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A CHANGE OF VARIABLE FORMULA WITH ITÔ CORRECTION TERM

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We consider the solution $u(x,t)$ to a stochastic heat equation. For fixed x, the process $F(t) = u(x,t)$ has a nontrivial quartic variation. It follows that F is not a semimartingale, so a stochastic integral with respect to F cannot be defined in the classical Itô sense. We show that for sufficiently differentiable functions $g(x,t)$, a stochastic integral $\int g(F(t),t)\,dF(t)$ exists as a limit of discrete, midpoint-style Riemann sums, where the limit is taken in distribution in the Skorokhod space of cadlag functions. Moreover, we show that this integral satisfies a change of variable formula with a correction term that is an ordinary Itô integral with respect to a Brownian motion that is independent of F.

1. Introduction. Recall that the classical Itô formula (i.e., change of variable formula) contains a "stochastic correction term" that is a Riemann integral. A purely intuitive conjecture is that the Itô integral itself may appear as a stochastic correction term in a change of variable formula when the underlying stochastic process has fourth order scaling properties. The first formula of this type was proven in [1]; however, the "fourth order scaling" process considered in that paper was a highly abstract object with little intuitive appeal. The present article presents a change of variable formula with Itô correction term for a family of processes with fourth order local scaling properties; see (1.5) and Corollary 6.4.

The process which is our primary focus is the solution, $u(x,t)$, to the stochastic heat equation $\partial_t u = \frac{1}{2}\,\partial_x^2 u + \dot{W}(x,t)$ with initial conditions $u(x,0) \equiv 0$, where $\dot{W}$ is a space–time white noise on $\mathbb{R} \times [0,\infty)$. That is,

$$u(x,t) = \int_{\mathbb{R}\times[0,t]} p(x-y,t-r)W(dy \times dr), \tag{1.1}$$

where $p(x,t) = (2\pi t)^{-1/2}e^{-x^2/2t}$ is the heat kernel. Let $F(t) = u(x,t)$, where $x \in \mathbb{R}$ is fixed. In the prequel to this paper [15], it was shown that F is a continuous,

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centered Gaussian process with covariance function

$$\rho(s,t) = EF(s)F(t) = (2\pi)^{-1/2}(|t+s|^{1/2} - |t-s|^{1/2}) \tag{1.2}$$

and that F has a nontrivial quartic variation. In particular,

$$\sum_{j=1}^{n} \left|F(j/n) - F\big((j-1)/n\big)\right|^4 \to \frac{6}{\pi}$$

in L^2. It follows that F is not a semimartingale, so a stochastic integral with respect to F cannot be defined in the classical Itô sense. In this paper, we complete the construction of a stochastic integral with respect to F which is a limit of discrete Riemann sums.

More generally, we shall construct a stochastic integral with respect to any process X of the form $X = cF + \xi$, where $c \in \mathbb{R}$ and ξ is a continuous stochastic process, independent of F, satisfying

$$\xi \in C^1((0,\infty)) \quad \text{and} \quad \overline{\lim_{t\to 0}}|\xi'(t)| < \infty \qquad \text{a.s.} \tag{1.3}$$

This allows us, for example, to consider solutions to (1.1) with nonzero initial conditions. Another example of such an X is fractional Brownian motion with Hurst parameter $1/4$; see Examples 6.7 and 6.8 for more details.

Note that ξ (and therefore X) need not be a Gaussian process. If it is Gaussian, however, its mean function will be $\mu_X(t) = EX(t) = \mu_\xi(t)$ and its covariance function will be $\rho_X(s,t) = c^2\rho(s,t) + \rho_\xi(s,t)$. Conversely, the results in this paper will apply to any Gaussian process X whose mean and covariance have the respective forms $\mu_X = \widetilde{\mu}$ and $\rho_X = c^2\rho + \widetilde{\rho}$, where $\widetilde{\mu}$ and $\widetilde{\rho}$ are the mean and covariance, respectively, of a Gaussian process satisfying (1.3).

We conjecture that the results in this paper hold when ξ is only required to be of bounded variation. We require ξ to be C^1, however, because of our particular method of proof; see the proofs of Corollaries 4.6 and 6.4 for further details.

For simplicity, we consider only evenly spaced partitions. That is, given a positive integer n, let $\Delta t = n^{-1}$, $t_j = j\Delta t$ and $\Delta X_j = X(t_j) - X(t_{j-1})$. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For $g \in C(\mathbb{R} \times [0,\infty))$, we consider the midpoint-style Riemann sums

$$I_n^X(g,t) = \sum_{j=1}^{\lfloor nt/2 \rfloor} g(X(t_{2j-1}), t_{2j-1})\big(X(t_{2j}) - X(t_{2j-2})\big). \tag{1.4}$$

When $X = F$, we will simply write I_n, rather than I_n^F.

In the construction of the classical Itô integral, the quadratic variation of the integrator plays a crucial role. Although the quadratic variation of X is infinite, the "alternating quadratic variation" of X is finite. That is, $Q_n^X(t) = \sum_{j=1}^{\lfloor nt/2 \rfloor}(\Delta X_{2j}^2 - \Delta X_{2j-1}^2)$ converges in law. If we denote the limit process by $\{X\}_t$, then it is a simple corollary of the main result in [15] that $\{X\}_t$ is a Brownian motion which is independent of X. More specifically, $(X, Q_n^X) \to (X, \kappa c^2 B)$, where B is a standard Brownian motion, independent of X, and $\kappa \approx 1.029$ [see (2.10) for the precise definition of κ]. The convergence here is in law in the Skorokhod space of cadlag functions from $[0, \infty)$ to $\mathbb{R}^2$, denoted by $D_{\mathbb{R}^2}[0, \infty)$.

We shall show that $I_n^X(g, t)$ also converges in law. If $\int_0^t g(X(s), s)\, dX(s)$ denotes a process with this limiting law, then our main result (Corollary 6.4) is the following change of variable formula:

$$g(X(t), t) = g(X(0), 0) + \int_0^t \partial_t g(X(s), s)\, ds + \int_0^t \partial_x g(X(s), s)\, dX(s) + \frac{1}{2} \int_0^t \partial_x^2 g(X(s), s)\, d\{X\}_s,$$

where the equality is in law as processes. This can be rewritten as

$$g(X(t), t) = g(X(0), 0) + \int_0^t \partial_t g(X(s), s)\, ds + \int_0^t \partial_x g(X(s), s)\, dX(s) + \frac{\kappa c^2}{2} \int_0^t \partial_x^2 g(X(s), s)\, dB(s), \tag{1.5}$$

where this last integral is a classical Itô integral with respect to a standard Brownian motion that is independent of X.

To state our results more completely, let Y be a semimartingale and define

$$I^{X,Y}(\partial_x g, t) = g(X(t), t) - g(X(0), 0) - \int_0^t \partial_t g(X(s), s)\, ds - \frac{\kappa}{2} \int_0^t \partial_x^2 g(X(s), s)\, dY(s). \tag{1.6}$$

We show that

$$(F, Q_n^F, I_n^X(\partial_x g, \cdot)) \to (F, \kappa B, I^{X, c^2 B}(\partial_x g, \cdot))$$

in law in $D_{\mathbb{R}^3}[0, \infty)$ whenever $g \in C_4^{9,1}(\mathbb{R} \times [0, \infty))$. [See (3.2)–(3.5) for the precise definition of the space $C_r^{k,1}$. Also see Remarks 6.5 and 6.6.]

The benefit of having the convergence of this triple, rather than just the Riemann sums, can be seen if one considers two separate sequences of sums: $\{I_n^{X_1}(g_1, \cdot)\}$ and $\{I_n^{X_2}(g_2, \cdot)\}$. As $n \to \infty$, these sequences will converge jointly in law. Separately, each limit will satisfy (1.5); moreover, the Brownian motions which appear in the two limits will be identical. In this sense, the Brownian motion in (1.5) depends only on F and not on ξ, c or g. Clearly, this can be extended to any finite collection of sequences of Riemann sums.

In the course of our analysis, we will also obtain the asymptotic behavior of the trapezoid-style sum

$$T_n^X(g,t) = \sum_{j=1}^{\lfloor nt \rfloor} \frac{g(X(t_{j-1}), t_{j-1}) + g(X(t_j), t_j)}{2} \Delta X_j. \tag{1.7}$$

We shall see (Corollary 4.6) that $T_n^X(\partial_x g, t) \to g(X(t), t) - g(X(0), 0) - \int_0^t \partial_t g(X(s), s)\, ds$ uniformly on compacts in probability (ucp) whenever $g \in C_3^{7,1}(\mathbb{R} \times [0, \infty))$. This result remains true even when $X = cF + \xi$, where ξ satisfies only (1.3), and is not necessarily independent of F.

It is instructive to contrast these results with those of Russo, Vallois and coauthors [5, 6, 13, 14], who, in the context of fractional Brownian motion, use a regularization procedure to transform these Riemann sums into integrals before passing to the limit; see also [2]. For instance, if g does not depend on t, then the regularized midpoint sum is

$$\frac{1}{2\varepsilon} \int_0^t g'(F(s)) \big(F(s+\varepsilon) - F\big((s-\varepsilon) \vee 0\big)\big)\, ds$$

and the regularized trapezoid sum is

$$\frac{1}{2\varepsilon} \int_0^t \big(g'(F(s)) + g'\big(F(s+\varepsilon)\big)\big)\big(F(s+\varepsilon) - F(s)\big)\, ds.$$

Using a change of variables, we can see that if g' is locally integrable, then the difference between these two integrals goes to zero almost surely as $\varepsilon \to 0$. Hence, under the regularization procedure, the midpoint and trapezoid sums exhibit the same limiting behavior: they converge ucp to integrals satisfying the classical change of variable formula from ordinary calculus. Under the discrete approach which we are following, however, we see new behavior for the midpoint sum: the emergence of a correction term which is a classical Itô integral against an independent Brownian motion.

It should be noted that all of our convergence results rely on the fact that F is a quartic variation process. That is,

$$C_1 \Delta t^{2H} \le E \Delta F_j^2 \le C_2 \Delta t^{2H}, \tag{1.8}$$

where $H = 1/4$. For example, the convergence of Q_n^F to a Brownian motion is made plausible by the fact that it is a sum of terms of the form $\Delta F_{2j}^2 - \Delta F_{2j-1}^2$, each of which is approximately mean zero with an approximate variance of Δt. If we replace F by a rougher process which satisfies (1.8) for some $H < 1/4$, then the midpoint sums will evidently diverge. On the other hand, the ucp convergence of the trapezoid sums $T_n(\partial_x g, t)$ remains plausible for any $H > 1/6$. This is consistent with the analogous results in [2, 5] for regularized sums.

The critical case for the trapezoid sum is $H = 1/6$. At the time of writing, we know of only one result in this case. If $g(x,t) = x^3$, then

$$T_n(\partial_x g, t) \approx F(t)^3 - F(0)^3 + \frac{1}{2}\sum_{j=1}^{\lfloor nt \rfloor} \Delta F_j^3.$$

[Here, and in what follows, $X_n(t) \approx Y_n(t)$ shall mean that $X_n - Y_n \to 0$ ucp.] If F is replaced by fractional Brownian motion with Hurst parameter $H = 1/6$, then this last sum converges in law to a Brownian motion; see [12], for example. It is natural to conjecture that a result analogous to (1.5) also holds in this case.

Our project is related to, and inspired by, several areas of stochastic analysis. Recently, a new approach to integration was developed by T. Lyons (with coauthors, students and other researchers). The new method is known as "rough paths"; an introduction can be found in [9]. Our approach is much more elementary since it is based on a form of Riemann sums. We consider it of interest to see how far the classical methods can be pushed and what they can yield. The Itô-type correction term in our change of variable formula has a certain elegance to it, and a certain logic, if we recall that our underlying process has quartic variation. Finally, our project can be considered a toy model for some numerical schemes. The fact that the correction term in the change of variable formula involves an independent Brownian motion may give some information about the form and size of errors in numerical schemes.

