Probab. Theory Relat. Fields
DOI 10.1007/s00440-013-0511-2

Joint convergence along different subsequences of the signed cubic variation of fractional Brownian motion

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Received: 4 October 2012 / Revised: 24 April 2013

Abstract The purpose of this paper is to study the convergence in distribution of two subsequences of the signed cubic variation of the fractional Brownian motion with Hurst parameter $H = 1/6$. We prove that, under some conditions on both subsequences, the limit is a two-dimensional Brownian motion whose components may be correlated and we find explicit formulae for its covariance function.

Keywords Fractional Brownian motion · Cubic variation · Convergence in law

Mathematics Subject Classification Primary 60G22; Secondary 60F17

1 Introduction

Suppose that $B = \{B(t), t \geq 0\}$ is a fractional Brownian motion with Hurst parameter $H = \frac{1}{2k}$, where k is an odd number. It is well-known that the sequence of sums,

$$W_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} (B(j/n) - B((j-1)/n))^k, \tag{1.1}$$

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Published online: 22 June 2013

converges in law to a Brownian motion $W = \{W(t), t \geq 0\}$, with variance $\sigma_k^2 t$, independent of B. This result can be seen as a corollary of the Breuer-Major theorem (see [1] or Chapter 7 of [4]), and a proof is also given by Nualart and Ortiz-Latorre in [8]. The Brownian motion W is called the k-signed variation of B. In the particular case $k = 3$, the variance, denoted by $\kappa^2 t$, is given in formula (2.1) below. A detailed analysis of the signed cubic variation of B has been recently developed by Swanson in [11], considering this variation as a class of sequences of processes. More generally, the study of sums such as (1.1) dates back to the work of Breuer and Major [1] in 1983 and Giraitis and Surgailis [3] in 1985. Further discussion of the literature related to sums such as this can be found in Nourdin et al. [5], as well as at the end of Chapter 7 of Nourdin and Peccati [4].

In the present paper, we take $H = 1/6$ and consider the case of the signed cubic variation,

$$W_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} (B(j/n) - B((j-1)/n))^3. \tag{1.2}$$

We are interested in the convergence in distribution of the sequence of two-dimensional processes $\{W_{a_n}(t), W_{b_n}(t)\}$, where $\{a_n\}$ and $\{b_n\}$ are two strictly increasing sequences of natural numbers. Under some conditions, the limit of this sequence is a two-dimensional Gaussian process X^ρ, independent of B, whose components are Brownian motions with variance $\kappa^2 t$, and with covariance $\int_0^t \rho(s) ds$ for some function ρ. The proof of this result is based on Theorem 2.6 (see Sect. 2.3 below), which implies that for a sequence of vectors whose components belong to a fixed Wiener chaos and each component converges in law to a Gaussian distribution, the convergence to a multidimensional Gaussian distribution follows from the convergence of the covariance matrix. This theorem can be found in the recent monograph by Nourdin and Peccati [4] (see Theorem 6.2.3) devoted to the normal approximation using Malliavin calculus combined with Stein's method. Theorem 2.6 has been first proved by Peccati and Tudor in [10], by means of stochastic calculus techniques, and Nualart and Ortiz-Latorre provide in [8] an alternative proof based on Malliavin calculus and on the use of characteristic functions.

The covariance function ρ depends on the asymptotic behavior of the sequences $\{a_n\}$ and $\{b_n\}$. Our main results are the following. We set $L_n = \frac{b_n}{a_n}$ and we assume that $L_n \to L \in [0, \infty]$.

- (i) If $L = 0$ or $L = \infty$, then $\rho(s) = 0$ for all s, and the components of X^ρ are independent Brownian motions.
- (ii) Suppose that $L_n = L \in (0, \infty)$ for all but finitely many n. Then, L is a rational number, and $\rho(s)$ is a constant which depends on L.
- (iii) If $L_n \neq L \in (0, \infty)$ for all but finitely many n and the greatest common divisor of a_n and b_n converges to infinity, then, again $\rho(s)$ is a constant which depends on L.
- (iv) If $L \in (0, \infty)$ and there exists $k \in \mathbb{N}$ such that $b_n - a_n = k \bmod a_n$ for all n, then $\rho(s)$ is not constant, and depends on L and k.

In the cases (ii)–(iv), an explicit value of $\rho(s)$ is given.

Our article is inspired by the relationship between higher (signed) variations of fractional Brownian motions and "change of variable" formulas for stochastic integrals with respect to these processes (see [2,6]). These results imply that approximations to variations of fractional Brownian motion have a direct relationship with numerical stochastic integration with respect to these processes. We hope that our study will shed light on the convergence and stability of numerical approximations to stochastic integrals, and perhaps will be relevant outside the narrow context of the present article. Additionally, we find the diversity of results presented in (i)–(iv) interesting from the purely intellectual point of view, irrespective of their potential applications.

The paper is organized as follows. In Sect. 2 we introduce some preliminary material that will be used in the paper. We present in this section some estimates for the covariance between two increments of the fractional Brownian motion, and we study the properties of a function $f_L(x)$, fundamental for our paper. Section 3 contains the main results and proofs, and in Sect. 4 we discuss some concrete examples.

As a final remark, let us consider generalizations of (1.2). Suppose that $f : \mathbb{R} \to \mathbb{R}$ is square integrable with respect to the standard normal distribution and has an expansion into a series of Hermite polynomials of the form $f(x) = \sum_{k=q}^{\infty} a_k H_k(x)$, where $q \geq 1$. If B is fractional Brownian motion with Hurst parameter $H < 1/(2q)$ and $\Delta B_j = B(j/n) - B((j-1)/n)$, then it can be shown using the Breuer-Major theorem that

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} f(n^H \Delta B_j)$$

converges in law to an independent Brownian motion. Theorem 3.1 can be extended to this general case but the computation of the asymptotic correlations (3.1) appears to be rather involved and requires methods and ideas beyond those developed in this paper, even in the particular case $H = 1/(2k)$ and $f(x) = x^k$ for odd k. In this paper, we consider the case when $H = 1/6$ and $f(x) = x^3$. It seems plausible that our results have natural extensions to the more general setting, but the study of asymptotic correlations with more general H and f will be reserved for future papers.

2 Preliminaries

If $x \in \mathbb{R}$, then $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and $\lceil x \rceil$ denotes the least integer greater than or equal to x. Note that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$, $\lceil x \rceil - 1 < x \leq \lceil x \rceil$, and $\lceil x \rceil = \lfloor x \rfloor + 1_{\mathbb{Z}^c}(x)$, for all $x \in \mathbb{R}$. Also note that for all $n \in \mathbb{Z}$ and all $x \in \mathbb{R}$, we have $x < n$ if and only if $\lfloor x \rfloor < n$, and $n < x$ if and only if $n < \lceil x \rceil$.

The Skorohod space of càdlàg functions from $[0, \infty)$ to $\mathbb{R}^d$ will be denoted by $D_{\mathbb{R}^d}[0, \infty)$, and convergence in law will be denoted by the symbol $\Rightarrow$.

Let $B = B^{1/6}$ be a two-sided fractional Brownian motion with Hurst parameter $H = 1/6$. That is, $\{B(t) : t \in \mathbb{R}\}$ is a centered Gaussian process with covariance function

$$R(s,t) = E[B(s)B(t)] = \frac{1}{2}(|t|^{1/3} + |s|^{1/3} - |t-s|^{1/3}),$$

for $s, t \in \mathbb{R}$.

Let $n \in \mathbb{N}$, $t_j = t_{j,n} = j/n$ and $\Delta B_j = \Delta B_{j,n} = B(t_j) - B(t_{j-1})$. If $k \in \mathbb{N}$, then we shall denote $(\Delta B_j)^k$ by ΔB_j^k. Let $W_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} \Delta B_j^3$. The signed cubic variation of B is defined in [11] as a class of sequences of processes, each of which is equivalent, in a certain sense, to the sequence $\{W_n\}$. The relevant fact for our present purposes is that the sequence $\{W_n\}$ converges in law to a Brownian motion independent of B. This was proven in [8], and the statement of the theorem is the following.

Theorem 2.1 *As $n \to \infty$, $(B, W_n) \Rightarrow (B, \kappa W)$ in $D_{\mathbb{R}^2}[0,\infty)$, where*

$$\kappa^2 = \frac{3}{4} \sum_{m \in \mathbb{Z}} (|m+1|^{1/3} + |m-1|^{1/3} - 2|m|^{1/3})^3, \tag{2.1}$$

and W is a standard Brownian motion, independent of B.

Since we are interested in the joint convergence of subsequences of $\{W_n\}$, we will be primarily concerned with the covariance of increments of this process, which can be expressed in terms of the covariance of increments of B. For this reason, let us define

$$\begin{aligned}
\Phi(s,t,u,v) &= 2E[(B(t)-B(s))(B(v)-B(u))] \\
&= 2(R(t,v) - R(t,u) - R(s,v) + R(s,u)) \\
&= t^{1/3} + v^{1/3} - |t-v|^{1/3} - t^{1/3} - u^{1/3} + |t-u|^{1/3} \\
&\quad - s^{1/3} - v^{1/3} + |s-v|^{1/3} + s^{1/3} + u^{1/3} - |s-u|^{1/3} \\
&= |t-u|^{1/3} + |s-v|^{1/3} - |s-u|^{1/3} - |t-v|^{1/3},
\end{aligned} \tag{2.2}$$

for $s, t, u, v \in \mathbb{R}$. Note that

$$\Phi(s,t,u,v) = \Phi(u,v,s,t), \tag{2.3}$$

$$\Phi(s,t,u,v) = \Phi(t, t+v-u, v, v+t-s), \tag{2.4}$$

$$\Phi(s+c, t+c, u+c, v+c) = \Phi(s,t,u,v), \tag{2.5}$$

$$\Phi(cs, ct, cu, cv) = |c|^{1/3}\Phi(s,t,u,v), \tag{2.6}$$

for all $c \geq 0$.

2.1 Estimates for the function Φ

As a first, coarse estimate of Φ, note that if $x, y \in \mathbb{R}$, then

$$||x|^{1/3} - |y|^{1/3}| \leq ||x| - |y||^{1/3} \leq |x-y|^{1/3}. \tag{2.7}$$

Thus,

$$|\Phi(s,t,u,v)| \le ||t-u|^{1/3} - |s-u|^{1/3}| + ||s-v|^{1/3} - |t-v|^{1/3}| \le 2|t-s|^{1/3}.$$

By (2.3),

$$|\Phi(s,t,u,v)| \le 2|v-u|^{1/3},$$

and it follows that

$$|\Phi(s,t,u,v)| \le 2(|t-s| \wedge |v-u|)^{1/3}, \tag{2.8}$$

for all $s, t, u, v \in \mathbb{R}$.

When more refined estimates are needed, we will rely on the following integral representations of Φ. If $u < v < s < t$, then

$$\begin{aligned}\Phi(s,t,u,v) &= (t-u)^{1/3} + (s-v)^{1/3} - (s-u)^{1/3} - (t-v)^{1/3}\\ &= \frac{1}{3}\int_s^t (y-u)^{-2/3}\,dy - \frac{1}{3}\int_s^t (y-v)^{-2/3}\,dy\\ &= -\frac{2}{9}\int_s^t\int_u^v (y-x)^{-5/3}\,dx\,dy\\ &= -\frac{2}{9}\int_0^{t-s}\int_0^{v-u} (s-v+x+y)^{-5/3}\,dx\,dy < 0.\end{aligned} \tag{2.9}$$

Also, if $u < s < t < v$, then

$$\begin{aligned}\Phi(s,t,u,v) &= (t-u)^{1/3} - (s-v)^{1/3} - (s-u)^{1/3} + (t-v)^{1/3}\\ &= \frac{1}{3}\int_s^t (y-u)^{-2/3}\,dy + \frac{1}{3}\int_s^t (y-v)^{-2/3}\,dy\\ &= \frac{1}{3}\int_s^t ((v-y)^{-2/3} + (y-u)^{-2/3})\,dy > 0.\end{aligned} \tag{2.10}$$

We will use these integral representations to generate several different estimates in Lemma 2.2 below.

