2(i). From the discussion following the statement of Theorem 2.2(i), we have that Theorem 2.2(i) with $k = 0$ is equivalent to Theorem 1.1(i). Thus, we may assume $a_n|\delta_n| \to k \in (0,\infty)$. Let $\{S_{n_m}\}$ be any subsequence of $\{S_n\}$. Recall from Step 1 that $\delta_n \neq 0$ for all $n \geq n_1$. Choose a subsequence $\{S_{n_{m(j)}}\}$ of $\{S_{n_m}\}$ such that $\operatorname{sgn}(\delta_{n_{m(j)}})$ does note depend on j. By Steps 3 and 4,

$$\lim_{j\to\infty} S_{n_{m(j)}}(t) = \int_0^t \frac{3}{4p}\sum_{\eta=1}^{q} f_L(\eta L + ks)\,ds.$$

Since every subsequence has a subsequence converging to this limit, it follows that

$$\lim_{n\to\infty} S_n(t) = \int_0^t \frac{3}{4p}\sum_{\eta=1}^{q} f_L(\eta L + ks)\,ds.$$

Note that $\eta L = \eta p/q$ and, since p and q are relatively prime,

$$\{\eta p/q : 1 \leq \eta \leq q\} = \{j/q : 1 \leq j \leq q\}.$$

Thus,

$$\lim_{n\to\infty} S_n(t) = \int_0^t \frac{3}{4p}\sum_{j=1}^{q} f_L\left(\frac{j}{q} + ks\right)\,ds.$$

By (2.2), this completes the proof of Theorem 2.2(i).

Step 6. We now prove Theorem 2.2(ii). From Step 2, it suffices to consider $L \notin \mathbb{Q}$. As in the proof of the equidistribution theorem, the idea is to approximate the function $f_{m,L}$ by its truncated Fourier series.

Fix $m \in \mathbb{Z}$ and let $\varepsilon > 0$ be arbitrary. Set

$$F_N(x) = \sum_{k=-N}^{N} c_k e^{2\pi i k x},$$

where

$$c_k = \int_0^1 f_{m,L}(y) e^{2\pi i k y} dy.$$

Since $\|f_{m,L}\|_\infty \le 8$ (see [1, (2.23)]), we have $|c_k| \le 8$.

The function $f_{m,L}$ is Hölder continuous of order $1/3$. Therefore, by Jackson's theorem, the sequence F_N converges uniformly on $[0,1]$ to $f_{m,L}$, and we may choose $N \in \mathbb{N}$ such that for all $x \in [0,1]$,

$$|F_N(x) - f_{m,L}(x)| < \varepsilon.$$

Recalling that $\{x\} = x - \lfloor x \rfloor$, we then have for any fixed $t > 0$,

$$\begin{aligned}
\widetilde{\beta}(m,n) &= \frac{1}{a_n} \sum_{j=1}^{\lfloor a_n t \rfloor} f_{m,L}(\{jL_n\}) = \frac{1}{a_n} \sum_{j=1}^{\lfloor a_n t \rfloor} F_N(\{jL_n\}) + O(\varepsilon) \\
&= \frac{1}{a_n} \sum_{k=-N}^{N} c_k \sum_{j=1}^{\lfloor a_n t \rfloor} e^{2\pi i k j L_n} + O(\varepsilon).
\end{aligned}$$

In the above and for the remainder of this proof, the coefficients implied by the big O notation depend only on t.