After the first draft of this paper had been finished, we received a preprint [10] from Nourdin and Réveillac, prepared independently of ours and using different methods. That paper contains a number of results, one of which, Theorem 1.2, is a special case of our Corollary 6.4. Namely, if $X = B^{1/4}$ (fractional Brownian motion with Hurst parameter $H = 1/4$), if g does not depend on t and if g satisfies an additional moment condition (see $\mathbf{H}_q$ in Section 3 of [10]), then [10] gives the convergence in distribution of the scalar-valued random variables $I_n^X(g', 1)$. While [10] is devoted exclusively to fractional Brownian motion, it is mentioned in a footnote that a Girsanov-type transformation can be used to extend the results from $B^{1/4}$ to F.

2. Preliminaries.

2.1. *Tools for cadlag processes.* Here, and in the remainder of this paper, C shall denote a constant whose value may change from line to line.

Let $D_{\mathbb{R}^d}[0,\infty)$ denote the space of cadlag functions from $[0,\infty)$ to $\mathbb{R}^d$ endowed with the Skorokhod topology. We use the notation $x(t-) = \lim_{s\uparrow t} x(s)$ and $\Delta x(t) = x(t) - x(t-)$. Note that if $F_n(t) = F(\lfloor nt \rfloor/n)$, then $\Delta F_n(t_j) = F(t_j) - F(t_{j-1})$. As in Section 1, we shall typically use ΔF_j as a shorthand notation for $\Delta F_n(t_j)$.

We note for future reference that if x is continuous, then $x_n \to x$ in the Skorokhod topology if and only if $x_n \to x$ uniformly on compacts. For our convergence results, we shall use the following moment condition for relative compactness, which is a consequence of Theorem 3.8.8 in [4].

THEOREM 2.1. *Let $\{X_n\}$ be a sequence of processes in $D_{\mathbb{R}^d}[0,\infty)$. Let $q(x) = |x| \wedge 1$. Suppose that for each $T > 0$, there exist $\nu > 0$, $\beta > 0$, $C > 0$ and $\theta > 1$ such that*:

(i) $E[q(X_n(t+h) - X_n(t))^{\beta/2} q(X_n(t) - X_n(t-h))^{\beta/2}] \le Ch^\theta$ *for all n and all* $0 \le t \le T+1$, $0 \le h \le t$;
(ii) $\lim_{\delta\to 0} \sup_n E[q(X_n(\delta) - X_n(0))^\beta] = 0$;
(iii) $\sup_n E[|X_n(T)|^\nu] < \infty$.

Then $\{X_n\}$ is relatively compact, that is, the distributions are relatively compact in the topology of weak convergence.

COROLLARY 2.2. *Let $\{X_n\}$ be a sequence of processes in $D_{\mathbb{R}^d}[0,\infty)$. Let $q(x) = |x| \wedge 1$. Let φ_1, φ_2 be nonnegative functions of n such that* $\sup_n n^{-1}\varphi_1(n) \times \varphi_2(n) < \infty$. *Suppose that for each $T > 0$, there exist $\nu > 0$, $\beta > 0$, $C > 0$ and $\theta > 1$ such that* $\sup_n E[|X_n(T)|^\nu] < \infty$ *and*

$$E\big[q\big(X_n(t) - X_n(s)\big)^\beta\big] \le C\bigg(\frac{\varphi_2(n)\lfloor\varphi_1(n)t\rfloor - \varphi_2(n)\lfloor\varphi_1(n)s\rfloor}{n}\bigg)^\theta \tag{2.1}$$

for all n and all $0 \le s, t \le T$. Then $\{X_n\}$ is relatively compact.

PROOF. We apply Theorem 2.1. By hypothesis, condition (iii) holds. Taking $s = 0$ and $t = \delta$ in (2.1) gives condition (ii). By Hölder's inequality,

$$\begin{aligned} &E\big[q\big(X_n(t+h) - X_n(t)\big)^{\beta/2} q\big(X_n(t) - X_n(t-h)\big)^{\beta/2}\big] \\ &\qquad \le C\bigg(\frac{\varphi_2(n)\lfloor\varphi_1(n)(t+h)\rfloor - \varphi_2(n)\lfloor\varphi_1(n)t\rfloor}{n}\bigg)^{\theta/2} \\ &\qquad\qquad \times \bigg(\frac{\varphi_2(n)\lfloor\varphi_1(n)t\rfloor - \varphi_2(n)\lfloor\varphi_1(n)(t-h)\rfloor}{n}\bigg)^{\theta/2}. \end{aligned}$$

If $\varphi_1(n)h < 1/2$, then the right-hand side of the above inequality is zero. Assume that $\varphi_1(n)h \ge 1/2$. Then

$$\begin{aligned} &E\big[q\big(X_n(t+h) - X_n(t)\big)^{\beta/2} q\big(X_n(t) - X_n(t-h)\big)^{\beta/2}\big] \\ &\qquad \le C\bigg(\frac{\varphi_2(n)\varphi_1(n)h + \varphi_2(n)}{n}\bigg)^\theta \le \widetilde{C}\bigg(h + \frac{1}{\varphi_1(n)}\bigg)^\theta \le \widetilde{C}(3h)^\theta, \end{aligned}$$

which verifies condition (i). $\square$

In general, the relative compactness in $D_{\mathbb{R}}[0,\infty)$ of $\{X_n\}$ and $\{Y_n\}$ does not imply the relative compactness of $\{X_n + Y_n\}$. This is because addition is not a continuous operation from $D_{\mathbb{R}}[0,\infty)^2$ to $D_{\mathbb{R}}[0,\infty)$. It is, however, a continuous operation from $D_{\mathbb{R}^2}[0,\infty)$ to $D_{\mathbb{R}}[0,\infty)$. To make use of this, we shall need the following well-known result and its subsequent corollary.

LEMMA 2.3. *Suppose that $x_n \to x$ in $D_{\mathbb{R}}[0,\infty)$ and $y_n \to y$ in $D_{\mathbb{R}}[0,\infty)$. If $\Delta x(t)\Delta y(t) = 0$ for all $t \geq 0$, then $x_n + y_n \to x + y$ in $D_{\mathbb{R}}[0,\infty)$.*

COROLLARY 2.4. *Suppose that the sequences $\{X_n\}$ and $\{Y_n\}$ are relatively compact in $D_{\mathbb{R}}[0,\infty)$. If every subsequential limit of $\{Y_n\}$ is continuous, then $\{X_n + Y_n\}$ is relatively compact.*

The following lemma is Problem 3.22(c) in [4].

LEMMA 2.5. *For fixed $d \geq 2$, $\{(X_n^1,\dots,X_n^d)\}$ is relatively compact in $D_{\mathbb{R}^d}[0,\infty)$ if and only if $\{X_n^k\}$ and $\{X_n^k + X_n^\ell\}$ are relatively compact in $D_{\mathbb{R}}[0,\infty)$ for all k and ℓ.*

We will also need the following lemma, which connects relative compactness and convergence in probability. This is Lemma A2.1 in [3].

LEMMA 2.6. *Let $\{X_n\}$, X be processes with sample paths in $D_{\mathbb{R}^d}[0,\infty)$ defined on the same probability space. Suppose that $\{X_n\}$ is relatively compact in $D_{\mathbb{R}^d}[0,\infty)$ and that for a dense set $H \subset [0,\infty)$, $X_n(t) \to X(t)$ in probability for all $t \in H$. Then $X_n \to X$ in probability in $D_{\mathbb{R}^d}[0,\infty)$. In particular, if X is continuous, then $X_n \to X$ ucp.*

Our primary tool is the following theorem, which is a special case of Theorem 2.2 in [7].

THEOREM 2.7. *For each n, let Y_n be a cadlag, $\mathbb{R}^m$-valued semimartingale with respect to a filtration $\{\mathcal{F}_t^n\}$. Suppose that $Y_n = M_n + A_n$, where M_n is an $\{\mathcal{F}_t^n\}$-local martingale and A_n is a finite variation process, and that*

$$\sup_n E\big[[M_n]_t + V_t(A_n)\big] < \infty \tag{2.2}$$

for each $t \geq 0$, where $V_t(A_n)$ is the total variation of A_n on $[0,t]$ and $[M_n]$ is the quadratic variation of M_n. Let X_n be a cadlag, $\{\mathcal{F}_t^n\}$-adapted, $\mathbb{R}^{k\times m}$-valued process and define

$$Z_n(t) = \int_0^t X_n(s-)\,dY_n(s).$$

Suppose that $(X_n, Y_n) \to (X, Y)$ *in law in* $D_{\mathbb{R}^{k\times m}\times\mathbb{R}^m}[0,\infty)$. *Then,* Y *is a semimartingale with respect to a filtration to which* X *and* Y *are adapted and* $(X_n, Y_n, Z_n) \to (X, Y, Z)$ *in law in* $D_{\mathbb{R}^{k\times m}\times\mathbb{R}^m\times\mathbb{R}^k}[0,\infty)$, *where*

$$Z(t) = \int_0^t X(s-)\, dY(s).$$

If $(X_n, Y_n) \to (X, Y)$ *in probability, then* $Z_n \to Z$ *in probability.*

REMARK 2.8. In the setting of Theorem 2.7, if $\{W_n\}$ is another sequence of cadlag, $\{\mathcal{F}_t^n\}$-adapted, $\mathbb{R}^\ell$-valued processes and (W_n, X_n, Y_n) converges to (W, X, Y) in law in $D_{\mathbb{R}^\ell\times\mathbb{R}^{k\times m}\times\mathbb{R}^m}[0,\infty)$, then (W_n, X_n, Y_n, Z_n) converges to (W, X, Y, Z) in law in $D_{\mathbb{R}^\ell\times\mathbb{R}^{k\times m}\times\mathbb{R}^m\times\mathbb{R}^k}[0,\infty)$. This can be seen by applying Theorem 2.7 to $(\overline{X}_n, \overline{Y}_n)$, where $\overline{X}_n$ is the block diagonal $(k+\ell)\times(m+1)$ matrix with upper-left entry W_n and lower-right entry X_n, and $\overline{Y}_n = (0, Y_n^T)^T$.

2.2. *Estimates from the prequel.* We now recall some of the basic estimates from [15].

By (2.6) in [15], for all $s \le t$,

$$\big|E|F(t) - F(s)|^2 - (2/\pi)^{1/2}|t-s|^{1/2}\big| \le \pi^{-1/2}(1+2^{1/2})^{-1}t^{-3/2}|t-s|^2.$$

Hence,

$$\pi^{-1/2}|t-s|^{1/2} \le E|F(t) - F(s)|^2 \le 2|t-s|^{1/2}. \tag{2.3}$$

In particular, if $\sigma_j^2 = E\Delta F_j^2$, then

$$|\sigma_j^2 - (2/\pi)^{1/2}\Delta t^{1/2}| \le t_j^{-3/2}\Delta t^2 = j^{-3/2}\Delta t^{1/2} \tag{2.4}$$

and

$$\pi^{-1/2}\Delta t^{1/2} \le \sigma_j^2 \le 2\Delta t^{1/2}. \tag{2.5}$$

Theorem 2.3 in [15] shows that F has a nontrivial quartic variation. A special case of this theorem is the fact that $\sum_{j=1}^{\lfloor nt\rfloor} \Delta F_j^4 \to 6t/\pi$ ucp. The proof can be easily adapted to show that

$$\sum_{\substack{j=1\\ j \text{ odd}}}^{\lfloor nt\rfloor} \Delta F_j^4 \to \frac{3}{\pi}t \quad\text{and}\quad \sum_{\substack{j=1\\ j \text{ even}}}^{\lfloor nt\rfloor} \Delta F_j^4 \to \frac{3}{\pi}t \tag{2.6}$$

ucp.