Lemma 2.2 *If $u < v < s < t$, then*

$$|\Phi(s,t,u,v)| \le \frac{2}{9}(t-s)(v-u)(s-v)^{-5/3}, \tag{2.11}$$

$$|\Phi(s,t,u,v)| \le (t-s)^{1/4}(v-u)^{11/12}(s-v)^{-5/6}, \tag{2.12}$$
$$|\Phi(s,t,u,v)| \le (t-s)^{11/12}(v-u)^{1/4}(s-v)^{-5/6}. \tag{2.13}$$

If $u < s < t < v$, *then*

$$|\Phi(s,t,u,v)| \le \frac{1}{3}(t-s)((v-t)^{-2/3} + (s-u)^{-2/3}). \tag{2.14}$$

Proof Suppose $u < v < s < t$. Inequality (2.11) follows directly from (2.9). By (2.9) and Lemma 5.7 (see the Appendix),

$$\begin{aligned}|\Phi(s,t,u,v)| &\le \frac{2}{9}(s-v)^{-5/6}\int_0^{v-u} x^{-1/12}\,dx \int_0^{t-s} y^{-3/4}\,dy\\ &= \frac{2}{9}(s-v)^{-5/6}\cdot\frac{12}{11}(v-u)^{11/12}\cdot 4(t-s)^{1/4},\end{aligned}$$

and this proves (2.12). Similarly,

$$\begin{aligned}|\Phi(s,t,u,v)| &\le \frac{2}{9}(s-v)^{-5/6}\int_0^{v-u} x^{-3/4}\,dx \int_0^{t-s} y^{-11/12}\,dy\\ &= \frac{2}{9}(s-v)^{-5/6}\cdot 4(v-u)^{1/4}\cdot\frac{12}{11}(t-s)^{11/12},\end{aligned}$$

proving (2.13). Finally, (2.14) follows directly from (2.10). □

Let $a = \{a_n\}_{n=1}^\infty$ and $b = \{b_n\}_{n=1}^\infty$ be strictly increasing sequences in $\mathbb{N}$, and let $L_n = b_n/a_n$. We define

$$\Phi_n^{a,b}(j,k) = \Phi\left(\frac{j-1}{a_n}, \frac{j}{a_n}, \frac{k-1}{b_n}, \frac{k}{b_n}\right) = E[\Delta B_{j,a_n}\Delta B_{k,b_n}], \tag{2.15}$$

for $j, k \in \mathbb{Z}$. When a and b are understood, we will simply write Φ_n instead of $\Phi_n^{a,b}$. By (2.3), we have $\Phi_n^{a,b}(j,k) = \Phi_n^{b,a}(k,j)$. Note that by (2.8),

$$|\Phi_n^{a,b}(j,k)|^3 \le 8(a_n^{-1} \wedge b_n^{-1}), \tag{2.16}$$

for any a, b, n, j, k. Applying Lemma 2.2 gives us Lemma 2.3.

Lemma 2.3 *If* $\frac{j}{a_n} < \frac{k-1}{b_n}$, *then*

$$|\Phi_n^{a,b}(j,k)| \le \frac{2}{9}a_n^{-1}b_n^{-1}\left(\frac{k-1}{b_n} - \frac{j}{a_n}\right)^{-5/3}, \tag{2.17}$$
$$|\Phi_n^{a,b}(j,k)| \le a_n^{-1/4}b_n^{-11/12}\left(\frac{k-1}{b_n} - \frac{j}{a_n}\right)^{-5/6}, \tag{2.18}$$

$$|\Phi_n^{a,b}(j,k)| \le a_n^{-11/12} b_n^{-1/4} \left(\frac{k-1}{b_n} - \frac{j}{a_n}\right)^{-5/6}. \tag{2.19}$$

If $\frac{k-1}{b_n} < \frac{j-1}{a_n}$ *and* $\frac{j}{a_n} < \frac{k}{b_n}$, *then*

$$|\Phi_n^{a,b}(j,k)| \le \frac{1}{3} a_n^{-1} \left(\left(\frac{k}{b_n} - \frac{j}{a_n}\right)^{-2/3} + \left(\frac{j-1}{a_n} - \frac{k-1}{b_n}\right)^{-2/3}\right). \tag{2.20}$$

2.2 The function f_L

An important function in our analysis is constructed as follows. If $m \in \mathbb{Z}$ and $L \in (0,\infty)$, define $f_{m,L} \in C[0,1]$ by

$$\begin{aligned} f_{m,L}(x) &= 8(E[(B(x+1)-B(x))(B(m+L)-B(m))])^3 \\ &= \Phi(x, x+1, m, m+L)^3 \\ &= (|x-m+1|^{1/3} + |x-m-L|^{1/3} - |x-m|^{1/3} - |x-m+1-L|^{1/3})^3. \end{aligned} \tag{2.21}$$

Although $f_{m,L}(x)$ is defined only for $x \in [0,1]$, the above formula for $\Phi(x, x+1, m, m+L)^3$ can be extended to all x using (2.5). We have

$$\Phi(x, x+1, m, m+L)^3 = f_{m-\lfloor x \rfloor, L}(x - \lfloor x \rfloor), \tag{2.22}$$

for any $m \in \mathbb{Z}$, $L \in (0,\infty)$, and $x \in \mathbb{R}$. Note that by (2.8),

$$\|f_{m,L}\|_\infty \le 8, \tag{2.23}$$

for any $m \in \mathbb{Z}$ and $L \in (0,\infty)$. Also, by (2.15), (2.6), (2.4), and (2.22),

$$\begin{aligned} \Phi_n^{a,b}(j,k)^3 &= \frac{1}{b_n} \Phi((j-1)L_n, jL_n, k-1, k)^3 = \frac{1}{b_n} \Phi(jL_n, jL_n+1, k, k+L_n)^3 \\ &= \frac{1}{b_n} f_{k-\lfloor jL_n \rfloor, L_n}(jL_n - \lfloor jL_n \rfloor). \end{aligned} \tag{2.24}$$

Lemma 2.4 *The series* $\sum_{m \in \mathbb{Z}} f_{m,L}$ *is absolutely convergent in* $C[0,1]$ *with the uniform norm.*

Proof Fix $L \in (0,\infty)$. Let $m \in \mathbb{Z}$ with $m < -L$. Then for any $x \in [0,1]$, we have $m < m+L < x < x+1$. Hence, by (2.13),

$$\begin{aligned} |f_{m,L}(x)| &= |\Phi(x, x+1, m, m+L)|^3 \\ &\le L^{3/4}(x-m-L)^{-5/2} \le L^{3/4}(-m-L)^{-5/2} = L^{3/4}|m+L|^{-5/2}. \end{aligned}$$

Thus, $\|f_{m,L}\|_\infty \le L^{3/4}|m+L|^{-5/2}$.

Next, let $m \in \mathbb{Z}$ with $m > 2$. Then for any $x \in [0, 1]$, we have $x < x + 1 < m < m + L$. Hence, by (2.3) and (2.12),

$$|f_{m,L}(x)| = |\Phi(x, x+1, m, m+L)|^3 \le L^{3/4}(m - x - 1)^{-5/2} \le L^{3/4}|m-2|^{-5/2}.$$

Thus, $\|f_{m,L}\|_\infty \le L^{3/4}|m-2|^{-5/2}$.

Therefore, using (2.23),

$$\sum_{m\in\mathbb{Z}} \|f_{m,L}\|_\infty \le L^{3/4} \sum_{m=-\infty}^{\lceil -L\rceil -1} |m+L|^{-5/2} + 8(3 - \lceil -L\rceil) + L^{3/4}\sum_{m=3}^{\infty} |m-2|^{-5/2} < \infty,$$

which shows that the series is absolutely convergent. □

By Lemma 2.4, we may define $f_L = \sum_{m\in\mathbb{Z}} f_{m,L} \in C[0, 1]$. Let us also define $\widehat{f}_{m,L} : \mathbb{R} \to \mathbb{R}$ by $\widehat{f}_{m,L}(x) = f_{m,L}(x - \lfloor x\rfloor)$ and $\widehat{f}_L : \mathbb{R} \to \mathbb{R}$ by $\widehat{f}_L(x) = f_L(x - \lfloor x\rfloor)$. By Lemma 2.4, $\widehat{f}_L = \sum_{m\in\mathbb{Z}} \widehat{f}_{m,L}$, and this series is absolutely and uniformly convergent on all of $\mathbb{R}$.

In Lemma 2.5, we catalog several properties of these functions that we will need later.

Lemma 2.5 *The following relations hold:*

(i) $|f_{m,L}(x) - f_{m,L'}(x)| \le 24|L - L'|^{1/3}$ *for all* m, L, L' *and* x,
(ii) *if* $L \in \mathbb{N}$, *then* $f_L(x) = f_L(1 - x)$ *for all* x,
(iii) $f_1(1/2) < 0.1$, *and*
(iv) $f_1(0) > 6.6$.

Proof Let $m \in \mathbb{Z}$, $L, L' \in (0, \infty)$, and $x \in [0, 1]$. By (2.23), we have

$$\begin{aligned}&\left|f_{m,L}(x) - f_{m,L'}(x)\right| \\ &\quad = \left|f_{m,L}(x)^{1/3} - f_{m,L'}(x)^{1/3}\right| \cdot \left|f_{m,L}(x)^{2/3} + f_{m,L}(x)^{1/3} f_{m,L'}(x)^{1/3} \right.\\ &\qquad \left. + f_{m,L'}(x)^{2/3}\right| \le 12\left|f_{m,L}(x)^{1/3} - f_{m,L'}(x)^{1/3}\right|.\end{aligned}$$

Also, by (2.21) and (2.7),

$$\begin{aligned}&|f_{m,L}(x)^{1/3} - f_{m,L'}(x)^{1/3}| \\ &\quad = ||x - m - L'|^{1/3} - |x - m - L|^{1/3} - |x - m + 1 - L'|^{1/3} \\ &\qquad + |x - m + 1 - L|^{1/3}| \le 2|L - L'|^{1/3}.\end{aligned}$$

Hence,

$$|f_{m,L}(x) - f_{m,L'}(x)| \le 24|L - L'|^{1/3},$$

and this proves (i).

For (ii), let $L \in \mathbb{N}$. For each $m \in \mathbb{Z}$, define $\widetilde{m} = 2 - L - m$. Then, for all $x \in [0, 1]$,

$$\begin{aligned} f_{m,L}(1-x) &= (|x+m-2|^{1/3} + |x+m+L-1|^{1/3} \\ &\quad -|x+m-1|^{1/3} - |x+m-2+L|^{1/3})^3 \\ &= (|x-\widetilde{m}-L|^{1/3} + |x-\widetilde{m}+1|^{1/3} \\ &\quad -|x-\widetilde{m}-L+1|^{1/3} - |x-\widetilde{m}|^{1/3})^3 = f_{\widetilde{m},L}(x). \end{aligned}$$

Since $f_L = \sum_{m\in\mathbb{Z}} f_{m,L}$ and $m \mapsto \widetilde{m}$ is a bijection from $\mathbb{Z}$ to $\mathbb{Z}$, this proves (ii).

By (2.21), (2.9), and (2.11), if $m < x - 1$, then $f_{m,1}(x) < 0$ and $|f_{m,1}(x)| \le \frac{2}{9}(x-m-1)^{-5}$. Similarly, using (2.3), (2.9), and (2.11), if $m > x+1$, then $f_{m,1}(x) < 0$ and $|f_{m,1}(x)| \le \frac{2}{9}(m-x-1)^{-5}$. It follows that

$$f_1(1/2) = \sum_{m\in\mathbb{Z}} f_{m,1}(1/2) < f_{0,1}(1/2) + f_{1,1}(1/2) = (3^{1/3}-1)^3.$$

Since $3 < 24389/8000 = (29/20)^3$, this gives $f_1(1/2) < (9/20)^3 = 729/8000 < 0.1$, proving (iii).

It also follows that

$$\begin{aligned} f_1(0) &= \sum_{m=-1}^{1} f_{m,1}(0) - \sum_{\substack{m\in\mathbb{Z} \\ |m|\ge 2}} |f_{m,1}(0)| \\ &= 8 - 2(2-2^{1/3})^3 - \sum_{\substack{m\in\mathbb{Z} \\ |m|\ge 2}} |f_{m,1}(0)| \\ &\ge 8 - 2(2-2^{1/3})^3 - \frac{2}{9} \sum_{\substack{m\in\mathbb{Z} \\ |m|\ge 2}} ||m|-1|^{-5} \\ &= 8 - 2(2-2^{1/3})^3 - \frac{4}{9} \sum_{m=1}^{\infty} m^{-5}. \end{aligned}$$

By Lemma 5.8,

$$\sum_{m=1}^{\infty} m^{-5} = 1 + \sum_{m=2}^{\infty} m^{-5} \le \frac{5}{4}.$$

Thus, since $2 > 125/64 = (5/4)^3$, we have

$$f_1(0) > \frac{67}{9} - 2\left(2 - \frac{5}{4}\right)^3 = \frac{1901}{288} > \frac{1900.8}{288} = 6.6,$$

and this proves (iv). □

2.3 Convergence in law of random vectors in a fixed Wiener chaos

We denote by $\mathcal{H}(B)$ the closed linear subspace of $L^2(\Omega)$ generated by the family of random variables $\{B(t), t \geq 0\}$. For each integer $q \geq 1$, we denote by $\mathcal{H}_q$ the q-Wiener chaos defined as the subspace of $L^2(\Omega)$ spanned by the random variables $\{h_q(F), F \in \mathcal{H}(B), E(F^2) = 1\}$, where

$$h_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q}(e^{-x^2/2})$$

is the qth Hermite polynomial. Notice that $\mathcal{H}_1 = \mathcal{H}(B)$.