Note that for any integer $M \ge 1$ and for any complex number α,

$$\sum_{j=1}^{M} \alpha^j = \begin{cases} \frac{\alpha(1-\alpha^M)}{1-\alpha} & \text{if } \alpha \ne 1, \\ M & \text{if } \alpha = 1. \end{cases} \tag{2.6}$$

Set $\sigma_{k,n} = \sum_{j=1}^{\lfloor a_n t \rfloor} e^{2\pi i k j L_n}$. Then, $\sigma_{k,n} = \lfloor a_n t \rfloor$ if $kL_n \in \mathbb{Z}$. If $kL_n \notin \mathbb{Z}$, then

$$|\sigma_{k,n}| = \left| e^{2\pi i k L_n} \frac{1 - e^{2\pi i k L_n \lfloor a_n t \rfloor}}{1 - e^{2\pi i k L_n}} \right| \le \frac{2}{|1 - e^{2\pi i k L_n}|}.$$

Since L_n converges to L which is irrational, there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ and for all $k \in \{-N, \dots, N\}$, we have $|1 - e^{2\pi i k L_n}| \ge \delta$. Therefore, $|\sigma_{k,n}| \le 2/\delta$ whenever $n \ge n_0$, $k \in \{-N, \dots, N\}$, and $kL_n \notin \mathbb{Z}$.

Recall that $L_n = p_n/q_n$, where p_n and q_n are relatively prime numbers. Hence, $kL_n \in \mathbb{Z}$ if and only if $q_n \mid k$. Therefore, we obtain

$$\begin{aligned}
\widetilde{\beta}(m,n) &= \frac{1}{a_n} \sum_{k=-N}^{N} c_k \sigma_{k,n} + O(\varepsilon) \\
&= \frac{1}{a_n} \sum_{\substack{k=-N \\ q_n \mid k}}^{N} c_k \lfloor a_n t \rfloor + R_n + O(\varepsilon),
\end{aligned}$$

where

$$|R_n| = \left| \frac{1}{a_n} \sum_{\substack{k=-N \\ q_n \nmid k}}^{N} c_k \sigma_{k,n} \right| \leq \frac{2}{a_n \delta} \sum_{k=-N}^{N} |c_k| = O(N a_n^{-1} \delta^{-1}).$$

By (2.6),

$$\frac{1}{q_n} \sum_{j=1}^{q_n} (e^{2\pi ik/q_n})^j = \begin{cases} 1 & \text{if } q_n \mid k, \\ 0 & \text{if } q_n \nmid k. \end{cases}$$

As a consequence, we can write

$$\begin{aligned} \widetilde{\beta}(m,n) &= \frac{\lfloor a_n t \rfloor}{a_n} \sum_{k=-N}^{N} c_k \frac{1}{q_n} \sum_{j=1}^{q_n} e^{2\pi ikj/q_n} + O(N a_n^{-1} \delta^{-1}) + O(\varepsilon) \\ &= \frac{\lfloor a_n t \rfloor}{a_n q_n} \sum_{j=1}^{q_n} F_N(j/q_n) + O(N a_n^{-1} \delta^{-1}) + O(\varepsilon) \\ &= \frac{\lfloor a_n t \rfloor}{a_n q_n} \sum_{j=1}^{q_n} f_{m,L}(j/q_n) + O(N a_n^{-1} \delta^{-1}) + O(\varepsilon). \end{aligned}$$

In [1], it is shown that $q_n \to \infty$ when $L \notin \mathbb{Q}$. Thus, letting n tend to infinity and using the fact that $f_{m,L}$ is Riemann integrable on $[0,1]$ gives

$$\limsup_{n\to\infty} \left| \widetilde{\beta}(m,n) - t \int_0^1 f_{m,L}(x)\,dx \right| = O(\varepsilon).$$

Since ε was arbitrary, and from (2.3) and (2.2), this completes the proof. □

Examples. Here are some examples that were not covered by the results of [1]. Suppose that $a_n = n^2$ and $b_n = (n+1)^2$. In this case $L_n \to 1$ and $a_n|\delta_n| = |b_n - a_n L| = 2n+1 \to \infty$. Therefore,

$$\rho(t) = \frac{3}{4} \int_0^1 f_1(x)\,dx.$$

If $a_n = 2n$ and $b_n = 3n+1$, then $L_n \to 3/2$ and $a_n|\delta_n| = |b_n - a_n L| = 1$ for all n. Therefore,

$$\rho(t) = \frac{1}{4}\left(f_{3/2}\left(\frac{1}{2} + t\right) + f_{3/2}(t) \right).$$

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