Let

$$\gamma_j = 2j^{1/2} - (j-1)^{1/2} - (j+1)^{1/2} \tag{2.7}$$

and note that $\sum_{j=1}^\infty \gamma_j = 1$. By (2.4) in [15], if $i < j$, then

$$|E[\Delta F_i \Delta F_j] + (2\pi)^{-1/2}\gamma_{j-i}\Delta t^{1/2}| \le (t_i + t_j)^{-3/2}\Delta t^2 = (i+j)^{-3/2}\Delta t^{1/2}.$$

Some related estimates are $0 < \gamma_j \le 2^{-1/2} j^{-3/2}$, which is (2.8) in [15], and

$$-2(t_j - t_i)^{-3/2} \Delta t^2 = -2(j-i)^{-3/2} \Delta t^{1/2} \le E[\Delta F_i \Delta F_j] < 0, \tag{2.8}$$

which precedes (2.10) in [15].

Let $\widehat{\sigma}_j = E[F(t_{j-1}) \Delta F_j]$. Since

$$\widehat{\sigma}_j + (2\pi)^{-1/2} \Delta t^{1/2} = \sum_{i=1}^{j-1} \bigl(E[\Delta F_i \Delta F_j] + (2\pi)^{-1/2} \gamma_{j-i} \Delta t^{1/2}\bigr) + (2\pi)^{-1/2} \Delta t^{1/2} \sum_{i=j}^{\infty} \gamma_i,$$

it follows that

$$|\widehat{\sigma}_j + (2\pi)^{-1/2} \Delta t^{1/2}| \le C j^{-1/2} \Delta t^{1/2}. \tag{2.9}$$

In particular, $|\widehat{\sigma}_j| \le C \Delta t^{1/2}$ and $|\widehat{\sigma}_j^2 - (2\pi)^{-1} \Delta t| \le C j^{-1/2} \Delta t$.

LEMMA 2.9. *If integers* c, i *and* j *satisfy* $0 \le c < i \le j$, *then*:

(i) $|E[(F(t_{i-1}) - F(t_c)) \Delta F_j]| \le C \Delta t^{1/2} ((j-i) \vee 1)^{-1/2}$;
(ii) $|E[(F(t_{j-1}) - F(t_c)) \Delta F_i]| \le C \Delta t^{1/2} [((j-i) \vee 1)^{-1/2} + (i-c)^{-1/2}]$;
(iii) $|E[F(t_{j-1}) \Delta F_i]| \le C \Delta t^{1/2} ((j-i) \vee 1)^{-1/2}$.

PROOF. By (2.8),

$$\bigl|E\bigl[\bigl(F(t_{i-1}) - F(t_c)\bigr) \Delta F_j\bigr]\bigr| \le \sum_{k=c+1}^{i-1} |E[\Delta F_k \Delta F_j]| \le C \Delta t^{1/2} \sum_{k=c+1}^{i-1} (j-k)^{-3/2}.$$

Hence,

$$\bigl|E\bigl[\bigl(F(t_{i-1}) - F(t_c)\bigr) \Delta F_j\bigr]\bigr| \le C \Delta t^{1/2} \sum_{k=j-i+1}^{\infty} k^{-3/2},$$

which proves the first claim.

For the second and third claims, it is easy to see that they hold when $i \ge j-1$. Assume $i < j-1$. Note that

$$\begin{aligned} E[F(t_{j-1}) \Delta F_i] &= \rho(t_i, t_{j-1}) - \rho(t_{i-1}, t_{j-1}) \\ &= \rho(t_{i-1} + \Delta t, t_{j-1}) - \rho(t_{i-1}, t_{j-1}) \\ &= \Delta t \, \partial_s \rho(t_{i-1} + \theta \Delta t, t_{j-1}) \end{aligned}$$

for some $\theta \in (0,1)$. Since $j > i$, $t_{i-1} + \theta \Delta t < t_{j-1}$. In the regime $s < t$, $\partial_s \rho(s,t) = (8\pi)^{-1/2}((t+s)^{-1/2} + (t-s)^{-1/2})$. Hence, $0 < \partial_s \rho(s,t) \le C(t-s)^{-1/2}$. It follows that

$$0 < E[F(t_{j-1}) \Delta F_i] \le C \Delta t |t_{j-1} - t_i|^{-1/2} = C \Delta t^{1/2} (j-i-1)^{-1/2}.$$

Since $j - i \geq 2$, this implies that $E[F(t_{j-1})\Delta F_i] \leq C\Delta t^{1/2}(j-i)^{-1/2}$, which proves the third claim. Combining this with the first claim gives

$$\begin{aligned}|E[(F(t_{j-1}) - F(t_c))\Delta F_i]| &\leq |E[F(t_{j-1})\Delta F_i]| + |E[F(t_c)\Delta F_i]| \\ &\leq C\Delta t^{1/2}[(j-i)^{-1/2} + (i-c)^{-1/2}],\end{aligned}$$

which proves the second claim. □

Recall γ_j, defined by (2.7). Let

$$\kappa = \left(\frac{4}{\pi} + \frac{2}{\pi}\sum_{j=1}^{\infty}\gamma_j^2(-1)^j\right)^{1/2} > 0 \tag{2.10}$$

(the quantity in the brackets is strictly positive by Proposition 4.7 of [15]) and define

$$B_n(t) = \kappa^{-1}\sum_{j=1}^{2\lfloor nt/2\rfloor}\Delta F_j^2(-1)^j. \tag{2.11}$$

(Note that this is simply $\kappa^{-1}Q_n^F$, in the notation of Section 1.) By Propositions 3.5 and 4.7 in [15],

$$E|B_n(t) - B_n(s)|^4 \leq C\left(\frac{2\lfloor nt/2\rfloor - 2\lfloor ns/2\rfloor}{n}\right)^2 \tag{2.12}$$

for all s and t. Recall that $F(t) = u(x,t)$, where u is given by (1.1). Let m denote Lebesgue measure and define the filtration

$$\mathcal{F}_t = \sigma\{W(A) : A \subset \mathbb{R} \times [0,t], m(A) < \infty\}. \tag{2.13}$$

Fix $\tau \geq 0$ and define $G(t) = F(t+\tau) - E[F(t+\tau) \mid \mathcal{F}_\tau]$. In the proof of Lemma 3.6 in [15], it was shown that G and F have the same law and that G is independent of $\mathcal{F}_\tau$. In particular, if $j > c$ and $\Delta\overline{F}_j = \Delta F_j - E[\Delta F_j \mid \mathcal{F}_{t_c}]$, then $\Delta\overline{F}_j$ is independent of $\mathcal{F}_{t_c}$ and equal in law to ΔF_{j-c}.

According to the equation displayed above (3.32) in [15], if $0 \leq \tau \leq s \leq t$, then

$$E|E[F(t) - F(s) \mid \mathcal{F}_\tau]|^2 \leq 2|t-s|^2|t-\tau|^{-3/2}. \tag{2.14}$$

In particular, $E|\Delta F_j - \Delta\overline{F}_j|^2 \leq 2\Delta t^2(t_j - t_c)^{-3/2} = 2\Delta t^{1/2}(j-c)^{-3/2}$, which, together with (2.5) and Hölder's inequality, implies that

$$\begin{aligned}E|\Delta F_j^2 - \Delta\overline{F}_j^2|^k &= E[|\Delta F_j + \Delta\overline{F}_j|^k|\Delta F_j - \Delta\overline{F}_j|^k] \\ &\leq C_k\Delta t^{5k/4}(t_j - t_c)^{-3k/4} \\ &= C_k\Delta t^{k/2}(j-c)^{-3k/4}.\end{aligned} \tag{2.15}$$

Finally, we recall the main result of interest to us, which is Proposition 4.7 in [15].

THEOREM 2.10. *Let $\{B_n\}$ be given by (2.11) and let B be a standard Brownian motion, independent of F. Then, $(F, B_n) \to (F, B)$ in law in $D_{\mathbb{R}^2}[0, \infty)$.*

2.3. *Tools for Gaussian random variables.* Let

$$h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}) \tag{2.16}$$

be the nth Hermite polynomial so that $\{h_n\}$ is an orthogonal basis of $L^2(\mu)$, where $\mu(dx) = (2\pi)^{-1/2} e^{-x^2/2} dx$; see Section 1.1.1 of [11] for details. Let $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the norm and inner product, respectively, in $L^2(\mu)$.

The first few Hermite polynomials are $h_0(x) = 1$, $h_1(x) = x$, $h_2(x) = x^2 - 1$ and $h_3(x) = x^3 - 3x$. We adopt the convention that $h_{-1}(x) = 0$. The Hermite polynomials satisfy the following identities for $n \geq 0$:

$$h_n'(x) = n h_{n-1}(x), \tag{2.17}$$

$$x h_n(x) = h_{n+1}(x) + n h_{n-1}(x), \tag{2.18}$$

$$h_n(-x) = (-1)^n h_n(x). \tag{2.19}$$

Any polynomial can be written as a linear combination of Hermite polynomials by using the formula

$$x^n = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} (2j-1)!! h_{n-2j}(x), \tag{2.20}$$

where $(2j-1)!! = (2j-1)(2j-3)(2j-5)\cdots 1$. Note that this can be rewritten as

$$x^n = \sum_{j=0}^{n} \binom{n}{j} E[Y^j] h_{n-j}(x), \tag{2.21}$$

where Y is a standard normal random variable.

In the remaining part of Section 2.3, X shall denote a standard normal random variable. If $r \in [-1, 1]$, then X_r, Y_r shall denote jointly normal random variables with mean zero, variance one and $E[X_r Y_r] = r$. By Lemma 1.1.1 in [11],

$$E[h_n(X_r) h_m(Y_r)] = \begin{cases} 0, & \text{if } n \neq m, \\ n! r^n, & \text{if } n = m. \end{cases} \tag{2.22}$$

In particular, $\|h_n\|^2 = E[h_n(X)^2] = n!$. Hence, if $g \in L^2(\mu)$, then

$$g = \sum_{n=0}^{\infty} \frac{1}{n!} \langle g, h_n \rangle h_n, \tag{2.23}$$

where the convergence is in $L^2(\mu)$.

If g and g' have polynomial growth and $n \geq 1$, then integration by parts gives

$$\begin{aligned}\langle g, h_n\rangle &= \frac{1}{\sqrt{2\pi}}\int g(x)h_n(x)e^{-x^2/2}\,dx = \frac{(-1)^n}{\sqrt{2\pi}}\int g(x)\,\frac{d^n}{dx^n}(e^{-x^2/2})\,dx \\ &= \frac{(-1)^{n-1}}{\sqrt{2\pi}}\int g'(x)\,\frac{d^{n-1}}{dx^{n-1}}(e^{-x^2/2})\,dx = \langle g', h_{n-1}\rangle.\end{aligned} \tag{2.24}$$

That is, $E[g(X)h_n(X)] = E[g'(X)h_{n-1}(X)]$. Using (2.22) and (2.23), we can generalize this as follows:

$$\begin{aligned}E[g(X_r)h_n(Y_r)] &= \sum_{m=0}^{\infty}\frac{1}{m!}\langle g, h_m\rangle E[h_m(X_r)h_n(Y_r)] \\ &= \langle g, h_n\rangle r^n = r\langle g', h_{n-1}\rangle r^{n-1} \\ &= rE[g'(X_r)h_{n-1}(Y_r)].\end{aligned} \tag{2.25}$$

The following two lemmas will be useful in Section 5.