We finish this section with a result on the convergence of vectors whose components belong to a fixed Wiener chaos. This theorem was first proved by Peccati and Tudor in [10], and appears as Theorem 6.2.3 in [4].

Theorem 2.6 *Let $d \geq 2$ and $q_1, \ldots, q_d \geq 1$ be some fixed integers. Consider the sequence of vectors $F_n = (F_{1,n}, \ldots, F_{d,n})$, where for each $i = 1, \ldots, d$, each component $F_{i,n}$ belongs to the Wiener chaos $\mathcal{H}_{q_i}$. Suppose that*

$$\lim_{n\to\infty} E[F_{i,n} F_{j,n}] = C(i, j), \quad 1 \leq i, j \leq d,$$

where C is a symmetric non-negative definite matrix. Then, the following two conditions are equivalent:

(i) *F_n converges in law to a d-dimensional Gaussian distribution $N(0, C)$.*
(ii) *For each $1 \leq i \leq d$, $F_{i,n}$ converges in law to $N(0, C(i, i))$.*

3 Main results and proofs

Recall from Sect. 2 that $W_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} \Delta B_j^3$, and that $(B, W_n) \Rightarrow (B, \kappa W)$, where W is a Brownian motion. We wish to investigate the joint convergence in law of (B, W_{a_n}, W_{b_n}), where $\{W_{a_n}\}$ and $\{W_{b_n}\}$ are two different subsequences of $\{W_n\}$. Our first theorem, Theorem 3.1, reduces this to an investigation of the asymptotic covariance.

Theorem 3.1 *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be strictly increasing sequences in $\mathbb{N}$. Let $\rho \in C[0, \infty)$, and suppose that*

$$\lim_{n\to\infty} E[W_{a_n}(s) W_{b_n}(t)] = \int_0^{s \wedge t} \rho(x)\, dx, \tag{3.1}$$

for all $0 \leq s, t < \infty$. Then $\|\rho\|_\infty \leq \kappa^2$, and we may define

$$\sigma(t) = \kappa \begin{pmatrix} \sqrt{1 - |\kappa^{-2}\rho(t)|^2} & \kappa^{-2}\rho(t) \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

Let W be a standard, two-dimensional Brownian motion, independent of B, and define

$$X^{\rho}(t) = \int_0^t \sigma(s)\, dW(s). \tag{3.3}$$

Then $(B, W_{a_n}, W_{b_n}) \to (B, X^{\rho})$ in law in $D_{\mathbb{R}^3}[0, \infty)$ as $n \to \infty$.

Remark 3.2 We know from [8] that $E|W_n(t) - W_n(s)|^2 \to \kappa^2|t-s|$. Thus, if (3.1) is satisfied for some continuous ρ, then by Hölder's inequality,

$$\int_s^t \rho(x)\, dx = \lim_{n\to\infty} E[(W_{a_n}(t) - W_{a_n}(s))(W_{b_n}(t) - W_{b_n}(s))] \le \kappa^2(t-s),$$

for all $s < t$. Since ρ is continuous, this implies $\|\rho\|_\infty \le \kappa^2$, so that $\sigma(t)$ is well-defined by (3.2). □

Remark 3.3 For any $j = 1, \dots, n$, the random variable ΔB_j^3 can be expressed as

$$\Delta B_j^3 = n^{-1/2} h_3(n^{1/6}\Delta B_j) + 3n^{-1/3}\Delta B_j,$$

where $h_3(x) = x^3 - 3x$ is the third Hermite polynomial. Define

$$\widetilde{W}_n(t) = W_n(t) - 3n^{-1/3} B(\lfloor nt \rfloor / n) = \sum_{j=1}^{\lfloor nt \rfloor} n^{-1/2} h_3(n^{1/6}\Delta B_j). \tag{3.4}$$

Then, for any $p \ge 2$ and any $t \ge 0$,

$$\lim_{n\to\infty} \sup_{0\le s\le t} E[|\widetilde{W}_n(s) - W_n(s)|^p] = 0. \tag{3.5}$$

□

Proof of Theorem 3.1 Taking into account (3.5), it suffices to establish the desired limit theorem for the sequence of processes $X_n = (X_n^1, X_n^2, X_n^3) := (B, \widetilde{W}_{a_n}, \widetilde{W}_{b_n})$.

We know (see, for instance, [8]) that this sequence is tight in $(D_{\mathbb{R}}[0, \infty))^3$. It is well-known (see Lemma 2.2 in [7], for example) that since the limit processes are continuous, this implies the sequence is tight in $D_{\mathbb{R}^3}[0, \infty)$. Thus to show the convergence in law it suffices to establish the convergence in law of the finite dimensional distributions. Consider a finite set of times $0 \le t_1 < t_2 < \cdots < t_M$ and the $3M$-dimensional random vector $(X_n(t_1), \dots, X_n(t_M))$. This sequence of vectors satisfies the following properties:

1. The components $X_n^i(t_j)$, $1 \le i \le 3$, $1 \le j \le M$, belong to the third Wiener chaos if $i = 2, 3$ and to the first Wiener chaos if $i = 1$.

2. The first component $X_n^1(t_j) = B(t_j)$ is Gaussian with a fixed law. On the other hand, we know from [8] that the other two components $X_n^2(t_j) = \widetilde{W}_{a_n}(t_j)$ and $X_n^3(t_j) = \widetilde{W}_{b_n}(t_j)$ converge in law as n tends to infinity to a Gaussian distribution with variance $\kappa^2 t_j$, which coincides with the common law of $X^{\rho,1}(t_j)$ and $X^{\rho,2}(t_j)$.

Set $X = (B, X^{\rho,1}, X^{\rho,2})$. Then, by Theorem 2.6, in order to show that

$$(X_n(t_1), \dots, X_n(t_M)) \Rightarrow (X(t_1), \dots, X(t_M)),$$

it suffices to show that for any $i \neq k$ and for any $s, t \geq 0$, we have

$$\lim_{n\to\infty} E[X_n^i(s)X_n^k(t)] = E[X^i(s)X^k(t)]. \tag{3.6}$$

If $i = 1$ and $k = 2, 3$, then $E[X^i(s)X^k(t)] = 0$ and (3.6) has been proved in [8]. For $i = 2$ and $k = 3$, then, taking into account (3.5) and using our assumption (3.1) we obtain

$$\lim_{n\to\infty} E[X_n^2(s)X_n^3(t)] = \lim_{n\to\infty} E[W_{a_n}(s)W_{b_n}(t)] = \int_0^{s\wedge t} \rho(u)du = E[X^2(s)X^3(t)],$$

and the proof is complete. □

Our first main result concerns the simplest of the situations we consider, where $\{W_{a_n}\}$ and $\{W_{b_n}\}$ are subsequences such that b_n/a_n converges to either 0 or ∞.

Theorem 3.4 *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be strictly increasing sequences in $\mathbb{N}$. Let $L_n = b_n/a_n$ and suppose that $L_n \to L \in \{0, \infty\}$. Then*

$$\lim_{n\to\infty} E[(W_{a_n}(t) - W_{a_n}(s))(W_{b_n}(t) - W_{b_n}(s))] = 0,$$

for all $0 \leq s \leq t$.

Proof From (3.5), it suffices to show that

$$\lim_{n\to\infty} E[(\widetilde{W}_{a_n}(t) - \widetilde{W}_{a_n}(s))(\widetilde{W}_{b_n}(t) - \widetilde{W}_{b_n}(s))] = 0,$$

where $\widetilde{W}$ has been defined in (3.4). By interchanging the roles of $\{a_n\}$ and $\{b_n\}$ if necessary, we may assume that $L = 0$. Fix $0 \leq s \leq t$ and note that

$$
\begin{aligned}
&E[(\widetilde{W}_{a_n}(t) - \widetilde{W}_{a_n}(s))(\widetilde{W}_{b_n}(t) - \widetilde{W}_{b_n}(s))] \\
&\quad = \sum_{j=\lfloor a_n s\rfloor+1}^{\lfloor a_n t\rfloor} \sum_{k=\lfloor b_n s\rfloor+1}^{\lfloor b_n t\rfloor} n^{-1} E[h_3(n^{1/6}\Delta B_{j,a_n}) h_3(n^{1/6}\Delta B_{k,b_n})] \\
&\quad = \sum_{j=\lfloor a_n s\rfloor+1}^{\lfloor a_n t\rfloor} \sum_{k=\lfloor b_n s\rfloor+1}^{\lfloor b_n t\rfloor} n^{-1} 3!(E[n^{1/6}\Delta B_{j,a_n} n^{1/6}\Delta B_{k,b_n}])^3 \\
&\quad = 3! \sum_{j=\lfloor a_n s\rfloor+1}^{\lfloor a_n t\rfloor} \sum_{k=\lfloor b_n s\rfloor+1}^{\lfloor b_n t\rfloor} (E[\Delta B_{j,a_n}\Delta B_{k,b_n}])^3 \\
&\quad = \frac{3}{4} \sum_{j=\lfloor a_n s\rfloor+1}^{\lfloor a_n t\rfloor} \sum_{k=\lfloor b_n s\rfloor+1}^{\lfloor b_n t\rfloor} \Phi_n(j,k)^3,
\end{aligned}
$$

where $\Phi_n(j,k) = \Phi_n^{a,b}(j,k)$ has been introduced in (2.15). Note that in the second equality above, we have used the fact that if X and Y are jointly Gaussian, each with mean zero and variance one, then $E[h_q(X)h_q(Y)] = q!(E[XY])^q$. See [9, Lemma 1.1.1].

Define

$$
S_n := \frac{3}{4} \sum_{j=1}^{\lfloor a_n t\rfloor} \sum_{k=1}^{\lfloor b_n t\rfloor} |\Phi_n(j,k)|^3 \geq |E[(\widetilde{W}_{a_n}(t) - \widetilde{W}_{a_n}(s))(\widetilde{W}_{b_n}(t) - \widetilde{W}_{b_n}(s))]|,
$$

so that it will suffice to show that $S_n \to 0$ as $n \to \infty$.

For each fixed $k \in \{1, \ldots, \lfloor b_n t\rfloor\}$, consider the following sets of indices:

$$
\begin{aligned}
A_1^k &= \left\{1 \leq j \leq \lfloor a_n t\rfloor : \frac{j-1}{a_n} \leq \frac{k-1}{b_n}\right\}, \\
A_2^k &= \left\{1 \leq j \leq \lfloor a_n t\rfloor : \frac{k-1}{b_n} < \frac{j-1}{a_n} < \frac{j}{a_n} < \frac{k}{b_n}\right\}, \\
A_3^k &= \left\{1 \leq j \leq \lfloor a_n t\rfloor : \frac{k}{b_n} \leq \frac{j}{a_n}\right\}.
\end{aligned}
$$

It is easily verified that $\bigcup_\ell A_\ell^k = \{1, \ldots, \lfloor a_n t\rfloor\}$, $A_1^k \cap A_2^k = \emptyset$, and $A_2^k \cap A_3^k = \emptyset$. Also, if $L_n < 1$, then $A_1^k \cap A_3^k = \emptyset$. Thus, for n sufficiently large, $\{A_1^k, A_2^k, A_3^k\}$ is a partition of $\{1, \ldots, \lfloor a_n t\rfloor\}$, and we may write

$$
S_n = S_n^{(1)} + S_n^{(2)} + S_n^{(3)},
$$

where

$$
S_n^{(i)} = \sum_{k=1}^{\lfloor b_n t\rfloor} \sum_{j \in A_i} |\Phi_n(j,k)|^3.
$$

Note that $(j-1)/a_n \le (k-1)/b_n$ if and only if $j \le \lfloor (k-1)/L_n \rfloor + 1$, and $j/a_n < (k-1)/b_n$ if and only if $j < \lceil (k-1)/L_n \rceil$. Also note that for n sufficiently large, $\lfloor (k-1)/L_n \rfloor + 1 \le \lfloor a_n t \rfloor$. Thus, by (2.16) and (2.19),

$$\begin{aligned}
\sum_{j\in A_1} |\Phi_n(j,k)|^3 &= \sum_{j=1}^{\lceil (k-1)/L_n\rceil -3} |\Phi_n(j,k)|^3 + \sum_{j=\lceil (k-1)/L_n\rceil -2}^{\lfloor (k-1)/L_n\rfloor +1} |\Phi_n(j,k)|^3 \\
&\le \sum_{j=1}^{\lceil (k-1)/L_n\rceil -3} a_n^{-11/4} b_n^{-3/4} \left(\frac{k-1}{b_n} - \frac{j}{a_n}\right)^{-5/2} + 32(a_n^{-1} \wedge b_n^{-1}) \\
&\le a_n^{-1/4} b_n^{-3/4} \sum_{j=1}^{\lceil (k-1)/L_n\rceil -3} \left(\frac{k-1}{L_n} - j\right)^{-5/2} + 32 a_n^{-1}.
\end{aligned}$$

Using Lemma 5.8, we have

$$\begin{aligned}
\sum_{j\in A_1} |\Phi_n(j,k)|^3 &\le \frac{2}{3} a_n^{-1/4} b_n^{-3/4} \left(\frac{k-1}{L_n} - \left\lceil \frac{k-1}{L_n} \right\rceil + 2\right)^{-3/2} + 32 a_n^{-1} \\
&\le \frac{2}{3} a_n^{-1/4} b_n^{-3/4} + 32 a_n^{-1},
\end{aligned}$$

giving

$$0 \le S_n^{(1)} \le b_n t \left(\frac{2}{3} a_n^{-1/4} b_n^{-3/4} + 32 a_n^{-1}\right) \le t \left(\frac{2}{3} L_n^{1/4} + 32 L_n\right) \to 0,$$

as $n \to \infty$.

For $S_n^{(3)}$, note that $k/b_n \le j/a_n$ if and only if $\lceil k/L_n \rceil \le j$. Also, $k/b_n < (j-1)/a_n$ if and only if $j > \lfloor k/L_n \rfloor + 1$. Since $\Phi_n^{a,b}(j,k) = \Phi_n^{b,a}(k,j)$, we apply (2.18) with j, k and a, b interchanged. Using also (2.16), we obtain

$$\begin{aligned}
\sum_{j\in A_3} |\Phi_n(j,k)|^3 &= \sum_{j=\lceil k/L_n\rceil}^{\lfloor k/L_n\rfloor +3} |\Phi_n(j,k)|^3 + \sum_{j=\lfloor k/L_n\rfloor +4}^{\lfloor a_n t\rfloor} |\Phi_n(j,k)|^3 \\
&\le 32(a_n^{-1} \wedge b_n^{-1}) + \sum_{j=\lfloor k/L_n\rfloor +4}^{\lfloor a_n t\rfloor} a_n^{-11/4} b_n^{-3/4} \left(\frac{j-1}{a_n} - \frac{k}{b_n}\right)^{-5/2} \\
&\le 32 a_n^{-1} + a_n^{-1/4} b_n^{-3/4} \sum_{j=\lfloor k/L_n\rfloor +4}^{\lfloor a_n t\rfloor} \left(j - 1 - \frac{k}{L_n}\right)^{-5/2}.
\end{aligned}$$

By Lemma 5.8, we have

$$\sum_{j\in A_3} |\Phi_n(j,k)|^3 \le 32a_n^{-1} + \frac{2}{3}a_n^{-1/4}b_n^{-3/4}\left(\left\lfloor \frac{k}{L_n}\right\rfloor + 2 - \frac{k}{L_n}\right)^{-3/2}$$
$$\le 32a_n^{-1} + \frac{2}{3}a_n^{-1/4}b_n^{-3/4},$$

giving, as above, $S_n^{(3)} \to 0$ as $n \to \infty$.

Finally, for $S_n^{(2)}$, note that for sufficiently large n, we have $L_n < 1$, which implies $b_n^{-1} - a_n^{-1} > 0$, giving

$$\left(\frac{k}{b_n} - \frac{j}{a_n}\right)^{-2/3} < \left(\frac{k-1}{b_n} - \frac{j-1}{a_n}\right)^{-2/3} = \left(\frac{j-1}{a_n} - \frac{k-1}{b_n}\right)^{-2/3}.$$

Hence, by (2.20),

$$\sum_{j\in A_2} |\Phi_n(j,k)|^3 = \sum_{j=\lfloor (k-1)/L_n\rfloor+2}^{\lceil k/L_n\rceil-1} |\Phi_n(j,k)|^3$$
$$\le 16(a_n^{-1}\wedge b_n^{-1}) + \frac{8}{27}a_n^{-3}\sum_{j=\lfloor (k-1)/L_n\rfloor+4}^{\lceil k/L_n\rceil-1}\left(\frac{j-1}{a_n} - \frac{k-1}{b_n}\right)^{-2}$$
$$\le 16a_n^{-1} + a_n^{-1}\sum_{j=\lfloor (k-1)/L_n\rfloor+4}^{\lceil k/L_n\rceil-1}\left(j-1-\frac{k-1}{L_n}\right)^{-2}.$$

By Lemma 5.8, we have

$$\sum_{j\in A_2} |\Phi_n(j,k)|^3 \le 16a_n^{-1} + a_n^{-1}\left(\left\lfloor \frac{k-1}{L_n}\right\rfloor + 2 - \frac{k-1}{L_n}\right)^{-1} \le 17a_n^{-1},$$

giving, as above, $S_n^{(2)} \to 0$ as $n \to \infty$. □

To use Theorem 3.1, we must verify hypothesis (3.1). Our next lemma, Lemma 3.5, simplifies this task, allowing us to check (3.1) only when $s = t$.

Lemma 3.5 *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be strictly increasing sequences in $\mathbb{N}$. Let $L_n = b_n/a_n$ and suppose that $L_n \to L \in [0,\infty]$. Then*

$$\lim_{n\to\infty} E[W_{a_n}(s)(W_{b_n}(t) - W_{b_n}(s))] = \lim_{n\to\infty} E[(W_{a_n}(t) - W_{a_n}(s))W_{b_n}(s)] = 0,$$

for any $0 \le s < t$.

Proof By interchanging the roles of $\{a_n\}$ and $\{b_n\}$ if necessary, we may assume that $L_n \to L \in (0,\infty]$. From (3.5), it suffices to show

$$\lim_{n\to\infty} E[\widetilde{W}_{a_n}(s)(\widetilde{W}_{b_n}(t)-\widetilde{W}_{b_n}(s))]=0, \tag{3.7}$$

and

$$\lim_{n\to\infty} E[(\widetilde{W}_{a_n}(t)-\widetilde{W}_{a_n}(s))\widetilde{W}_{b_n}(s)]=0, \tag{3.8}$$

where $\widetilde{W}$ has been defined in (3.4). We begin by proving (3.7).

As in the proof of Theorem 3.4, we have

$$E[\widetilde{W}_{a_n}(s)(\widetilde{W}_{b_n}(t)-\widetilde{W}_{b_n}(s))]=\frac{3}{4}\sum_{j=1}^{\lfloor a_n s\rfloor}\sum_{k=\lfloor b_n s\rfloor+1}^{\lfloor b_n t\rfloor}\Phi_n(j,k)^3. \tag{3.9}$$

We claim that for all $i\geq 0$,

$$\sum_{k=\lfloor b_n s\rfloor+1}^{\lfloor b_n t\rfloor}\Phi_n(\lfloor a_n s\rfloor-i,k)^3\to 0, \tag{3.10}$$

as $n\to\infty$. By (2.16), it is enough to show that

$$\sum_{k=\lfloor b_n s\rfloor+3}^{\lfloor b_n t\rfloor}\Phi_n(\lfloor a_n s\rfloor-i,k)^3\to 0.$$

For this, fix n and let $j=\lfloor a_n s\rfloor-i$. Note that since $\lfloor x\rfloor\leq x<\lfloor x\rfloor+1$, we have

$$\frac{j}{a_n}\leq\frac{a_n s-i}{a_n}\leq s=\frac{b_n s}{b_n}<\frac{\lfloor b_n s\rfloor+1}{b_n}\leq\frac{k-2}{b_n}, \tag{3.11}$$

for any $k\geq\lfloor b_n s\rfloor+3$. Hence, by (2.18) we have

$$|\Phi_n(j,k)|\leq a_n^{-1/4}b_n^{-11/12}\left(\frac{k-1}{b_n}-\frac{j}{a_n}\right)^{-5/6}. \tag{3.12}$$

Using (3.11), this gives

$$\begin{aligned}\sum_{k=\lfloor b_n s\rfloor+3}^{\lfloor b_n t\rfloor}|\Phi_n(\lfloor a_n s\rfloor-i,k)|^3&\leq a_n^{-3/4}b_n^{-11/4}\sum_{k=\lfloor b_n s\rfloor+3}^{\lfloor b_n t\rfloor}\left(\frac{k-1}{b_n}-\frac{\lfloor b_n s\rfloor+1}{b_n}\right)^{-5/2}\\&=a_n^{-3/4}b_n^{-1/4}\sum_{k=\lfloor b_n s\rfloor+3}^{\lfloor b_n t\rfloor}(k-\lfloor b_n s\rfloor-2)^{-5/2}\\&\leq a_n^{-3/4}b_n^{-1/4}\sum_{k=1}^{\infty}k^{-5/2}\to 0,\end{aligned}$$

as $n\to\infty$, and this prove (3.10).

Now, since $b_n/a_n \to L \in (0,\infty]$, there exists an integer $\ell \geq 2$ such that $b_n/a_n \geq 1/(\ell-1)$ for all n. We next claim that for all $i \geq 0$,

$$\sum_{j=1}^{\lfloor a_n s \rfloor - \ell} \Phi_n(j, \lfloor b_n s \rfloor + i)^3 \to 0, \tag{3.13}$$

as $n \to \infty$. Again, fix n and let $k = \lfloor b_n s \rfloor + i$. Then, for all $j \leq \lfloor a_n s \rfloor - \ell$, we have

$$\begin{aligned} \frac{j}{a_n} &< \frac{\lfloor a_n s \rfloor - \ell + 1}{a_n} \leq s - \frac{\ell - 1}{a_n} \leq s - \frac{1}{b_n} = \frac{b_n s - 1}{b_n} < \frac{\lfloor b_n s \rfloor}{b_n} \\ &= \frac{k-i}{b_n} \leq \frac{k-1}{b_n}. \end{aligned} \tag{3.14}$$

Since $j/a_n < (k-1)/b_n$, from (2.19) we conclude

$$|\Phi_n(j,k)| \leq a_n^{-11/12} b_n^{-1/4} \left(\frac{k-1}{b_n} - \frac{j}{a_n} \right)^{-5/6}.$$

Using (3.14), this gives

$$\begin{aligned} \sum_{j=1}^{\lfloor a_n s \rfloor - \ell} |\Phi_n(j, \lfloor b_n s \rfloor + i)|^3 &\leq a_n^{-11/4} b_n^{-3/4} \sum_{j=1}^{\lfloor a_n s \rfloor - \ell} \left(\frac{\lfloor a_n s \rfloor - \ell + 1}{a_n} - \frac{j}{a_n} \right)^{-5/2} \\ &= a_n^{-1/4} b_n^{-3/4} \sum_{j=1}^{\lfloor a_n s \rfloor - \ell} (\lfloor a_n s \rfloor - \ell + 1 - j)^{-5/2} \\ &\leq a_n^{-1/4} b_n^{-3/4} \sum_{j=1}^{\infty} j^{-5/2} \to 0, \end{aligned}$$

as $n \to \infty$, and this proves (3.13).

Finally, (3.7) will be proved once we show that the double sum in (3.9) converges to zero. Let us write

$$\begin{aligned} \sum_{j=1}^{\lfloor a_n s \rfloor} \sum_{k=\lfloor b_n s \rfloor + 1}^{\lfloor b_n t \rfloor} \Phi_n(j,k)^3 &= \sum_{i=0}^{\ell - 1} \sum_{k=\lfloor b_n s \rfloor + 1}^{\lfloor b_n t \rfloor} \Phi_n(\lfloor a_n s \rfloor - i, k)^3 \\ &\quad + \sum_{j=1}^{\lfloor a_n s \rfloor - \ell} \sum_{k=\lfloor b_n s \rfloor + 1}^{\lfloor b_n t \rfloor} \Phi_n(j,k)^3 \\ &= \sum_{i=0}^{\ell - 1} \sum_{k=\lfloor b_n s \rfloor + 1}^{\lfloor b_n t \rfloor} \Phi_n(\lfloor a_n s \rfloor - i, k)^3 \end{aligned}$$

$$+\sum_{i=1}^{3}\sum_{j=1}^{\lfloor a_n s\rfloor-\ell} \Phi_n(j, \lfloor b_n s\rfloor + i)^3$$
$$+\sum_{j=1}^{\lfloor a_n s\rfloor-\ell}\sum_{k=\lfloor b_n s\rfloor+4}^{\lfloor b_n t\rfloor} \Phi_n(j,k)^3.$$

By (3.10) and (3.13), the first two double sums above converge to zero. Hence, it will suffice to show that

$$\varepsilon_n := \sum_{j=1}^{\lfloor a_n s\rfloor-\ell}\sum_{k=\lfloor b_n s\rfloor+4}^{\lfloor b_n t\rfloor} \Phi_n(j,k)^3 \to 0,$$

as $n \to \infty$.