LEMMA 2.11. *Suppose g, h, g', h' all have polynomial growth. If $f(r) = E[g(X_r)h(Y_r)]$, then $f'(r) = E[g'(X_r)h'(Y_r)]$ for all $r \in (-1, 1)$.*

PROOF. By (2.23) and (2.22), $f(r) = \sum_{n=0}^{\infty}\frac{1}{n!}\langle g, h_n\rangle\langle h, h_n\rangle r^n$, which, by (2.24), gives

$$\begin{aligned}f'(r) &= \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\langle g, h_n\rangle\langle h, h_n\rangle r^{n-1} \\ &= \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\langle g', h_{n-1}\rangle\langle h', h_{n-1}\rangle r^{n-1} \\ &= E[g'(X_r)h'(X_r)].\end{aligned}$$

□

LEMMA 2.12. *Suppose g, g', g'', h, h', h'' have polynomial growth. Let $U = aX_r$ and $V = bY_r$. If $\varphi(a, b, r) = E[g(U)h(V)]$, then*

$$\frac{\partial\varphi}{\partial a}(a, b, r) = aE[g''(U)h(V)] + brE[g'(U)h'(V)]$$

for all real a, b and all $r \in (-1, 1)$.

PROOF. By (2.23) and (2.22), $\varphi(a, b, r) = \sum_{n=0}^{\infty}\frac{1}{n!}\langle g(a\cdot), h_n\rangle\langle h(b\cdot), h_n\rangle r^n$. Fix $a_0 \in \mathbb{R}$. To justify differentiating under the summation at a_0, we must show that there exists an $\varepsilon > 0$ and a sequence $C_n(b, r)$ such that

$$\left|\frac{\partial}{\partial a}\left[\frac{1}{n!}\langle g(a\cdot), h_n\rangle\langle h(b\cdot), h_n\rangle r^n\right]\right| \leq C_n(b, r)$$

for all $|a - a_0| < \varepsilon$, and $\sum_{n=0}^{\infty} C_n(b,r) < \infty$. For this, we use (2.18) and (2.24) to compute

$$\begin{aligned}
&\frac{\partial}{\partial a}\left[\frac{1}{n!}\langle g(a\cdot), h_n\rangle\langle h(b\cdot), h_n\rangle r^n\right] \\
&\quad = \frac{1}{n!}\langle g'(a\cdot), h_{n+1}\rangle\langle h(b\cdot), h_n\rangle r^n \\
&\qquad + \frac{1}{(n-1)!}\langle g'(a\cdot), h_{n-1}\rangle\langle h(b\cdot), h_n\rangle r^n \\
&\quad = \frac{a}{n!}\langle g''(a\cdot), h_n\rangle\langle h(b\cdot), h_n\rangle r^n \\
&\qquad + \frac{b}{(n-1)!}\langle g'(a\cdot), h_{n-1}\rangle\langle h'(b\cdot), h_{n-1}\rangle r^n.
\end{aligned}$$

Since $|\langle \cdot, h_n/\sqrt{n!}\rangle| \le \|\cdot\|$, we may take $C_n(b,r) = Mr^n$ for an appropriately chosen constant M, provided that $|r| < 1$. We may therefore differentiate under the summation at a_0. Since a_0 was arbitrary, we have

$$\begin{aligned}
\frac{\partial\varphi}{\partial a}(a,b,r) &= a\sum_{n=0}^{\infty}\frac{1}{n!}\langle g''(a\cdot), h_n\rangle\langle h(b\cdot), h_n\rangle r^n \\
&\quad + b\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\langle g'(a\cdot), h_{n-1}\rangle\langle h'(b\cdot), h_{n-1}\rangle r^n \\
&= aE[g''(U)h(V)] + brE[g'(U)h'(V)]
\end{aligned}$$

for all a, b, r with $|r| < 1$. □

2.4. *Multi-indices and Taylor's theorem.* We recall here the standard multi-index notation. A multi-index is a vector $\alpha \in \mathbb{Z}_+^d$, where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We use e^j to denote the multi-index with $e_j^j = 1$ and $e_i^j = 0$ for $i \ne j$. If $\alpha \in \mathbb{Z}_+^d$ and $x \in \mathbb{R}^d$, then

$$|\alpha| = \sum_{j=1}^{d}\alpha_j, \qquad \alpha! = \prod_{j=1}^{d}\alpha_j!,$$

$$\partial_j = \frac{\partial}{\partial x_j}, \qquad \partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}, \qquad x^\alpha = \prod_{j=1}^{d}x_j^{\alpha_j}.$$

Note that by convention, $0^0 = 1$. Also note that $|x^\alpha| = y^\alpha$, where $y_j = |x_j|$ for all j.

Taylor's theorem with integral remainder states that if $g \in C^{k+1}(\mathbb{R})$, then

$$g(b) = \sum_{j=0}^{k} g^{(j)}(a)\frac{(b-a)^j}{j!} + \frac{1}{k!}\int_a^b (b-u)^k g^{(k+1)}(u)\,du. \tag{2.26}$$

Taylor's theorem in higher dimensions is the following.

THEOREM 2.13. *If* $g \in C^{k+1}(\mathbb{R}^d)$, *then*

$$g(b) = \sum_{|\alpha| \le k} \partial^\alpha g(a) \frac{(b-a)^\alpha}{\alpha!} + R,$$

where

$$R = (k+1) \sum_{|\alpha|=k+1} \frac{(b-a)^\alpha}{\alpha!} \int_0^1 (1-u)^k \partial^\alpha g\bigl(a + u(b-a)\bigr)\,du.$$

In particular,

$$|R| \le (k+1) \sum_{|\alpha|=k+1} M_\alpha |(b-a)^\alpha|,$$

where $M_\alpha = \sup\{|\partial^\alpha g(a + u(b-a))| : 0 \le u \le 1\}$.

For integers a and b with $a \ge 0$, we adopt the convention that

$$\binom{a}{b} = \begin{cases} \dfrac{a!}{b!(a-b)!}, & \text{if } 0 \le b \le a, \\ 0, & \text{if } b < 0 \text{ or } b > a. \end{cases}$$

We define

$$\binom{\gamma}{\alpha} = \prod_{j=1}^d \binom{\gamma_j}{\alpha_j}$$

for any multi-indices γ and α. Later in the paper, we shall need the following two combinatorial lemmas.

LEMMA 2.14. *Let* a, b *and* c *be integers. If* $a \ge 0$ *and* $0 \le c \le a$, *then*

$$\sum_{j=0}^c \binom{a-c}{b-j} \binom{c}{j} = \binom{a}{b}.$$

PROOF. The proof is by induction on a. For $a = 0$, the lemma is trivial. Suppose the lemma holds for $a - 1$. Since the lemma clearly holds for $c = 0$ or $c = a$, we may assume $0 < c \le a - 1$. In that case,

$$\begin{aligned} \binom{a}{b} &= \binom{a-1}{b} + \binom{a-1}{b-1} \\ &= \sum_{j=0}^c \left[\binom{a-1-c}{b-j} + \binom{a-1-c}{b-1-j}\right] \binom{c}{j} \\ &= \sum_{j=0}^c \binom{a-c}{b-j} \binom{c}{j}. \end{aligned}$$

□

Suppose α and γ are multi-indices. We will write $\alpha \le \gamma$ if $\alpha_j \le \gamma_j$ for all j.

LEMMA 2.15. *If γ is a multi-index in $\mathbb{Z}_+^d$ and $m \ge 0$, then*

$$\sum_{\substack{|\alpha|=m\\ \alpha\le\gamma}} \binom{\gamma}{\alpha} = \binom{|\gamma|}{m}.$$

PROOF. We shall prove this by induction on d. If $d = 1$, then the lemma is trivial. Suppose the lemma is true for $d - 1$. Let γ be a multi-index in $\mathbb{Z}_+^d$ and fix m with $0 \le m \le |\gamma|$. For multi-indices α and γ, let $\widehat{\alpha} = (\alpha_1, \dots, \alpha_{d-1})$ and $\widehat{\gamma} = (\gamma_1, \dots, \gamma_{d-1})$. Then,

$$\begin{aligned}\sum_{\substack{|\alpha|=m\\ \alpha\le\gamma}} \binom{\gamma}{\alpha} &= \sum_{\alpha_d=0}^{m\wedge\gamma_d} \sum_{\substack{|\widehat{\alpha}|=m-\alpha_d\\ \widehat{\alpha}\le\widehat{\gamma}}} \binom{\widehat{\gamma}}{\widehat{\alpha}}\binom{\gamma_d}{\alpha_d}\\ &= \sum_{\alpha_d=0}^{m\wedge\gamma_d} \binom{|\widehat{\gamma}|}{m-\alpha_d}\binom{\gamma_d}{\alpha_d}\\ &= \sum_{\alpha_d=0}^{\gamma_d} \binom{|\gamma|-\gamma_d}{m-\alpha_d}\binom{\gamma_d}{\alpha_d}.\end{aligned}$$

Applying Lemma 2.14 completes the proof. □

3. Fourth order integrals.

THEOREM 3.1. *Suppose $g : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is continuous. For each n, let $\{s_j^*\}$ and $\{t_j^*\}$ be collections of points with $s_j^*, t_j^* \in [t_{j-1}, t_j]$. Then,*

$$\begin{aligned}\lim_{n\to\infty} \sum_{\substack{j=1\\ j \text{ odd}}}^{\lfloor nt\rfloor} g(F(s_j^*), t_j^*)\Delta F_j^4 &= \lim_{n\to\infty} \sum_{\substack{j=1\\ j \text{ even}}}^{\lfloor nt\rfloor} g(F(s_j^*), t_j^*)\Delta F_j^4\\ &= \frac{3}{\pi}\int_0^t g(F(s), s)\,ds,\end{aligned} \tag{3.1}$$

where the convergence is ucp.

PROOF. We prove only the first limit. The proof for the other limit is nearly identical. Let

$$X_n(t) = \sum_{j=1}^{\infty} g(F(s_j^*), t_j^*) 1_{[t_{j-1},t_j)}(t)$$

and

$$A_n(t) = \sum_{\substack{j=1 \\ j \text{ odd}}}^{\lfloor nt \rfloor} \Delta F_j^4$$

so that

$$\sum_{\substack{j=1 \\ j \text{ odd}}}^{\lfloor nt \rfloor} g(F(s_j^*), t_j^*)\Delta F_j^4 = \int_0^t X_n(s-)\, dA_n(s).$$

By (2.6), $A_n(t) \to 3t/\pi$ ucp. Also, by the continuity of g and F, $X_n \to g(F(\cdot), \cdot)$ ucp. Finally, note that the expected total variation $V_t(A_n)$ of A_n on $[0, t]$ is uniformly bounded in n. That is,

$$E[V_t(A_n)] = \sum_{\substack{j=1 \\ j \text{ odd}}}^{\lfloor nt \rfloor} E\Delta F_j^4 \le C \sum_{j=1}^{\lfloor nt \rfloor} \Delta t \le Ct.$$

By Theorem 2.7, (3.1) holds with the convergence being in probability in $D_{\mathbb{R}}[0, \infty)$. Since the limit is continuous, (3.1) holds ucp. □

If r and k are nonnegative integers with $r \le k$, then we shall use the notation $g \in C_r^{k,1}(\mathbb{R} \times [0, \infty))$ to mean that

(3.2) $$g : \mathbb{R} \times [0, \infty) \to \mathbb{R} \text{ is continuous,}$$

(3.3) $$\partial_x^j g \text{ exists and is continuous on } \mathbb{R} \times [0, \infty) \qquad \text{for all } 0 \le j \le k,$$

(3.4) $$\partial_t \partial_x^j g \text{ exists and is continuous on } \mathbb{R} \times (0, \infty) \qquad \text{for all } 0 \le j \le r,$$

(3.5) $$\overline{\lim_{t \to 0}} \sup_{x \in K} |\partial_t \partial_x^j g(x, t)|\, dt < \infty \qquad \text{for all compact } K \subset \mathbb{R} \text{ and all } 0 \le j \le r.$$

Note that $g \in C_r^{k,1}$ implies $\partial_x^j g \in C_{r-j}^{k-j,1}$ whenever $r \ge j$. For functions of one spatial dimension, we shall henceforth use standard prime notation to denote spatial derivatives. For example, $g'' = \partial_x^2 g$ and $g^{(4)} = \partial_x^4 g$.