As before, for all $j \le \lfloor a_n s\rfloor - \ell$ and all $k \ge \lfloor b_n s\rfloor + 4$, we have

$$\frac{j}{a_n} \le s - \frac{\ell}{a_n} < \frac{b_n s - 1}{b_n} \le \frac{\lfloor b_n s\rfloor}{b_n} \le \frac{k-4}{b_n},$$

and the estimate (2.17) implies

$$|\Phi_n(j,k)| \le \frac{2}{9} a_n^{-1} b_n^{-1} \left(\frac{k-1}{b_n} - \frac{j}{a_n}\right)^{-5/3},$$

so that

$$|\varepsilon_n| \le a_n^{-3} b_n^{-3} \sum_{j=1}^{\lfloor a_n s\rfloor-\ell}\sum_{k=\lfloor b_n s\rfloor+4}^{\lfloor b_n t\rfloor} \left(\frac{k-1}{b_n} - \frac{j}{a_n}\right)^{-5}$$
$$= a_n^2 b_n^{-3} \sum_{k=\lfloor b_n s\rfloor+4}^{\lfloor b_n t\rfloor}\sum_{j=1}^{\lfloor a_n s\rfloor-\ell} \left(\frac{(k-1)a_n}{b_n} - j\right)^{-5}.$$

To apply Lemma 5.8, we check that

$$\lfloor a_n s\rfloor - \ell + 1 < a_n s + \frac{a_n}{b_n} = \frac{(b_n s + 1)a_n}{b_n} < \frac{(\lfloor b_n s\rfloor + 2)a_n}{b_n} < \frac{(k-1)a_n}{b_n}.$$

Thus,

$$|\varepsilon_n| \le a_n^2 b_n^{-3} \sum_{k=\lfloor b_n s\rfloor+4}^{\lfloor b_n t\rfloor} \left(\frac{(k-1)a_n}{b_n} - (\lfloor a_n s\rfloor - \ell + 1)\right)^{-4}$$
$$= a_n^{-2} b_n \sum_{k=\lfloor b_n s\rfloor+4}^{\lfloor b_n t\rfloor} \left(k - 1 - \frac{(\lfloor a_n s\rfloor - \ell + 1)b_n}{a_n}\right)^{-4}.$$

To apply Lemma 5.8 once again, we check that

$$1 + \frac{(\lfloor a_n s \rfloor - \ell + 1)b_n}{a_n} < 1 + \frac{\lfloor a_n s \rfloor \, b_n}{a_n} \le 1 + b_n s < \lfloor b_n s \rfloor + 3,$$

so that

$$\begin{aligned} |\varepsilon_n| &\le a_n^{-2} b_n \left(\lfloor b_n s \rfloor + 2 - \frac{(\lfloor a_n s \rfloor - \ell + 1)b_n}{a_n} \right)^{-3} \\ &= a_n^{-2} b_n^{-2} \left(\frac{\lfloor b_n s \rfloor + 2}{b_n} - \frac{\lfloor a_n s \rfloor - \ell + 1}{a_n} \right)^{-3}. \end{aligned}$$

Recalling that $\ell \ge 2$, this gives

$$|\varepsilon_n| < a_n^{-2} b_n^{-2} \left(\frac{b_n s + 1}{b_n} - \frac{a_n s - 1}{a_n} \right)^{-3} = a_n^{-2} b_n^{-2} \left(\frac{1}{a_n} + \frac{1}{b_n} \right)^{-3}.$$

Finally, by Lemma 5.7, we have

$$|\varepsilon_n| \le a_n^{-2} b_n^{-2} \left(\frac{1}{a_n} \right)^{-3/2} \left(\frac{1}{b_n} \right)^{-3/2} = a_n^{-1/2} b_n^{-1/2} \to 0,$$

as $n \to \infty$, and this concludes the proof of (3.7).

For (3.8), note that

$$\begin{aligned} E[(\widetilde{W}_{a_n}(t) - \widetilde{W}_{a_n}(s))\widetilde{W}_{b_n}(s)] = \; & E[\widetilde{W}_{a_n}(t)\widetilde{W}_{b_n}(t)] - E[\widetilde{W}_{a_n}(s)\widetilde{W}_{b_n}(s)] \\ & - E[(\widetilde{W}_{a_n}(t) - \widetilde{W}_{a_n}(s))(\widetilde{W}_{b_n}(t) - \widetilde{W}_{b_n}(s))] \\ & - E[\widetilde{W}_{a_n}(s)(\widetilde{W}_{b_n}(t) - \widetilde{W}_{b_n}(s))]. \end{aligned}$$

By Theorem 3.4 and Remark 3.3, the first three expectations on the right-hand side tend to zero; and by (3.7), the fourth expectation on the right-hand side tends to zero. This proves (3.8) and completes the proof of the lemma. □

As a consequence of Theorem 3.1, Lemma 3.5, and Theorem 3.4 we obtain the following limit result in the case $L \in \{0, \infty\}$.

Corollary 3.6 *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be strictly increasing sequences in $\mathbb{N}$. Let $L_n = b_n/a_n$ and suppose that $L_n \to L \in \{0, \infty\}$. Then $(B, W_{a_n}, W_{b_n}) \to (B, \kappa W^1, \kappa W^2)$ in $D_{\mathbb{R}^3}[0, \infty)$ as $n \to \infty$, where W^1 and W^2 are independent, standard one-dimensional Brownian motions.*

When $\{W_{a_n}\}$ and $\{W_{b_n}\}$ are such that $b_n/a_n \to L \in (0, \infty)$, the situation is much more delicate than in Theorem 3.4. We show that the function $\rho(t)$ is non zero, and in some cases it is not constant.

Theorem 3.7 *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be strictly increasing sequences in $\mathbb{N}$. Let $L_n = b_n/a_n$ and suppose that $L_n \to L \in (0, \infty)$. Let $I = \{n : L_n = L\}$ and $c_n = \gcd(a_n, b_n)$.*

(i) *If I^c is finite, then $L \in \mathbb{Q}$ and, for all $t \geq 0$,*

$$\lim_{n\to\infty} E[W_{a_n}(t)W_{b_n}(t)] = \frac{3t}{4p}\sum_{j=1}^{q} f_L(j/q),$$

where $L = p/q$ and $p, q \in \mathbb{N}$ are relatively prime.

(ii) *If I is finite, then*

$$\lim_{n\to\infty} E[W_{a_n}(1)W_{b_n}(1)] = \frac{3}{4L}\int_0^1 f_L(x)\,dx.$$

(iii) *If I is finite and $c_n \to \infty$, then*

$$\lim_{n\to\infty} E[W_{a_n}(t)W_{b_n}(t)] = \frac{3t}{4L}\int_0^1 f_L(x)\,dx,$$

for all $t \geq 0$.

(iv) *If there exists $k \in \mathbb{N}$ such that $b_n = k \mod a_n$ for all n, then*

$$\lim_{n\to\infty} E[W_{a_n}(t)W_{b_n}(t)] = \frac{3}{4L}\int_0^t \widehat{f_L}(kx)\,dx,$$

for all $t \geq 0$.

Remark 3.8 In Theorem 3.7(iv), we assume that there exists $k \in \mathbb{N}$ such that $b_n = k \mod a_n$ for all n. Note that this implies k/c_n is an integer for all n. In particular, $\{c_n\}$ is a bounded sequence of integers. Moreover, since $\{c_n\}$ is bounded, this implies that I is finite. Comparing parts (ii) and (iv) of Theorem 3.7, it follows that

$$\frac{3}{4L}\int_0^1 \widehat{f_L}(kx)\,dx = \frac{3}{4L}\int_0^1 f_L(x)\,dx,$$

for all $k \in \mathbb{N}$. In fact, more can be said. Letting $y = kx$, we have

$$\begin{aligned}
\frac{3}{4L}\int_0^t \widehat{f}_L(kx)\,dx &= \frac{3}{4Lk}\int_0^{kt} \widehat{f}_L(y)\,dy \\
&= \frac{3}{4Lk}\left(\left(\sum_{j=1}^{\lfloor kt\rfloor}\int_{j-1}^{j} \widehat{f}_L(x)\,dx\right) + \int_{\lfloor kt\rfloor}^{kt} \widehat{f}_L(x)\,dx\right) \\
&= \frac{3}{4Lk}\left(\lfloor kt\rfloor \int_0^1 f_L(x)\,dx + \int_0^{kt-\lfloor kt\rfloor} f_L(x)\,dx\right) \\
&= \frac{3}{4L}\left(\frac{\lfloor kt\rfloor}{k}\int_0^1 f_L(x)\,dx + \frac{1}{k}\int_0^{kt-\lfloor kt\rfloor} f_L(x)\,dx\right).
\end{aligned}$$

Hence,

$$\frac{3}{4L}\int_0^t \widehat{f}_L(kx)\,dx = \frac{3t}{4L}\int_0^1 f_L(x)\,dx,$$

whenever $kt \in \mathbb{N}$. □

Proof of Theorem 3.7 Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be strictly increasing sequences in $\mathbb{N}$. Let $L_n = b_n/a_n$ and suppose that $L_n \to L \in (0,\infty)$. Recall $\widetilde{W}_n(t) = W_n(t) - 3n^{-1/3}B(\lfloor nt\rfloor/n)$, and note that it will suffice to prove the corresponding limits for $\widetilde{W}$ rather than W.

Fix $t \geq 0$. Since $W_n(t) = 0$ if $\lfloor nt\rfloor = 0$, we may assume $t > 0$ and n is sufficiently large so that $\lfloor a_n t\rfloor > 0$ and $\lfloor b_n t\rfloor > 0$. As in (3.9), we have

$$S_n(t) := E[\widetilde{W}_{a_n}(t)\widetilde{W}_{b_n}(t)] = \frac{3}{4}\sum_{j=1}^{\lfloor a_n t\rfloor}\sum_{k=1}^{\lfloor b_n t\rfloor}\Phi_n(j,k)^3.$$

Making the change of index $m = k - \lfloor jL_n\rfloor$, we then have

$$S_n(t) = \frac{3}{4}\sum_{j=1}^{\lfloor a_n t\rfloor}\sum_{m=1-\lfloor jL_n\rfloor}^{\lfloor b_n t\rfloor-\lfloor jL_n\rfloor}\Phi_n(j, m+\lfloor jL_n\rfloor)^3.$$

Note that by (2.24),

$$\Phi_n(j, m+\lfloor jL_n\rfloor)^3 = \frac{1}{b_n} f_{m,L_n}(jL_n - \lfloor jL_n\rfloor). \tag{3.15}$$

If $1 \leq j \leq \lfloor a_n t\rfloor$, then

$$\begin{aligned}
m \leq \lfloor b_n t\rfloor - \lfloor jL_n\rfloor &\iff \lfloor jL_n\rfloor < \lfloor b_n t\rfloor - m + 1 \\
&\iff jL_n < \lfloor b_n t\rfloor - m + 1
\end{aligned}$$

$$\begin{aligned}
&\iff j < \frac{\lfloor b_n t \rfloor - m + 1}{L_n} \\
&\iff j < \left\lceil \frac{\lfloor b_n t \rfloor - m + 1}{L_n} \right\rceil,
\end{aligned}$$

and also

$$\begin{aligned}
m \geq 1 - \lfloor j L_n \rfloor &\iff \lfloor j L_n \rfloor \geq 1 - m \\
&\iff j L_n \geq 1 - m \\
&\iff j \geq \frac{1 - m}{L_n} \\
&\iff j \geq \left\lceil \frac{1 - m}{L_n} \right\rceil.
\end{aligned}$$

Hence, when we reverse the order of summation, we obtain

$$S_n(t) = \frac{3}{4} \sum_{m=1-\lfloor \lfloor a_n t \rfloor L_n \rfloor}^{\lfloor b_n t \rfloor - \lfloor L_n \rfloor} \sum_{j=\ell_{m,n}}^{u_{m,n}} \Phi_n(j, m + \lfloor j L_n \rfloor)^3,$$

where

$$\ell_{m,n} = \left\lceil \frac{1 - m}{L_n} \right\rceil \vee 1, \tag{3.16}$$

$$u_{m,n} = \left(\left\lceil \frac{\lfloor b_n t \rfloor - m + 1}{L_n} \right\rceil - 1 \right) \wedge \lfloor a_n t \rfloor. \tag{3.17}$$

Let us define

$$\beta(m, n) = \sum_{j=\ell_{m,n}}^{u_{m,n}} \Phi_n(j, m + \lfloor j L_n \rfloor)^3,$$

so that we may write

$$S_n(t) = \frac{3}{4} \sum_{m \in \mathbb{Z}} \beta(m, n) \mathbf{1}_{[1-\lfloor \lfloor a_n t \rfloor L_n \rfloor, \lfloor b_n t \rfloor - \lfloor L_n \rfloor]}(m).$$

We wish to apply dominated convergence to this sum.

Choose an integer $M \geq 2$ such that $L_n \leq M$ for all $n \in \mathbb{N}$. Define

$$C_m = \begin{cases} 8 & \text{if } |m| \leq M, \\ 27(|m| - M)^{-3} & \text{if } |m| > M. \end{cases}$$

We claim that $|\beta(m,n)| \le C_m t$ for all m and n. Once we prove this claim, we may use dominated convergence to conclude that

$$\lim_{n\to\infty} S_n(t) = \frac{3}{4} \sum_{m\in\mathbb{Z}} \lim_{n\to\infty} \beta(m,n), \tag{3.18}$$

provided the limit on the right-hand side exists for each fixed m.

To prove the claim, first note that $1 \le \ell_{m,n} \le u_{m,n} \le \lfloor a_n t \rfloor$, so that

$$|\beta(m,n)| \le \sum_{j=1}^{\lfloor a_n t\rfloor} |\Phi_n(j, m + \lfloor jL_n \rfloor)|^3.$$

Thus, by (2.16), we have $|\beta(m,n)| \le 8(a_n^{-1} \wedge b_n^{-1}) \lfloor a_n t \rfloor \le 8t$ for all m and n. We therefore need only consider $|m| > M$.

First suppose $m > M$. Then

$$\frac{m + \lfloor jL_n \rfloor - 1}{b_n} > \frac{m + jL_n - 2}{b_n} > \frac{jL_n}{b_n} = \frac{j}{a_n}.$$

Hence, by (2.15), (2.3), and (2.9),

$$\begin{aligned}
|\Phi_n(j, m + \lfloor jL_n \rfloor)| &= \left| \frac{2}{9} \int_0^{a_n^{-1}} \int_0^{b_n^{-1}} \left(\frac{m + \lfloor jL_n \rfloor - 1}{b_n} - \frac{j}{a_n} + x + y \right)^{-5/3} dx\, dy \right| \\
&\le \int_0^{a_n^{-1}} \int_0^{b_n^{-1}} \left(\frac{m + jL_n - 2}{b_n} - \frac{j}{a_n} + y \right)^{-5/3} dx\, dy \\
&= \int_0^{a_n^{-1}} \int_0^{b_n^{-1}} \left(\frac{m-2}{b_n} + y \right)^{-5/3} dx\, dy.
\end{aligned}$$

By Lemma 5.7,

$$|\Phi_n(j, m + \lfloor jL_n \rfloor)| \le \int_0^{a_n^{-1}} \int_0^{b_n^{-1}} \left(\frac{m-2}{b_n} \right)^{-1} y^{-2/3}\, dx\, dy = 3a_n^{-1/3}(m-2)^{-1}.$$

Thus, $|\beta(m,n)| \le 27a_n^{-1} \sum_{j=1}^{\lfloor a_n t \rfloor} (m-2)^{-3} \le 27t(m-2)^{-3} \le C_m t$.

Next, suppose $m < -M$. Then

$$\frac{m + \lfloor jL_n \rfloor}{b_n} \le \frac{m + jL_n}{b_n} < \frac{-L_n + jL_n}{b_n} = \frac{j-1}{a_n}.$$

Hence, by (2.15) and (2.9),

$$
\begin{aligned}
|\Phi_n(j, m + \lfloor jL_n \rfloor)| &= \left| \frac{2}{9} \int_0^{a_n^{-1}} \int_0^{b_n^{-1}} \left(\frac{j-1}{a_n} - \frac{m + \lfloor jL_n \rfloor}{b_n} + x + y \right)^{-5/3} dx\, dy \right| \\
&\leq \int_0^{a_n^{-1}} \int_0^{b_n^{-1}} \left(\frac{j-1}{a_n} - \frac{m + jL_n}{b_n} + y \right)^{-5/3} dx\, dy \\
&= \int_0^{a_n^{-1}} \int_0^{b_n^{-1}} \left(-\frac{1}{a_n} - \frac{m}{b_n} + y \right)^{-5/3} dx\, dy.
\end{aligned}
$$

By Lemma 5.7,

$$
\begin{aligned}
|\Phi_n(j, m + \lfloor jL_n \rfloor)| &\leq \int_0^{a_n^{-1}} \int_0^{b_n^{-1}} \left(-\frac{1}{a_n} - \frac{m}{b_n} \right)^{-1} y^{-2/3}\, dx\, dy = 3a_n^{-1/3}(-L_n - m)^{-1} \\
&= 3a_n^{-1/3}(|m| - L_n)^{-1} \leq 3a_n^{-1/3}(|m| - M)^{-1}.
\end{aligned}
$$

Thus, $|\beta(m, n)| \leq 27a_n^{-1} \sum_{j=1}^{\lfloor a_n t \rfloor} (|m| - M)^{-3} \leq C_m t$. This proves our claim and establishes (3.18), provided the limit on the right-hand side exists for each fixed m.

Recalling (2.21), let us now define

$$
\widetilde{\beta}(m, n) = \frac{1}{L_n a_n} \sum_{j=1}^{\lfloor a_n t \rfloor} f_{m,L}(jL_n - \lfloor jL_n \rfloor). \tag{3.19}
$$

We will first show that

$$
\lim_{n \to \infty} |\widetilde{\beta}(m, n) - \beta(m, n)| = 0, \tag{3.20}
$$

for each fixed $m \in \mathbb{Z}$. Since $1 \leq \ell_{m,n} \leq u_{m,n} \leq \lfloor a_n t \rfloor$, we have $(\widetilde{\beta} - \beta)(m, n) = A_{m,n} + B_{m,n}$, where

$$
\begin{aligned}
A_{m,n} &= \frac{1}{L_n a_n} \sum_{j=1}^{\ell_{m,n}-1} f_{m,L}(jL_n - \lfloor jL_n \rfloor) + \frac{1}{L_n a_n} \sum_{j=u_{m,n}+1}^{\lfloor a_n t \rfloor} f_{m,L}(jL_n - \lfloor jL_n \rfloor), \\
B_{m,n} &= \sum_{j=\ell_{m,n}}^{u_{m,n}} \left(\frac{1}{L_n a_n} f_{m,L}(jL_n - \lfloor jL_n \rfloor) - \Phi_n(j, m + \lfloor jL_n \rfloor)^3 \right).
\end{aligned}
$$

By (2.23), we have

$$|f_{m,L}(x)| \le 8 \quad \text{for all } m,\ L, \text{ and } x. \tag{3.21}$$

Thus,

$$|A_{m,n}| \le \frac{8}{b_n}(\ell_{m,n} - 1 + \lfloor a_n t \rfloor - u_{m,n}).$$

From (3.16), we see that $\limsup_{n\to\infty} \ell_{m,n} < \infty$. From (3.17), we have

$$\begin{aligned} u_{m,n} &\ge \left(\frac{\lfloor b_n t\rfloor - m + 1}{L_n} - 1\right) \wedge \lfloor a_n t\rfloor \\ &= \lfloor a_n t\rfloor + \left(\left(\frac{\lfloor b_n t\rfloor - m + 1}{L_n} - 1 - \lfloor a_n t\rfloor\right) \wedge 0\right) \\ &= \lfloor a_n t\rfloor - \left(\left(\lfloor a_n t\rfloor - \frac{\lfloor b_n t\rfloor}{L_n} + \frac{m-1}{L_n} + 1\right) \vee 0\right). \end{aligned}$$

Since

$$\lfloor a_n t\rfloor - \frac{\lfloor b_n t\rfloor}{L_n} < a_n t - \frac{b_n t - 1}{L_n} = \frac{1}{L_n},$$

this gives

$$\lfloor a_n t\rfloor - u_{m,n} \le \left(\frac{m}{L_n} + 1\right) \vee 0,$$

and this shows that $\limsup_{n\to\infty}(\lfloor a_n t\rfloor - u_{m,n}) < \infty$. Hence, $A_{m,n} \to 0$ as $n \to \infty$.

For $B_{m,n}$, we may use (3.15) to write

$$B_{m,n} = \frac{1}{b_n}\sum_{j=\ell_{m,n}}^{u_{m,n}} (f_{m,L} - f_{m,L_n})(jL_n - \lfloor jL_n\rfloor).$$

By Lemma 2.5 (i),

$$|f_{m,L}(x) - f_{m,L_n}(x)| \le 24|L_n - L|^{1/3},$$

for all x. This gives

$$|B_{m,n}| \le \frac{24}{b_n}\sum_{j=1}^{\lfloor a_n t\rfloor} |L_n - L|^{1/3} \le \frac{24t}{L_n}|L_n - L|^{1/3} \to 0,$$

as $n \to \infty$, and we have proved (3.20).

Finally, we calculate $\lim_{n\to\infty}\widetilde{\beta}(m,n)$. We begin by rewriting $\widetilde{\beta}(m,n)$ in the following way. For each n, choose $p_n, q_n, c_n \in \mathbb{N}$ such that $a_n = c_n q_n$, $b_n = c_n p_n$, and p_n and q_n are relatively prime. In general, if $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, then let $[p]_q$ denote the unique integer such that $0 \le [p]_q < q$ and $p \equiv [p]_q \mod q$. Note that

$$[p]_q = q\left(\frac{p}{q} - \left\lfloor \frac{p}{q} \right\rfloor\right).$$

Thus,

$$jL_n - \lfloor jL_n \rfloor = \frac{jb_n}{a_n} - \left\lfloor \frac{jb_n}{a_n} \right\rfloor = \frac{jp_n}{q_n} - \left\lfloor \frac{jp_n}{q_n} \right\rfloor = \frac{1}{q_n}[jp_n]_{q_n}. \tag{3.22}$$

Hence, by (3.19),

$$\begin{aligned}\widetilde{\beta}(m,n) &= \frac{1}{L_n a_n}\sum_{j=1}^{\lfloor a_n t\rfloor} f_{m,L}([jp_n]_{q_n}/q_n)\\ &= \frac{1}{L_n c_n q_n}\sum_{j=1}^{\lfloor c_n q_n t\rfloor} f_{m,L}([jp_n]_{q_n}/q_n).\end{aligned}$$

Let α_n, r_n be the unique integers such that $\lfloor c_n q_n t\rfloor = \alpha_n q_n + r_n$ and $0 \le r_n < q_n$. Note that $\alpha_n \ge 0$ and $r_n = [\lfloor c_n q_n t\rfloor]_{q_n}$. Since $h \in \mathbb{Z}$ implies $[p + hq]_q = [p]_q$, we have

$$\begin{aligned}\widetilde{\beta}(m,n) &= \frac{1}{L_n c_n q_n} \times \Bigg(\sum_{h=0}^{\alpha_n - 1}\sum_{j=1}^{q_n} f_{m,L}([(j + hq_n)p_n]_{q_n}/q_n)\\ &\qquad + \sum_{j=1}^{r_n} f_{m,L}([(j + \alpha_n q_n)p_n]_{q_n}/q_n)\Bigg)\\ &= \frac{\alpha_n}{c_n}\frac{1}{L_n q_n}\sum_{j=1}^{q_n} f_{m,L}([jp_n]_{q_n}/q_n) + \frac{1}{L_n a_n}\sum_{j=1}^{r_n} f_{m,L}([jp_n]_{q_n}/q_n).\end{aligned}$$

Also note that if p and q are relatively prime, then

$$\{[jp]_q : 1 \le j \le q\} = \{0, 1, 2, \ldots, q-1\}.$$

Therefore,

$$\widetilde{\beta}(m,n) = \frac{\alpha_n}{c_n}\frac{1}{L_n q_n}\sum_{j=0}^{q_n - 1} f_{m,L}(j/q_n) + \frac{1}{L_n a_n}\sum_{j=1}^{r_n} f_{m,L}([jp_n]_{q_n}/q_n). \tag{3.23}$$

Now let $I = \{n : L_n = L\}$. First assume I is finite and $t = 1$. Then $r_n = 0$ and $\alpha_n = c_n$, so that

$$\widetilde{\beta}(m,n) = \frac{1}{L_n q_n} \sum_{j=0}^{q_n - 1} f_{m,L}(j/q_n).$$

We first prove that $\lim_{n\to\infty} q_n = \infty$. Let $M > 0$ be arbitrary. Let $S = \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}, q \le M\}$. Choose $\varepsilon > 0$ small enough so that $S \cap (L-\varepsilon, L+\varepsilon) \subset \{L\}$. Choose $n_0 \in \mathbb{N}$ large enough so that $I \subset \{1, \dots, n_0\}$, and also $|L_n - L| < \varepsilon$ for all $n > n_0$. Let $n > n_0$ be arbitrary. Then

$$\frac{p_n}{q_n} = \frac{b_n}{a_n} = L_n \in (L-\varepsilon, L+\varepsilon)\backslash\{L\}.$$

Hence, $p_n/q_n \notin S$, which implies $q_n > M$, and this shows that $\lim_{n\to\infty} q_n = \infty$.