Typically, we shall need (3.4) and (3.5) only when $j = 0$. There are a few places, however, where $j > 0$ is needed. We need $j = 3$ in the derivation of (3.10), which is used in the proofs of both Theorem 3.3 and Corollary 4.5; we need $j = 2$ in the proof of Lemma 5.8; we need $j = 4$ in the proof of Theorem 6.2. Note that $\partial_t \partial_x^j g$ need not be continuous at $t = 0$. In particular, $\partial_t \partial_x^j g$ need not be bounded on sets of the form $K \times (0, \varepsilon]$.

Recall that $X_n(t) \approx Y_n(t)$ means that $X_n - Y_n \to 0$ ucp.

THEOREM 3.2. *If* $g \in C_0^{5,1}(\mathbb{R} \times [0, \infty))$, *then*

$$I_n(g', t) \approx g(F(t), t) - g(F(0), 0) - \int_0^t \partial_t g(F(s), s)\, ds$$
$$- \frac{1}{2} \sum_{j=1}^{\lfloor nt/2 \rfloor} g''(F(t_{2j-1}), t_{2j-1})(\Delta F_{2j}^2 - \Delta F_{2j-1}^2)$$
$$- \frac{1}{6} \sum_{j=1}^{\lfloor nt/2 \rfloor} g'''(F(t_{2j-1}), t_{2j-1})(\Delta F_{2j}^3 + \Delta F_{2j-1}^3),$$

where $I_n(g, t)$ *is given by* (1.4).

PROOF. By (2.26),

$$g(x + h_1, t) - g(x + h_2, t)$$
$$= \sum_{j=1}^{4} \frac{1}{j!} g^{(j)}(x, t)(h_1^j - h_2^j) + R(x, h_1, t) - R(x, h_2, t),$$

where

$$R(x, h, t) = \frac{1}{4!} \int_0^h (h - u)^4 g^{(5)}(x + u, t)\, du.$$

Taking $x = F(t_{2j-1})$, $h_1 = \Delta F_{2j}$ and $h_2 = -\Delta F_{2j-1}$, we have

$$g(F(t_{2j}), t_{2j-1}) - g(F(t_{2j-2}), t_{2j-1})$$
$$= \sum_{j=1}^{4} \frac{1}{j!} g^{(j)}(F(t_{2j-1}), t_{2j-1})\big(\Delta F_{2j}^j - (-1)^j \Delta F_{2j-1}^j\big)$$
$$+ R(F(t_{2j-1}), \Delta F_{2j}, t_{2j-1})$$
$$- R(F(t_{2j-1}), -\Delta F_{2j-1}, t_{2j-1}).$$

Let $N(t) = 2\lfloor nt/2 \rfloor / n$. That is, if $t \in [t_{2j-2}, t_{2j})$, then $N(t) = t_{2j-2}$. Let $F_n(t) = F(N(t))$. Then,

$$g(F(t_{2j}), t_{2j}) - g(F(t_{2j}), t_{2j-1}) = \int_{t_{2j-1}}^{t_{2j}} \partial_t g\big(F_n(s + \Delta t), s\big)\, ds,$$

$$g(F(t_{2j-2}), t_{2j-1}) - g(F(t_{2j-2}), t_{2j-2}) = \int_{t_{2j-2}}^{t_{2j-1}} \partial_t g(F_n(s), s)\, ds$$
$$= \int_{t_{2j-2}}^{t_{2j-1}} \partial_t g\big(F_n(s + \Delta t), s\big)\, ds.$$

Thus,

$$
\begin{aligned}
g(F(t),t) = g(F(0),0) &+ \sum_{j=1}^{\lfloor nt/2\rfloor} \{g(F(t_{2j}),t_{2j}) - g(F(t_{2j-2}),t_{2j-2})\} \\
&+ g(F(t),t) - g(F_n(t),N(t)) \\
&= g(F(0),0) + \int_0^{N(t)} \partial_t g\bigl(F_n(s+\Delta t),s\bigr)\,ds + I_n(g',t) \\
&+ \frac{1}{2}\sum_{j=1}^{\lfloor nt/2\rfloor} g''(F(t_{2j-1}),t_{2j-1})(\Delta F_{2j}^2 - \Delta F_{2j-1}^2) \\
&+ \frac{1}{6}\sum_{j=1}^{\lfloor nt/2\rfloor} g'''(F(t_{2j-1}),t_{2j-1})(\Delta F_{2j}^3 + \Delta F_{2j-1}^3) \\
&+ \varepsilon_n(g,t),
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_n(g,t) = \frac{1}{24}&\sum_{j=1}^{\lfloor nt/2\rfloor} g^{(4)}(F(t_{2j-1}),t_{2j-1})(\Delta F_{2j}^4 - \Delta F_{2j-1}^4) \\
&+ \sum_{j=1}^{\lfloor nt/2\rfloor} \{R(F(t_{2j-1}),\Delta F_{2j},t_{2j-1}) \\
&\qquad - R(F(t_{2j-1}),-\Delta F_{2j-1},t_{2j-1})\} \\
&+ g(F(t),t) - g(F_n(t),N(t)).
\end{aligned} \tag{3.6}
$$

By (3.4), (3.5), the continuity of F and dominated convergence,

$$\int_0^{N(t)} \partial_t g\bigl(F_n(s+\Delta t),s\bigr)\,ds \to \int_0^t \partial_t g(F(s),s)\,ds$$

uniformly on compacts, with probability one. Therefore, it will suffice to show that $\varepsilon_n(g,t)\to 0$ ucp.

First, assume that g has compact support. By the continuity of g and the almost sure continuity of F, $g(F(t),t) - g(F_n(t),N(t)) \to 0$ ucp. Since $g^{(5)}$ is bounded, $|R(x,h,t)| \le C|h|^5$. Thus,

$$
\begin{aligned}
&\left|\sum_{j=1}^{\lfloor nt/2\rfloor} \{R(F(t_{2j-1}),\Delta F_{2j},t_{2j-1}) - R(F(t_{2j-1}),-\Delta F_{2j-1},t_{2j-1})\}\right| \\
&\qquad \le C\sum_{j=1}^{\lfloor nt/2\rfloor} |\Delta F_j|^5
\end{aligned}
$$

and

$$E\left[\sup_{0\le t\le T}\sum_{j=1}^{\lfloor nt/2\rfloor}|\Delta F_j|^5\right]=\sum_{j=1}^{\lfloor nT/2\rfloor}E|\Delta F_j|^5=C\sum_{j=1}^{\lfloor nT/2\rfloor}\sigma_j^5$$
$$\le CnT\Delta t^{5/4}=CT\Delta t^{1/4}.$$

It follows that

$$\sum_{j=1}^{\lfloor nt/2\rfloor}\{R(F(t_{2j-1}),\Delta F_{2j},t_{2j-1})-R(F(t_{2j-1}),-\Delta F_{2j-1},t_{2j-1})\}\to 0$$

ucp. An application of Theorem 3.1 to the first sum in (3.6) completes the proof that $\varepsilon_n(g,t)\to 0$ ucp, in the case where g has compact support.

To deal with the general case, we use the following truncation argument, which we will make use of several times throughout this paper. Fix $T>0$ and $\eta>0$. Choose $L>T$ so large that

$$P\Big(\sup_{0\le t\le T}|F(t)|\ge L\Big)<\eta.$$

Let $\varphi\in C^\infty(\mathbb{R})$ have compact support with $\varphi\equiv 1$ on $[-L,L]$. Define $h(x,t)=g(x,t)\varphi(x)\varphi(t)$. Then, $h\in C_0^{5,1}(\mathbb{R}\times[0,\infty))$, h has compact support and $h=g$ on $[-L,L]\times[0,T]$. By the above, we may choose n_0 such that

$$P\Big(\sup_{0\le t\le T}|\varepsilon_n(h,t)|>\eta\Big)<\eta$$

for all $n\ge n_0$. Hence,

$$\begin{aligned}P\Big(\sup_{0\le t\le T}|\varepsilon_n(g,t)|>\eta\Big)&\\ &\le P\Big(\sup_{0\le t\le T}|F(t)|\ge L\Big)+P\Big(\sup_{0\le t\le T}|\varepsilon_n(h,t)|>\eta\Big)\\ &<2\eta\end{aligned}$$

for all $n\ge n_0$, which shows that $\varepsilon_n(g,t)\to 0$ ucp and completes the proof. □

THEOREM 3.3. *If $g\in C_3^{5,1}(\mathbb{R}\times[0,\infty))$, then*

$$T_n(g',t)\approx g(F(t),t)-g(F(0),0)-\int_0^t\partial_t g(F(s),s)\,ds$$
$$+\frac{1}{24}\sum_{j=1}^{\lfloor nt\rfloor}g'''(F(t_j),t_j)(\Delta F_{j+1}^3+\Delta F_j^3),$$

where $T_n(g,t)$ is given by (1.7).

PROOF. As in the proof of Theorem 3.2, we may assume g has compact support. Define

$$\widehat{I}_n(g,t) = \sum_{j=1}^{\lfloor nt/2 \rfloor} g(F(t_{2j}), t_{2j})\big(F(t_{2j+1}) - F(t_{2j-1})\big).$$

The proof of Theorem 3.2 can be easily adapted to show that

$$\begin{aligned} \widehat{I}_n(g',t) \approx{}& g(F(t),t) - g(F(0),0) - \int_0^t \partial_t g(F(s),s)\,ds \\ & - \frac{1}{2}\sum_{j=1}^{\lfloor nt/2 \rfloor} g''(F(t_{2j}), t_{2j})(\Delta F_{2j+1}^2 - \Delta F_{2j}^2) \\ & - \frac{1}{6}\sum_{j=1}^{\lfloor nt/2 \rfloor} g'''(F(t_{2j}), t_{2j})(\Delta F_{2j+1}^3 + \Delta F_{2j}^3). \end{aligned} \tag{3.7}$$

Note that

$$\begin{aligned} &I_n(g',t) + \widehat{I}_n(g',t) \\ &\quad = \sum_{\substack{j=1 \\ j \text{ odd}}}^{2\lfloor nt/2 \rfloor} g'(F(t_j), t_j)(\Delta F_{j+1} + \Delta F_j) + \sum_{\substack{j=1 \\ j \text{ even}}}^{2\lfloor nt/2 \rfloor} g'(F(t_j), t_j)(\Delta F_{j+1} + \Delta F_j) \\ &\quad = \sum_{j=1}^{2\lfloor nt/2 \rfloor} g'(F(t_j), t_j)(\Delta F_{j+1} + \Delta F_j). \end{aligned}$$

Also, note that

$$T_n(g',t) = \frac{1}{2}\left(\sum_{j=0}^{\lfloor nt \rfloor - 1} g'(F(t_j), t_j)\Delta F_{j+1} + \sum_{j=0}^{\lfloor nt \rfloor} g'(F(t_j), t_j)\Delta F_j\right).$$

By the continuity of F and g', this shows that

$$T_n(g',t) \approx \frac{I_n(g',t) + \widehat{I}_n(g',t)}{2}.$$

By (3.7) and Theorem 3.2, we have

$$\begin{aligned} T_n(g',t) \approx{}& g(F(t),t) - g(F(0),0) - \int_0^t \partial_t g(F(s),s)\,ds \\ & + \frac{1}{4}\sum_{j=1}^{\lfloor nt \rfloor}\big(g''(F(t_j), t_j) - g''(F(t_{j-1}), t_{j-1})\big)\Delta F_j^2 \\ & - \frac{1}{12}\sum_{j=1}^{\lfloor nt \rfloor} g'''(F(t_j), t_j)(\Delta F_{j+1}^3 + \Delta F_j^3). \end{aligned} \tag{3.8}$$