Since $f_{m,L}$ is continuous, it now follows that

$$\begin{aligned}\lim_{n\to\infty} \widetilde{\beta}(m,n) &= \lim_{n\to\infty} \frac{1}{L_n q_n} \sum_{j=0}^{q_n-1} f_{m,L}(j/q_n)\\ &= \frac{1}{L} \int_0^1 f_{m,L}(x)\,dx.\end{aligned}$$

By (3.18) and (3.20), we therefore have

$$\lim_{n\to\infty} S_n(1) = \frac{3}{4} \sum_{m\in\mathbb{Z}} \frac{1}{L} \int_0^1 f_{m,L}(x)\,dx.$$

By Lemma 2.4, we may interchange the summation and integration, and this proves Part (ii) of the theorem.

Next, assume I^c is finite and $t > 0$. In this case, there exists n_0 such that $L_n = L$ for all $n \ge n_0$. In particular, $L \in \mathbb{Q}$, so we may write $L = p/q$, where $p, q \in \mathbb{N}$ are relatively prime. In this case, $q_n = q$ for all $n \ge n_0$. Therefore, by (3.23), for all $n \ge n_0$,

$$\widetilde{\beta}(m,n) = \frac{\alpha_n}{c_n} \frac{1}{L_n q} \sum_{j=0}^{q-1} f_{m,L}(j/q) + \varepsilon_n,$$

where, by (3.21),

$$|\varepsilon_n| = \left| \frac{1}{L_n a_n} \sum_{j=1}^{r_n} f_{m,L}([jp]_q/q) \right| \le \frac{8 r_n}{L_n a_n} < \frac{8q}{L_n a_n} \to 0, \tag{3.24}$$

as $n \to \infty$. Also,

$$\left|\frac{\alpha_n}{c_n} - t\right| = \left|\frac{\lfloor c_n q t\rfloor - c_n q t - r_n}{c_n q}\right| \leq \frac{1+q}{a_n} \to 0,$$

as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \widetilde{\beta}(m,n) = \frac{t}{Lq}\sum_{j=0}^{q-1} f_{m,L}(j/q).$$

As above, using (3.18) and (3.20), we have

$$\lim_{n\to\infty} S_n(t) = \frac{3t}{4Lq}\sum_{m\in\mathbb{Z}}\sum_{j=0}^{q-1} f_{m,L}(j/q) = \frac{3t}{4p}\sum_{j=0}^{q-1}\sum_{m\in\mathbb{Z}} f_{m,L}(j/q).$$

Since $f_{m,L}(1) = f_{m-1,L}(0)$, we may write

$$\lim_{n\to\infty} S_n(t) = \frac{3t}{4p}\sum_{j=1}^{q}\sum_{m\in\mathbb{Z}} f_{m,L}(j/q) = \frac{3t}{4p}\sum_{j=1}^{q} f_L(j/q),$$

and this proves Part (i) of the theorem.

Next, assume I is finite and $c_n \to \infty$. Note that

$$r_n = [\lfloor a_n t\rfloor]_{q_n} = q_n\left(\frac{\lfloor a_n t\rfloor}{q_n} - \left\lfloor\frac{\lfloor a_n t\rfloor}{q_n}\right\rfloor\right).$$

Thus,

$$\alpha_n = \frac{\lfloor a_n t\rfloor - r_n}{q_n} = \left\lfloor\frac{\lfloor a_n t\rfloor}{q_n}\right\rfloor.$$

It follows that $\alpha_n \leq a_n t/q_n = c_n t$, and also

$$\alpha_n > \frac{\lfloor a_n t\rfloor}{q_n} - 1 > \frac{a_n t - 1}{q_n} - 1 = c_n t - \frac{1}{q_n} - 1.$$

Hence,

$$t - \left(\frac{1}{a_n} + \frac{1}{c_n}\right) < \frac{\alpha_n}{c_n} \leq t.$$

Since both $c_n \to \infty$ and $a_n \to \infty$, this shows that $\alpha_n/c_n \to t$ as $n \to \infty$.

Also, as in (3.24),

$$\left|\frac{1}{L_n a_n}\sum_{j=1}^{r_n} f_{m,L}([jp]_q/q)\right| < \frac{8q_n}{L_n a_n} = \frac{8}{L_n c_n} \to 0,$$

as $n \to \infty$. Therefore, using (3.23) and the argument immediately following (3.23), we have

$$\lim_{n\to\infty} \widetilde{\beta}(m,n) = \frac{t}{L}\int_0^1 f_{m,L}(x)\,dx.$$

By (3.18) and (3.20), we therefore have

$$\lim_{n\to\infty} S_n(t) = \frac{3t}{4}\sum_{m\in\mathbb{Z}}\frac{1}{L}\int_0^1 f_{m,L}(x)\,dx.$$

By Lemma 2.4, we may interchange the summation and integration, and this proves Part (iii) of the theorem.

Finally, assume there exists $k \in \mathbb{N}$ such that $b_n = k \mod a_n$ for all n. As in (3.22), we may write $jL_n - \lfloor jL_n \rfloor = [jb_n]_{a_n}/a_n$. Hence, by (3.19),

$$\widetilde{\beta}(m,n) = \frac{1}{L_n a_n}\sum_{j=1}^{\lfloor a_n t\rfloor} f_{m,L}([jb_n]_{a_n}/a_n).$$

For n sufficiently large, $k < a_n$, so that

$$k = [b_n - a_n]_{a_n} = a_n\left(\frac{b_n - a_n}{a_n} - \left\lfloor \frac{b_n - a_n}{a_n}\right\rfloor\right).$$

Define $k_n = (b_n - k)/a_n$. Then $b_n = k_n a_n + k$ and

$$k_n = \frac{b_n}{a_n} - \left(\frac{b_n - a_n}{a_n} - \left\lfloor \frac{b_n - a_n}{a_n}\right\rfloor\right) = 1 + \left\lfloor \frac{b_n - a_n}{a_n}\right\rfloor \in \mathbb{N}.$$

Thus,

$$[jb_n]_{a_n} = [j(k_n a_n + k)]_{a_n} = [jk]_{a_n} = a_n\left(\frac{jk}{a_n} - \left\lfloor \frac{jk}{a_n}\right\rfloor\right),$$

giving

$$\widetilde{\beta}(m,n) = \frac{1}{L_n a_n}\sum_{j=1}^{\lfloor a_n t\rfloor} \widehat{f}_{m,L}(jk/a_n).$$

Since $a_n \to \infty$ and $\widehat{f}_{m,L}$ is continuous, we have

$$\lim_{n\to\infty} \widetilde{\beta}(m,n) = \frac{1}{L}\int_0^t \widehat{f}_{m,L}(kx)\,dx.$$

By (3.18) and (3.20), we therefore have

$$\lim_{n\to\infty} S_n(t) = \frac{3}{4}\sum_{m\in\mathbb{Z}} \frac{1}{L}\int_0^t \widehat{f}_{m,L}(kx)\,dx.$$

By Lemma 2.4, we may interchange the summation and integration, and this proves Part (iv) of the theorem. □

As a consequence of Theorem 3.7 we can establish the following result on the convergence in distribution of the sequence (B, W_{a_n}, W_{b_n}).

Corollary 3.9 *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be strictly increasing sequences in $\mathbb{N}$. Let $L_n = b_n/a_n$ and suppose that $L_n \to L \in (0,\infty)$. Let $I = \{n : L_n = L\}$ and $c_n = \gcd(a_n, b_n)$. Given $\rho \in C[0,\infty)$, let σ be given by (3.2) and X^ρ by (3.3).*

(i) *If I^c is finite, then $(B, W_{a_n}, W_{b_n}) \Rightarrow (B, X^\rho)$ in $D_{\mathbb{R}^3}[0,\infty)$ as $n \to \infty$, where*

$$\rho(t) = \frac{3}{4p}\sum_{j=1}^{q} f_L(j/q), \tag{3.25}$$

for all $t \ge 0$. Here, $L \in \mathbb{Q}$ and p and q are determined by $L = p/q$, where $p, q \in \mathbb{N}$ are relatively prime.

(ii) *If I is finite and $c_n \to \infty$, then $(B, W_{a_n}, W_{b_n}) \to (B, X^\rho)$ in law in $D_{\mathbb{R}^3}[0,\infty)$ as $n \to \infty$, where*

$$\rho(t) = \frac{3}{4L}\int_0^1 f_L(x)\,dx,$$

for all $t \ge 0$.

(iii) *If there exists $k \in \mathbb{N}$ such that $b_n = k \mod a_n$ for all n, then $(B, W_{a_n}, W_{b_n}) \to (B, X^\rho)$ in law in $D_{\mathbb{R}^3}[0,\infty)$ as $n \to \infty$, where*

$$\rho(t) = \frac{3}{4L}\widehat{f}_L(kt),$$

for all $t \ge 0$.

Proof First assume I^c is finite. Let $0 \leq s \leq t < \infty$. Let ρ be given by (3.25). By Theorem 3.7 and Remark 3.3,

$$\lim_{n\to\infty} E[\widetilde{W}_{a_n}(s)\widetilde{W}_{b_n}(s)] = \int_0^s \rho(u)\,du.$$

By Lemma 3.5, this gives

$$\lim_{n\to\infty} E[\widetilde{W}_{a_n}(s)\widetilde{W}_{b_n}(t)] = \int_0^s \rho(u)\,du.$$

Part (i) of the theorem now follows from Theorem 3.1 and Remark 3.3. The proofs of Parts (ii) and (iii) are similar. □

4 Remarks and examples

Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be strictly increasing sequences in $\mathbb{N}$. For each $t > 0$, let

$$\gamma^{a,b}(t) = \lim_{n\to\infty} \frac{E[W_{a_n}(t)W_{b_n}(t)]}{\kappa^2 t},$$

provided this limit exists. Then $\gamma^{a,b}(t)$ is the asymptotic correlation of $W_{a_n}(t)$ and $W_{b_n}(t)$ as $n \to \infty$. Under the hypotheses of Corollary 3.9, we have

$$\gamma^{a,b}(t) = \frac{1}{\kappa^2 t}\int_0^t \rho(x)\,dx. \tag{4.1}$$

Note that $\gamma^{a,b}$ is a constant function if and only if ρ is constant, as in Corollary 3.9 (i) and (ii). Also note that, since $\widehat{f_L}(k\,\cdot)$ is not a constant function by Lemma 2.5, Corollary 3.9 (iii) shows that there are circumstances under which $\gamma^{a,b}$ is not constant.

Example 4.1 If $a_n = b_n = n$ for all n, then $E[W_{a_n}(t)W_{b_n}(t)] \to \kappa^2 t$ as $n \to \infty$, so that $\gamma^{a,b} \equiv 1$. By (2.1) and (2.21), we observe that

$$\kappa^2 = \frac{3}{4}\sum_{m\in\mathbb{Z}} f_{m,1}(0) = \frac{3}{4} f_1(0). \tag{4.2}$$

Equivalently, we may write

$$\kappa^2 = 6\sum_{m\in\mathbb{Z}} (E[(B(1)-B(0))(B(m+1)-B(m))])^3.$$

For $L \in \mathbb{N}$, let us define

$$\kappa_L^2 = \frac{6}{L} \sum_{m \in \mathbb{Z}} (E[(B(1) - B(0))(B(m+L) - B(m))])^3.$$

Then using (2.2), (2.21), and Lemma 2.5 (ii), we have

$$\kappa_L^2 = \frac{3}{4L} \sum_{m \in \mathbb{Z}} \Phi(0, 1, m, m+L)^3 = \frac{3}{4L} f_L(0) = \frac{3}{4L} f_L(1).$$

If $a_n = n$ and $b_n = Ln$, then using Theorem 3.7 (i) with $p = L$ and $q = 1$, as well as Lemma 2.5 (ii) and (2.21), we obtain $\lim_{n\to\infty} E[W_n(t)W_{Ln}(t)] = \kappa_L^2 t$, giving $\gamma^{a,b}(t) \equiv \kappa_L^2/\kappa^2$.

Numerical calculations suggest that, in this family of examples, $\gamma^{a,b}$ decreases fairly quickly with L. For example, when $L = 2$, we have $\gamma^{a,b} \approx 0.201928$, and when $L = 5$, we have $\gamma^{a,b} \approx 0.043837$. □

The scaling property of fBm manifests itself in the present investigation via the following result.

Lemma 4.2 *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be strictly increasing sequences in $\mathbb{N}$, and $r \in (0, \infty)$. Assume $a_n^* = ra_n$ and $b_n^* = rb_n$ are integers for all n. Fix $t > 0$. Then*

$$\lim_{n\to\infty} E[W_{a_n}(rt)W_{b_n}(rt)] = r \lim_{n\to\infty} E[W_{a_n^*}(t)W_{b_n^*}(t)],$$

provided that one of the two limits exist.