Since $g \in C_3^{5,1}(\mathbb{R} \times [0, \infty))$, we may use the Taylor expansion $f(b) - f(a) = \frac{1}{2}(f'(a) + f'(b))(b - a) + O(|b - a|^3)$ with $f = g''$ to obtain

$$\begin{aligned}
&g''(F(t_j), t_j) - g''(F(t_{j-1}), t_{j-1}) \\
&\qquad = \int_{t_{j-1}}^{t_j} \partial_t g''(F(t_j), s)\, ds \\
&\qquad\quad + \frac{1}{2}\big(g'''(F(t_{j-1}), t_{j-1}) + g'''(F(t_j), t_{j-1})\big)\Delta F_j + R \\
&\qquad = \int_{t_{j-1}}^{t_j} \partial_t g''(F(t_j), s)\, ds - \frac{1}{2}\Delta F_j \int_{t_{j-1}}^{t_j} \partial_t g'''(F(t_j), s)\, ds \\
&\qquad\quad + \frac{1}{2}\big(g'''(F(t_{j-1}), t_{j-1}) + g'''(F(t_j), t_j)\big)\Delta F_j + R,
\end{aligned} \tag{3.9}$$

where $|R| \le C|\Delta F_j|^3$. Since g has compact support, we may use (3.5) with $K = \mathbb{R}$ and $j = 3$ to conclude that the above integrals are bounded by $C\Delta t$. This yields

$$\begin{aligned}
&\sum_{j=1}^{\lfloor nt \rfloor} \big(g''(F(t_j), t_j) - g''(F(t_{j-1}), t_{j-1})\big)\Delta F_j^2 \\
&\qquad = \sum_{j=1}^{\lfloor nt \rfloor} \frac{1}{2}\big(g'''(F(t_j), t_j) + g'''(F(t_{j-1}), t_{j-1})\big)\Delta F_j^3 + \widetilde{R},
\end{aligned} \tag{3.10}$$

where $|\widetilde{R}| \le C \sum (\Delta t \Delta F_j^2 + \Delta t |\Delta F_j|^3 + |\Delta F_j|^5)$. We can combine this formula with (3.8) to complete the proof. $\square$

4. Third order integrals. To analyze the third order integrals, we will need a Taylor expansion of a different kind. That is, we will need an expansion for the expectation of functions of jointly Gaussian random variables. For this Gaussian version of Taylor's theorem, we first introduce some terminology. We shall say that a function $g : \mathbb{R}^d \to \mathbb{R}$ has *polynomial growth* if there exist positive constants K and r such that

$$|g(x)| \le K(1 + |x|^r)$$

for all $x \in \mathbb{R}^d$. If k is nonnegative integer, we shall say that a function g has *polynomial growth of order k* if $g \in C^k(\mathbb{R}^d)$ and there exist positive constants K and r such that

$$|\partial^\alpha g(x)| \le K(1 + |x|^r)$$

for all $x \in \mathbb{R}^d$ and all $|\alpha| \le k$.

THEOREM 4.1. *Let k be a nonnegative integer. Suppose $h:\mathbb{R}\to\mathbb{R}$ is measurable and has polynomial growth, and $f\in C^{k+1}(\mathbb{R}^d)$ has polynomial growth of order $k+1$, both with common constants K and r. Suppose, also, that $\partial^\alpha f$ has polynomial growth with constants K_α and r for all $|\alpha|\le k+1$. Let $\xi\in\mathbb{R}^d$ and $Y\in\mathbb{R}$ be jointly normal with mean zero. Suppose that $EY^2=1$ and $E\xi_j^2\le\nu$ for some $\nu>0$. Define $\rho\in\mathbb{R}^d$ by $\rho_j=E[\xi_j Y]$. Then,*

$$E[f(\xi)h(Y)]=\sum_{|\alpha|\le k}\frac{1}{\alpha!}\rho^\alpha E[\partial^\alpha f(\xi-\rho Y)]E\big[Y^{|\alpha|}h(Y)\big]+R,$$

where $|R|\le C\sum_{|\alpha|=k+1}K_\alpha|\rho^\alpha|$ and C depends only on K, r, ν, k and d. In particular, $|R|\le C|\rho|^{k+1}$.

PROOF. Let $U=\xi-\rho Y$ and define $\varphi:\mathbb{R}^d\to\mathbb{R}$ by $\varphi(x)=E[f(U+xY)h(Y)]$. Since h and f have polynomial growth and all derivatives of f up to order $k+1$ have polynomial growth, we may differentiate under the expectation and conclude that $\varphi\in C^{k+1}(\mathbb{R}^d)$. Hence, by Theorem 2.13 and the fact that U and Y are independent,

$$\begin{aligned}E[f(\xi)h(Y)]=\varphi(\rho)&=\sum_{|\alpha|\le k}\frac{1}{\alpha!}\rho^\alpha\partial^\alpha\varphi(0)+R\\&=\sum_{|\alpha|\le k}\frac{1}{\alpha!}\rho^\alpha E[\partial^\alpha f(U)]E\big[Y^{|\alpha|}h(Y)\big]+R,\end{aligned}$$

where

$$|R|\le(k+1)\sum_{|\alpha|=k+1}M_\alpha|\rho^\alpha|$$

and $M_\alpha=\sup\{|\partial^\alpha\varphi(u\rho)|:0\le u\le 1\}$. Note that

$$\partial^\alpha\varphi(u\rho)=E[\partial^\alpha f(U+u\rho Y)Y^{k+1}h(Y)]=E\big[\partial^\alpha f\big(\xi-\rho(1-u)Y\big)Y^{k+1}h(Y)\big].$$

Hence,

$$\begin{aligned}|\partial^\alpha\varphi(u\rho)|&\le K_\alpha K E\big[\big(1+|\xi-\rho(1-u)Y|^r\big)|Y|^{k+1}(1+|Y|^r)\big]\\&\le K_\alpha K E[(1+2^r|\xi|^r+2^r|\rho|^r|Y|^r)(|Y|^{k+1}+|Y|^{k+1+r})].\end{aligned}$$

Since $|\rho|^2\le\nu d$, this shows that $M_\alpha\le CK_\alpha$ and completes the proof. □

COROLLARY 4.2. *Recall the Hermite polynomials $h_n(x)$ from (2.16). Under the hypotheses of Theorem 4.1,*

$$E[f(\xi)h(Y)]=\sum_{|\alpha|\le k}\frac{1}{\alpha!}\rho^\alpha E[\partial^\alpha f(\xi)]E\big[h_{|\alpha|}(Y)h(Y)\big]+R,$$

where $|R|\le C\sum_{|\alpha|=k+1}K_\alpha|\rho^\alpha|$ and C depends only on K, r, ν, k and d. In particular, $|R|\le C|\rho|^{k+1}$.

PROOF. Recursively define the sequences $\{a_j^{(n)}\}_{j=0}^\infty$ by $a_j^{(0)} = E[Y^j h(Y)]$ and

$$a_j^{(n+1)} = \begin{cases} a_j^{(n)}, & \text{if } j \le n, \\ a_j^{(n)} - \binom{j}{n} a_n^{(n)} E[Y^{j-n}], & \text{if } j \ge n+1. \end{cases} \tag{4.1}$$

We will show that for all $0 \le n \le k+1$,

$$\begin{aligned} E[f(\xi)h(Y)] &= \sum_{|\alpha| \le n-1} \frac{1}{\alpha!} \rho^\alpha E[\partial^\alpha f(\xi)] a_{|\alpha|}^{(n)} \\ &\quad + \sum_{n \le |\alpha| \le k} \frac{1}{\alpha!} \rho^\alpha E[\partial^\alpha f(\xi - \rho Y)] a_{|\alpha|}^{(n)} + R, \end{aligned} \tag{4.2}$$

where $|R| \le C \sum_{|\alpha|=k+1} K_\alpha |\rho^\alpha|$ and C depends only on K, r, ν, k and d. The proof is by induction on n. The case $n = 0$ is given by Theorem 4.1. Suppose (4.2) holds for some $n < k+1$. Fix α such that $|\alpha| = n$. Let c_k denote $E[Y^k]$. Applying Theorem 4.1 to $\partial^\alpha f$ with $h(y) = 1$ gives

$$\begin{aligned} E[\partial^\alpha f(\xi)] &= \sum_{|\beta| \le k-n} \frac{1}{\beta!} \rho^\beta E[\partial^{\alpha+\beta} f(\xi - \rho Y)] c_{|\beta|} + \widehat{R}_\alpha \\ &= E[\partial^\alpha f(\xi - \rho Y)] + \sum_{1 \le |\beta| \le k-n} \frac{1}{\beta!} \rho^\beta E[\partial^{\alpha+\beta} f(\xi - \rho Y)] c_{|\beta|} + \widehat{R}_\alpha, \end{aligned}$$

where $|\widehat{R}_\alpha| \le C \sum_{|\beta|=k+1-n} K_{\alpha+\beta} |\rho^\beta|$. Hence, by (4.2),

$$\begin{aligned} E[f(\xi)h(Y)] &= \sum_{|\alpha| \le n} \frac{1}{\alpha!} \rho^\alpha E[\partial^\alpha f(\xi)] a_{|\alpha|}^{(n)} \\ &\quad + \sum_{n+1 \le |\alpha| \le k} \frac{1}{\alpha!} \rho^\alpha E[\partial^\alpha f(\xi - \rho Y)] a_{|\alpha|}^{(n)} \\ &\quad - S + R^*, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} |R^*| &\le |R| + C \sum_{|\alpha|=n} |\rho^\alpha| |\widehat{R}_\alpha| \\ &\le |R| + C \sum_{|\alpha|=n} |\rho^\alpha| \sum_{|\beta|=k+1-n} K_{\alpha+\beta} |\rho^\beta| \\ &\le C \sum_{|\alpha|=n} K_\alpha |\rho^\alpha| \end{aligned}$$

and

$$S = \sum_{|\alpha|=n} \sum_{1 \le |\beta| \le k-n} \frac{1}{\alpha!\beta!} \rho^{\alpha+\beta} E[\partial^{\alpha+\beta} f(\xi - \rho Y)] a_n^{(n)} c_{|\beta|}.$$

Making the change of index $\gamma = \alpha + \beta$ and using Lemma 2.15 gives

$$S = \sum_{n+1 \le |\gamma| \le k} \sum_{\substack{|\alpha|=n \\ \alpha \le \gamma}} \binom{\gamma}{\alpha} \frac{1}{\gamma!} \rho^\gamma E[\partial^\gamma f(\xi - \rho Y)] a_n^{(n)} c_{|\gamma|-n}$$

$$= \sum_{n+1 \le |\gamma| \le k} \binom{|\gamma|}{n} \frac{1}{\gamma!} \rho^\gamma E[\partial^\gamma f(\xi - \rho Y)] a_n^{(n)} c_{|\gamma|-n}.$$

Substituting this into (4.3) and using (4.1) shows that

$$\begin{aligned} E[f(\xi)h(Y)] &= \sum_{|\alpha| \le n} \frac{1}{\alpha!} \rho^\alpha E[\partial^\alpha f(\xi)] a_{|\alpha|}^{(n+1)} \\ &\quad + \sum_{n+1 \le |\alpha| \le k} \frac{1}{\alpha!} \rho^\alpha E[\partial^\alpha f(\xi - \rho Y)] a_{|\alpha|}^{(n+1)} + R^*, \end{aligned} \tag{4.4}$$

which completes the induction.

By (4.2) with $n = k + 1$, it remains only to show that

$$a_j^{(n)} = E[h_j(Y)h(Y)] \qquad \text{for all } j \le n. \tag{4.5}$$

The proof is by induction on n. For $n = 0$, the claim is trivial. Suppose (4.5) holds for all $n \le N$. If $j \le N$, then (4.1) implies $a_j^{(N+1)} = a_j^{(N)} = E[h_j(Y)h(Y)]$. If $j = N + 1$, then

$$a_{N+1}^{(N+1)} = a_{N+1}^{(N)} - \binom{N+1}{N} a_N^{(N)} E[Y].$$

Using induction, this gives

$$a_{N+1}^{(N+1)} = a_{N+1}^{(0)} - \sum_{j=0}^{N} \binom{N+1}{j} a_j^{(j)} E[Y^{N+1-j}]$$

$$= E[Y^{N+1}h(Y)] - \sum_{j=0}^{N} \binom{N+1}{j} E[h_j(Y)h(Y)] E[Y^{N+1-j}]$$

$$= E\left[\left\{Y^{N+1} - \sum_{j=0}^{N} \binom{N+1}{j} E[Y^{N+1-j}] h_j(Y)\right\} h(Y)\right].$$

By (2.21),

$$Y^{N+1} = \sum_{j=0}^{N+1} \binom{N+1}{j} E[Y^j] h_{N+1-j}(Y) = \sum_{j=0}^{N+1} \binom{N+1}{j} E[Y^{N+1-j}] h_j(Y).$$

Hence, $a_{N+1}^{(N+1)} = E[h_{N+1}(Y)h(Y)]$, completing the proof of (4.5). □

THEOREM 4.3. *If* $g \in C_0^{4,1}(\mathbb{R} \times [0,\infty))$, *then*

$$\begin{aligned}\lim_{n\to\infty} \sum_{\substack{j=1\\ j \text{ odd}}}^{\lfloor nt \rfloor} g(F(t_{j-1}), t_{j-1})\Delta F_j^3 &= \lim_{n\to\infty} \sum_{\substack{j=1\\ j \text{ even}}}^{\lfloor nt \rfloor} g(F(t_{j-1}), t_{j-1})\Delta F_j^3 \\ &= -\frac{3}{2\pi}\int_0^t g'(F(s), s)\, ds \end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\lim_{n\to\infty} \sum_{\substack{j=1\\ j \text{ odd}}}^{\lfloor nt \rfloor} g(F(t_j), t_j)\Delta F_j^3 &= \lim_{n\to\infty} \sum_{\substack{j=1\\ j \text{ even}}}^{\lfloor nt \rfloor} g(F(t_j), t_j)\Delta F_j^3 \\ &= \frac{3}{2\pi}\int_0^t g'(F(s), s)\, ds, \end{aligned} \tag{4.7}$$

where the convergence is ucp.

REMARK 4.4. The nonzero limits result from the dependence between $F(t_{j-1})$ and ΔF_j in (4.6), and $F(t_j)$ and ΔF_j in (4.7). Note that

$$E[F(t_{j-1})\Delta F_j] = \Delta t\, \partial_t \rho(t_{j-1}, t_{j-1} + \varepsilon)$$

for some $0 < \varepsilon < \Delta t$. Similarly, $E[F(t_j)\Delta F_j] = \Delta t\, \partial_t \rho(t_j, t_j - \varepsilon)$. If X is a centered, quartic variation Gaussian process, then

$$\begin{aligned}\rho(s,t) &= \tfrac{1}{2}\big(EX(t)^2 + EX(s)^2 - E|X(t) - X(s)|^2\big) \\ &\approx \tfrac{1}{2}\big(EX(t)^2 + EX(s)^2 - |t-s|^{1/2}\big),\end{aligned}$$

which means the leading term in $\partial_t \rho(s,t)$ is $-|t-s|^{-1/2}\operatorname{sgn}(t-s)$. Hence, it is not surprising that the limits in (4.6) and (4.7) are of equal magnitude and opposite sign.

PROOF OF THEOREM 4.3. We prove only the case for odd indices. The proof for even indices is nearly identical. To simplify notation, we will not explicitly indicate that the indices are odd in the subscript of the summation symbol (this convention applies only in this proof).

Using the truncation argument in the proof of Theorem 3.2, we may assume that g has compact support. Fix $T > 0$. Let $0 \le s \le t \le T$ be arbitrary. Recall σ_j and $\widehat{\sigma}_j$ from Section 2.2. Let

$$\begin{aligned} Z_n(t) &= \sum_{j=1}^{\lfloor nt \rfloor} g(F(t_{j-1}), t_{j-1})\Delta F_j^3, \\ X_n &= X_n(s,t) = Z_n(t) - Z_n(s), \end{aligned}$$

$$Y_n = Y_n(s,t) = 3 \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} g'(F(t_{j-1}), t_{j-1}) \widehat{\sigma}_j \sigma_j^2.$$

We may write

$$\begin{aligned} E|X_n - Y_n|^2 & \\ &= E\Bigg| \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} g(F(t_{j-1}), t_{j-1}) \Delta F_j^3 \\ &\qquad - 3 \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} g'(F(t_{j-1}), t_{j-1}) \widehat{\sigma}_j \sigma_j^2 \Bigg|^2 \\ &= (S_1 - S_2) - (S_2 - S_3), \end{aligned} \tag{4.8}$$

where

$$S_1 = \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} E[g(F(t_{i-1}), t_{i-1}) \Delta F_i^3 g(F(t_{j-1}), t_{j-1}) \Delta F_j^3],$$

$$S_2 = 3 \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} E[g(F(t_{i-1}), t_{i-1}) \Delta F_i^3 g'(F(t_{j-1}), t_{j-1})] \widehat{\sigma}_j \sigma_j^2,$$

$$S_3 = 9 \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} E[g'(F(t_{i-1}), t_{i-1}) g'(F(t_{j-1}), t_{j-1})] \widehat{\sigma}_i \sigma_i^2 \widehat{\sigma}_j \sigma_j^2.$$

Let $\xi_1 = F(t_{i-1})$, $\xi_2 = \sigma_i^{-1} \Delta F_i$, $\xi_3 = F(t_{j-1})$, $Y = \sigma_j^{-1} \Delta F_j$ and $\rho_k = E[\xi_k Y]$. Define $f \in C^3(\mathbb{R}^3)$ by $f(x) = g(x_1, t_{i-1}) x_2^3 g(x_3, t_{j-1})$ and define $h(x) = x^3$. By Corollary 4.2 with $k = 2$,

$$\begin{aligned} E[f(\xi) Y^3] &= \sum_{|\alpha| \le 2} \frac{1}{\alpha!} \rho^\alpha E[\partial^\alpha f(\xi)] E\bigl[h_{|\alpha|}(Y) Y^3\bigr] + R \\ &= 3 \sum_{|\alpha|=1} \frac{1}{\alpha!} \rho^\alpha E[\partial^\alpha f(\xi)] + R, \end{aligned}$$

where $|R| \le C|\rho|^3$. Hence,

$$\begin{aligned} |S_1 - S_2| &= \Bigg| \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sigma_i^3 \sigma_j^3 \bigl(E[f(\xi) Y^3] - 3\rho_3 E[\partial_3 f(\xi)]\bigr) \Bigg| \\ &\le C \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sigma_i^3 \sigma_j^3 (|\rho_1| + |\rho_2| + |\rho_3|^3) \end{aligned}$$

$$\leq C \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \big(\Delta t^{5/4} |EF(t_{i-1})\Delta F_j| + \Delta t |E\Delta F_i \Delta F_j| + \Delta t^{3/4} |\widehat{\sigma}_j|^3\big).$$

By (2.8), (2.9), Lemma 2.9(i) with $c = 0$ and Lemma 2.9(iii),

$$\begin{aligned}
|S_1 - S_2| &\leq C \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \big(\Delta t^{7/4}(|j-i| \vee 1)^{-1/2} + \Delta t^{3/2}(|j-i| \vee 1)^{-3/2} + \Delta t^{9/4}\big) \\
&\leq C \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \Delta t^{5/4} \leq C\left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}\right)\Delta t^{1/4}.
\end{aligned}$$

To estimate $S_2 - S_3$, let $\xi_1 = F(t_{i-1})$, $\xi_2 = F(t_{j-1})$, $Y = \sigma_i^{-1}\Delta F_i$ and $\rho_k = E[\xi_k Y]$. Define $f \in C^3(\mathbb{R}^2)$ by $f(x) = g(x_1, t_{j-1})g'(x_2, t_{j-1})$ and $h(x) = x^3$. As above,

$$\begin{aligned}
|S_2 - S_3| &= 3\left|\sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \widehat{\sigma}_j \sigma_j^2 \sigma_i^3 \big(E[f(\xi)Y^3] - 3\rho_1 E[\partial_1 f(\xi)]\big)\right| \\
&\leq C \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} |\widehat{\sigma}_j| \sigma_j^2 \sigma_i^3 (|\rho_2| + |\rho_1|^3) \\
&\leq C \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \big(\Delta t^2(|j-i| \vee 1)^{-1/2} + \Delta t^{5/2}\big) \\
&\leq C \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \Delta t^{3/2} \leq C\left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}\right)\Delta t^{1/2}.
\end{aligned}$$

Combining these results, we have

$$E|X_n - Y_n|^2 \leq C\left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}\right)\Delta t^{1/4} \leq C\left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}\right)^{5/4}.$$

Note that

$$\begin{aligned}
EY_n^2 &\leq C \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} |\widehat{\sigma}_i \sigma_i^2 \widehat{\sigma}_j \sigma_j^2| \\
&\leq C \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \Delta t^2 = C\left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}\right)^2.
\end{aligned}$$

Since $t - s \le T$, this shows that

$$E|Z_n(t) - Z_n(s)|^2 = EX_n^2 \le C\left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}\right)^{5/4}.$$

Taking $s = 0$ verifies condition (iii) of Theorem 2.1. Hence, by Corollary 2.2, $\{Z_n\}$ is relatively compact. Since $X_n - Y_n \to 0$ in L^2, it will suffice, by Lemma 2.6, to show that

$$Y_n(0,t) = 3\sum_{j=1}^{\lfloor nt \rfloor} g'(F(t_{j-1}), t_{j-1})\widehat{\sigma}_j \sigma_j^2 \to -\frac{3}{2\pi}\int_0^t g'(F(s), s)\, ds$$

in probability. For this, observe that by (2.4) and (2.9),

$$\begin{aligned} |\widehat{\sigma}_j \sigma_j^2 + \pi^{-1}\Delta t| &\le |\widehat{\sigma}_j + (2\pi)^{-1/2}\Delta t^{1/2}|\sigma_j^2 \\ &\qquad + (2\pi)^{-1/2}\Delta t^{1/2}|(2/\pi)^{1/2}\Delta t^{1/2} - \sigma_j^2| \\ &\le Cj^{-1/2}\Delta t. \end{aligned}$$

Hence,

$$\left|\sum_{j=1}^{\lfloor nt \rfloor} g'(F(t_{j-1}), t_{j-1})\widehat{\sigma}_j \sigma_j^2 + \frac{1}{\pi}\sum_{j=1}^{\lfloor nt \rfloor} g'(F(t_{j-1}), t_{j-1})\Delta t\right| \le C\Delta t^{1/2} \to 0.$$

Since

$$\sum_{\substack{j=1 \\ j \text{ odd}}}^{\lfloor nt \rfloor} g'(F(t_{j-1}), t_{j-1})\Delta t \to \frac{1}{2}\int_0^t g'(F(s), s)\, ds$$

almost surely, this completes the proof of (4.6).