Proof Recall $\widetilde{W}_n(t) = W_n(t) - 3n^{-1/3}B(\lfloor nt \rfloor/n)$, and note that it will suffice to prove the lemma for $\widetilde{W}$ rather than W. As in (3.9), we have

$$E[\widetilde{W}_{a_n}(rt)\widetilde{W}_{b_n}(rt)] = \frac{3}{4} \sum_{j=1}^{\lfloor a_n rt \rfloor} \sum_{k=1}^{\lfloor b_n rt \rfloor} \Phi_n^{a,b}(j,k)^3.$$

Note that $\Phi_n^{a^*,b^*} = r^{-1/3}\Phi_n^{a,b}$. Thus,

$$\begin{aligned} E[\widetilde{W}_{a_n}(rt)\widetilde{W}_{b_n}(rt)] &= \frac{3r}{4} \sum_{j=1}^{\lfloor a_n rt \rfloor} \sum_{k=1}^{\lfloor b_n rt \rfloor} \Phi_n^{a^*,b^*}(j,k)^3 \\ &= \frac{3r}{4} \sum_{j=1}^{\lfloor a_n^* t \rfloor} \sum_{k=1}^{\lfloor b_n^* t \rfloor} \Phi_n^{a^*,b^*}(j,k)^3 = rE[\widetilde{W}_{a_n^*}(t)\widetilde{W}_{b_n^*}(t)]. \end{aligned}$$

Letting $n \to \infty$ completes the proof. □

Example 4.3 At first glance, Lemma 4.2 may seem to suggest that $\gamma^{a,b}$ should always be the constant function $\gamma^{a,b}(t) \equiv \kappa^{-2} E[W_{a_n}(1) W_{b_n}(1)]$. But, of course, we know this to be false from Corollary 3.9(iii). A simple example illustrating this is the following.

Fix $L, k \in \mathbb{N}$. Let $a_n = n$ and $b_n = Ln + k$. Note that

$$L_n = \frac{b_n}{a_n} = L + \frac{k}{n} \to L,$$

as $n \to \infty$. By Corollary 3.9 (iii), (4.1), and (4.2),

$$\gamma^{a,b}(t) = \frac{1}{L f_1(0) t} \int_0^t \widehat{f_L}(kx)\,dx. \tag{4.3}$$

Lemma 2.5 shows that this is not a constant function, at least when $L = 1$. In this example, Lemma 4.2 implies that for $r \in \mathbb{N}$,

$$\lim_{n\to\infty} E[W_n(rt) W_{Ln+k}(rt)] = r \lim_{n\to\infty} E[W_{rn}(t) W_{rLn+rk}(t)].$$

This does not contradict (4.3), since $\{(W_{rn}, W_{rLn+rk})\}_{n=1}^{\infty}$ is not a subsequence of $\{(W_n, W_{Ln+k})\}_{n=1}^{\infty}$. Rather, it is a subsequence of $\{(W_n, W_{Ln+rk})\}_{n=1}^{\infty}$. Hence, Lemma 4.2 is, in this case, illustrating the equality,

$$\int_0^{rt} \widehat{f_L}(kx)\,dx = r \int_0^t \widehat{f_L}(rkx)\,dx,$$

which is easily verified by a simple change of variable.

Another interesting feature illustrated here is the following. If we fix $L = 1$, then we have a family of examples indexed by k that all share the same limiting ratio, L, yet produce different asymptotic correlation functions. Indeed, if it were the case that $\widehat{f_1}(k_1 \cdot) = \widehat{f_1}(k_2 \cdot)$ for some $k_1 < k_2$, then we would have $\widehat{f_1}(1/2) = \widehat{f_1}(k_1^n / 2k_2^n)$ for all n, and by continuity, $\widehat{f_1}(1/2) = \widehat{f_1}(0)$, contradicting Lemma 2.5.

Note that, by the continuity of f_L, we have $\gamma^{a,b}(t) \to f_L(0)/(L f_1(0))$ as $t \downarrow 0$. Numerical calculations for the case $L = k = 1$ suggest that $\gamma_{a,b}$ is a positive function with $\gamma^{a,b}(0.8) \approx 0.0750475$, so that the asymptotic correlation between $W_n(t)$ and $W_{n+1}(t)$ varies dramatically with t. □

Example 4.4 As an example illustrating Corollary 3.9 (ii), let $L \in \mathbb{N}$ and consider $a_n = n^2$ and $b_n = Ln^2 + n$. Then $L_n = b_n/a_n = L + 1/n \to L$. Since $c_n = \gcd(a_n, b_n) = n$, Corollary 3.9 (ii), (4.1), and (4.2) give

$$\gamma^{a,b}(t) \equiv \frac{1}{L f_1(0)} \int_0^1 f_L(x)\,dx.$$

Numerical calculations suggest that for $L = 1$ and $L = 2$, $\gamma^{a,b} \approx 0.101932$ and $\gamma^{a,b} \approx 0.0468229$, respectively. Note that these numbers are several times smaller than the corresponding numbers for the sequences $a_n = n^2$ and $b_n = Ln^2$, which are covered by Example 4.1. □

Example 4.5 Our penultimate example illustrates a situation where $c_n = \gcd(a_n, b_n)$ is constant, yet the asymptotic correlation $\gamma^{a,b}$ does not exist.

Fix $k \in \mathbb{N}$. Let $a_n = kn^2$ and

$$b_n = \begin{cases} kn^2 + 2k & \text{if } n \text{ is odd,} \\ kn^2 + k & \text{if } n \text{ is even.} \end{cases}$$

Then $c_n = \gcd(a_n, b_n) = k$ for all n. By Theorem 3.7 (iv),

$$\begin{aligned} \lim_{\substack{n\to\infty \\ n \text{ odd}}} E[W_{a_n}(t) W_{b_n}(t)] &= \lim_{m\to\infty} E[W_{k(2m+1)^2}(t) W_{k(2m+1)^2+2k}(t)] \\ &= \frac{3}{4} \int_0^t \widehat{f}_1(2kx)\, dx. \end{aligned}$$

On the other hand,

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} E[W_{a_n}(t) W_{b_n}(t)] = \lim_{m\to\infty} E[W_{4km^2}(t) W_{4km^2+k}(t)] = \frac{3}{4} \int_0^t \widehat{f}_1(kx)\, dx.$$

Since these are different functions, the sequence (W_{a_n}, W_{b_n}) does not converge and $\gamma^{a,b}$ does not exist. □

Example 4.6 Finally, we collect what might be called some non-examples, a few cases which are not covered by our present results. The first is $a_n = n^2$ and $b_n = (n+1)^2$. In this case, $L_n \to 1$, but $\gcd(a_n, b_n) = 1$ for all n, and $b_n - a_n = 2n + 1 \mod a_n$. By Theorem 3.7 (ii), we know that $\gamma^{a,b}(1)$ exists, but the existence and value of $\gamma^{a,b}(t)$ for $t \neq 1$ is not covered by our results.

The second non-example is $a_n = 2n$ and $b_n = 3n + 1$. In this case, $L_n \to 3/2$, but $\gcd(a_n, b_n) \leq 2$ for all n, and $b_n - a_n = n + 1 \mod a_n$. Again our results fail to give a complete picture of the function $\gamma^{a,b}$.

Our last non-example is the following. Let $\alpha \in (1, 2)$ be an irrational number whose decimal expansion contains only the digits 1, 3, 7, and 9. In other words, $\alpha = 1 + \sum_{j=1}^{\infty} c_j 10^{-j}$, where $c_j \in \{1, 3, 7, 9\}$ for all j. Let $s_n = \sum_{j=1}^{n} c_j 10^{-j}$, and define $a_n = 10^n$ and $b_n = 10^n(1 + s_n)$. In this case, $L_n \to \alpha$, but $\gcd(a_n, b_n) = 1$ for all n, and $b_n - a_n = 10^n s_n \mod a_n$. Again our results tell us only the existence and value of $\gamma^{a,b}(1)$.

There are, of course, many examples such as these which are not covered by Theorem 3.7. Developing a more general set of results that describe the asymptotic behavior of the correlation of $W_{a_n}(t)$ and $W_{b_n}(t)$ in these examples is an open problem to be studied in subsequent work. □

Acknowledgments The authors are grateful for the careful reading and helpful suggestions of an anonymous referee. Krzysztof Burdzy was partially supported by Grant DMS-1206276 from the NSF and by Grant N N201 397137 from the MNiSW, Poland. David Nualart was partially supported by Grant DMS-1208625 from the NSF.

Appendix

In this section we include a couple of technical results that are used along the paper.

Lemma 5.7 *If a_j and p_j are positive real numbers, then*

$$\left(\sum_{j=1}^{n} a_j\right)^{-\sum_{j=1}^{n} p_j} \leq \prod_{j=1}^{n} a_j^{-p_j}.$$

Proof For every $k = 1, \ldots, n$ we have

$$\left(\sum_{j=1}^{n} a_j\right)^{-p_k} \leq a_k^{-p_k},$$

and the desired result follows by taking the product of these terms in k. □

Lemma 5.8 *Let $a, b \in \mathbb{Z}$ with $a \leq b$. Let $C \in \mathbb{R}$ and $p > 1$. If $b + 1 < C$, then*

$$\sum_{j=a}^{b} (C - j)^{-p} \leq \frac{1}{p-1}(C - (b+1))^{-(p-1)}.$$

If $C < a - 1$, then

$$\sum_{k=a}^{b} (k - C)^{-p} \leq \frac{1}{p-1}(a - 1 - C)^{-(p-1)}.$$

Proof If $b + 1 < C$, then

$$\begin{aligned}
\frac{1}{p-1}(C - (b+1))^{-(p-1)} &\geq \frac{1}{p-1}((C - (b+1))^{-(p-1)} - (C - a)^{-(p-1)}) \\
&= \int_{a}^{b+1} (C - x)^{-p}\, dx = \sum_{j=a}^{b} \int_{j}^{j+1} (C - x)^{-p}\, dx \\
&\geq \sum_{j=a}^{b} (C - j)^{-p}.
\end{aligned}$$

If $C < a - 1$, then

$$\begin{aligned}\frac{1}{p-1}(a-1-C)^{-(p-1)} &\geq \frac{1}{p-1}((a-1-C)^{-(p-1)} - (b-C)^{-(p-1)}) \\ &= \int_{a-1}^{b} (x-C)^{-p}\,dx = \sum_{k=a}^{b} \int_{k-1}^{k} (x-C)^{-p}\,dx \\ &\geq \sum_{k=a}^{b} (k-C)^{-p}.\end{aligned}$$

□

References

1. Breuer, P., Major, P.: Central limit theorems for nonlinear functionals of Gaussian fields. J. Multivar. Anal. **13**(3), 425–441 (1983)
2. Burdzy, K., Swanson, J.: A change of variable formula with Itô correction term. Ann. Probab. **38**(5), 1817–1869 (2010)
3. Giraitis, L., Surgailis, D.: CLT and other limit theorems for functionals of Gaussian processes. Z. Wahrsch. Verw. Gebiete **70**(2), 191–212 (1985)
4. Nourdin, I., Peccati, G.: Normal Approximations with Malliavin Calculus: From Stein's Method to Universality. Cambridge University Press, Cambridge (2012)
5. Nourdin, I., Peccati, G.: Quantitative Breuer-Major theorems. Stochastic Process. Appl. **121**(4), 793–812 (2011)
6. Nourdin, I., Réveillac, A.: Asymptotic behavior of weighted quadratic variations of fractional Brownian motion: the critical case $H = 1/4$. Ann. Probab. **37**(6), 2200–2230 (2009)
7. Nourdin, I., Réveillac, A., Swanson, J.: The weak Stratonovich integral with respect to fractional Brownian motion with Hurst parameter 1/6. Electron. J. Probab. **15**(70), 2117–2162 (2010)
8. Nualart, D., Ortiz-Latorre, S.: Central limit theorems for multiple stochastic integrals and Malliavin calculus. Stochastic Process. Appl. **118**(4), 614–628 (2008)
9. Nualart, D.: The Malliavin Calculus and Related Topics. Probability and its Applications (New York), 2nd edn. Springer, Berlin (2006)
10. Peccati, G, Tudor, C.A.: Gaussian Limits for Vector-Valued Multiple Stochastic Integrals. In: Séminaire de Probabilités XXXVIII, vol. 1857 of lecture notes in Math., pp. 247–262. Springer, Berlin (2005)
11. Swanson, J.: The calculus of differentials for the weak Stratonovich integral. In: Viens, F., Feng, J., Hu, Y., Nualart, E. (eds.) Malliavin Calculus and Stochastic Analysis. Springer Proceedings in Mathematics & Statistics, vol. 34, pp. 95–111. Springer, US (2013)