For (4.7), note that we may use (3.5) with $K = \mathbb{R}$ and $j = 0$ to obtain

$$\begin{aligned} &g(F(t_j), t_j) - g(F(t_{j-1}), t_{j-1}) \\ &\qquad = \int_{t_{j-1}}^{t_j} \partial_t g(F(t_j), s)\, ds + g(F(t_j), t_{j-1}) - g(F(t_{j-1}), t_{j-1}) \\ &\qquad = g'(F(t_{j-1}), t_{j-1})\Delta F_j + R, \end{aligned}$$

where $|R| \le C(\Delta t + \Delta F_j^2)$. Hence,

$$\begin{aligned} \sum_{j=1}^{\lfloor nt \rfloor} g(F(t_j), t_j)\Delta F_j^3 &= \sum_{j=1}^{\lfloor nt \rfloor} g(F(t_{j-1}), t_{j-1})\Delta F_j^3 \\ &\qquad + \sum_{j=1}^{\lfloor nt \rfloor} g'(F(t_{j-1}), t_{j-1})\Delta F_j^4 + \widetilde{R}, \end{aligned} \tag{4.9}$$

where $|\widetilde{R}| \to 0$ ucp. Applying (4.6) and Theorem 3.1 completes the proof. $\square$

As a reminder, $X_n(t) \approx Y_n(t)$ means that $X_n - Y_n \to 0$ ucp. Let

$$J_n(g,t) = \sum_{j=1}^{2\lfloor nt/2 \rfloor} g(F(t_{j-1}), t_{j-1}) \Delta F_j^2 (-1)^j. \tag{4.10}$$

COROLLARY 4.5. *If $g \in C_3^{7,1}(\mathbb{R} \times [0,\infty))$, then*

$$I_n(g',t) \approx g(F(t),t) - g(F(0),0) - \int_0^t \partial_t g(F(s),s)\,ds - \frac{1}{2} J_n(g'',t),$$

where $I_n(g,t)$ and $J_n(g,t)$ are given by (1.4) and (4.10), respectively. Moreover,

$$T_n^F(g',t) \approx g(F(t),t) - g(F(0),0) - \int_0^t \partial_t g(F(s),s)\,ds,$$

where T_n^F is given by (1.7).

PROOF. By Theorems 3.2, 3.3 and 4.3, it will suffice to show that

$$\sum_{j=1}^{\lfloor nt/2 \rfloor} g''(F(t_{2j-1}), t_{2j-1})(\Delta F_{2j}^2 - \Delta F_{2j-1}^2) \approx J_n(g'',t).$$

As before, we may assume that g has compact support. Note that

$$\begin{aligned}
&\sum_{j=1}^{\lfloor nt/2 \rfloor} g''(F(t_{2j-1}), t_{2j-1})(\Delta F_{2j}^2 - \Delta F_{2j-1}^2) \\
&\quad = \sum_{\substack{j=1 \\ j \text{ even}}}^{2\lfloor nt/2 \rfloor} g''(F(t_{j-1}), t_{j-1}) \Delta F_j^2 - \sum_{\substack{j=1 \\ j \text{ odd}}}^{2\lfloor nt/2 \rfloor} g''(F(t_j), t_j) \Delta F_j^2 \\
&\quad = J_n(g'',t) - \sum_{\substack{j=1 \\ j \text{ odd}}}^{2\lfloor nt/2 \rfloor} \{g''(F(t_j), t_j) - g''(F(t_{j-1}), t_{j-1})\} \Delta F_j^2.
\end{aligned}$$

The proof is completed by using (3.10) and applying Theorem 4.3. $\square$

COROLLARY 4.6. *If $g \in C_3^{7,1}(\mathbb{R} \times [0,\infty))$, then*

$$T_n^X(g',t) \approx g(X(t),t) - g(X(0),0) - \int_0^t \partial_t g(X(s),s)\,ds,$$

where T_n^X is given by (1.7). This result remains true even when $X = cF + \xi$, where ξ satisfies only (1.3), and is not necessarily independent of F.

PROOF. By passing to a subsequence, we may assume that Corollary 4.5 holds almost surely. It will therefore suffice to prove Corollary 4.6 under the assumption that ξ is deterministic.

The claim is trivial when $c = 0$. Suppose $c \neq 0$. Let $h = h_\xi$ be given by $h(x,t) = g(cx + \xi(t), t)$. We claim that $h \in C_3^{7,1}(\mathbb{R} \times [0,\infty))$. Note that $h^{(j)}(F(t),t) = c^j g^{(j)}(X(t),t)$ for all $j \le 7$. It is straightforward to verify (3.2) and (3.3). Conditions (3.4) and (3.5) follow from the fact that

$$\partial_t h^{(j)}(x,t) = c^j g^{(j+1)}\big(cx + \xi(t), t\big)\xi'(t) + c^j \partial_t g^{(j)}\big(cx + \xi(t), t\big)$$

for all $j \le 3$.

Observe that

$$T_n^X(g',t) = T_n^F(h',t) + c^{-1} \sum_{j=1}^{\lfloor nt \rfloor} \frac{h'(F(t_{j-1}), t_{j-1}) + h'(F(t_j), t_j)}{2} \Delta\xi_j.$$

By our hypotheses on ξ, and the continuity of h' and F, the above summation converges to $\int_0^t h'(F(s),s)\xi'(s)\,ds$, uniformly on compacts with probability one. Thus, by Corollary 4.5, we have

$$\begin{aligned} T_n^X(g',t) &\approx h(F(t),t) - h(F(0),0) \\ &\qquad - \int_0^t \partial_t h(F(s),s)\,ds + c^{-1}\int_0^t h'(F(s),s)\xi'(s)\,ds \\ &= g(X(t),t) - g(X(0),0) - \int_0^t \partial_t g(X(s),s)\,ds, \end{aligned}$$

which completes the proof. $\square$

5. Relative compactness. The main result of this section is Theorem 5.1 below, from which the relative compactness of $\{J_n(g,\cdot)\}$ will follow as a corollary. [Recall that $\{J_n(g,\cdot)\}$ is defined in (4.10).] Later in Section 6, we will again need Theorem 5.1, when we show that J_n converges weakly to an ordinary Itô integral.

THEOREM 5.1. *Let $g \in C_2^{7,1}(\mathbb{R} \times [0,\infty))$ have compact support. Fix $T > 0$ and let c and d be integers such that $0 \le t_c < t_d \le T$. Then,*

$$E\left|\sum_{j=c+1}^{d} \{g(F(t_{j-1}), t_{j-1}) - g(F(t_c), t_c)\}\Delta F_j^2 (-1)^j\right|^2 \le C|t_d - t_c|^{3/2},$$

where C depends only on g and T.

Consider the simple case $c = 0$ and $g(x,t) = x$. In that case, the above expectation is

$$E\left|\sum_{j=1}^{d} F(t_{j-1})\Delta F_j^2(-1)^j\right|^2 = \sum_{i=1}^{d}\sum_{j=1}^{d} E[F(t_{i-1})\Delta F_i^2 F(t_{j-1})\Delta F_j^2](-1)^{i+j}.$$

Using Corollary 4.2, we can remove the ΔF^2 factors from inside the expectation. The leading term in the resulting expansion would be roughly

$$\begin{aligned}\Delta t \sum_{i=1}^{d}\sum_{j=1}^{d} E[F(t_{i-1})F(t_{j-1})](-1)^{i+j} \\ = \Delta t \sum_{i,j \text{ even}} E\big[\big(F(t_{i-1}) - F(t_{i-2})\big)\big(F(t_{j-1}) - F(t_{j-2})\big)\big].\end{aligned}$$

We could now use (2.8) to analyze these expectations and prove the theorem in this simple case.

If we are to follow this strategy, then we will need an estimate analogous to (2.8) which applies to functions of F. The estimate in (2.8) was originally arrived at through direct computations with the covariance function. Unfortunately, such direct computations are not tractable for a general function of F. There is, however, an alternative derivation of (2.8). Specifically, if we observe that $|\partial_{st}\rho(s,t)| \le C|t-s|^{-3/2}$, where ∂_{st} is the mixed second partial derivative, then we may conclude that $|E[\Delta F_i \Delta F_j]| \le C\Delta t^2|t_j - t_i|^{-3/2}$. Based on these heuristics, we begin with the following.

LEMMA 5.2. *Let X be a centered Gaussian process with continuous covariance function $\rho(s,t)$ and define $V(t) = \rho(t,t)$. Suppose that ρ is a C^2 function away from the set $\{s=0\} \cup \{t=0\} \cup \{s=t\}$ and that $V(t)$ is a positive C^1 function on $\{t>0\}$. Suppose that $\varphi \in C^2(\mathbb{R})$ has polynomial growth of order 2 with constants K and r, and define $V_\varphi(t) = E[\varphi(X(t))]$. Then,*

$$V_\varphi'(t) = \tfrac{1}{2}V'(t)E[\varphi''(X(t))].$$

In particular, $|V_\varphi'(t)| \le C|V'(t)|$ for all $0 < t \le T$, where C depends only on K, r and T.

PROOF. Let $\sigma(t) = V(t)^{1/2}$ and note that σ is a positive C^1 function on $\{t>0\}$. Fix $t>0$ and let $X = \sigma(t)^{-1}X(t)$ so that X is a standard normal random variable and $V_\varphi(t) = E[\varphi(\sigma(t)X)]$. Since φ' has polynomial growth, we may differentiate under the expectation, giving

$$V_\varphi'(t) = \sigma'(t)E[X\varphi'(\sigma(t)X)] = \frac{V'(t)}{2\sigma(t)}E[\varphi'(\sigma(t)X)h_1(X)],$$

where h_n is given by (2.16). By (2.25), we have

$$V_\varphi'(t) = \frac{V'(t)}{2\sigma(t)}E[\sigma(t)\varphi''(\sigma(t)X)h_0(X)] = \frac{1}{2}V'(t)E[\varphi''(X(t))]. \qquad \square$$

PROPOSITION 5.3. *Let X, ρ, and V be as in Lemma 5.2. Let $g, h \in C^2(\mathbb{R})$ have polynomial growth of order 2 with common constants K and r, and define $f(s,t) = E[g(X(s))h(X(t))]$. Then,*

$$\begin{aligned}\partial_s f(s,t) &= \tfrac{1}{2}V'(s)E[g''(X(s))h(X(t))] \\ &\quad + \partial_s \rho(s,t)E[g'(X(s))h'(X(t))] \quad \text{and}\end{aligned} \tag{5.1}$$

$$\partial_t f(s,t) = \tfrac{1}{2}V'(t)E[g(X(s))h''(X(t))] + \partial_t \rho(s,t)E[g'(X(s))h'(X(t))] \tag{5.2}$$

whenever $0 < s, t \le T$ and $s \ne t$. In particular,

$$|\partial_s f(s,t)| \le C\big(|V'(s)| + |\partial_s \rho(s,t)|\big)$$

and

$$|\partial_t f(s,t)| \le C\big(|V'(t)| + |\partial_t \rho(s,t)|\big),$$

where C depends only on K, r and T.

PROOF. By symmetry, we only need to prove (5.1). Let $\sigma(t) = V(t)^{1/2}$ and note that σ is a positive C^1 function on $\{t > 0\}$. Let $r = r(s,t) = \sigma(s)^{-1}\sigma(t)^{-1} \times \rho(s,t)$ and define $X_r = \sigma(s)^{-1}X(s)$ and $Y_r = \sigma(t)^{-1}X(t)$. Note that X_r and Y_r are jointly normal with mean zero, variance one and $E[X_r Y_r] = r$.

Let φ be as in Lemma 2.12. Then $f(s,t) = \varphi(\sigma(s), \sigma(t), r(s,t))$. Hence, by Lemmas 2.11 and 2.12,

$$\begin{aligned}\partial_s f(s,t) &= \sigma'(s)\sigma(s)E[g''(X(s))h(X(t))] \\ &\quad + \sigma'(s)\sigma(t)r(s,t)E[g'(X(s))h'(X(t))] \\ &\quad + \partial_s r(s,t)\sigma(s)\sigma(t)E[g'(X(s))h'(X(t))].\end{aligned}$$

Note that $\sigma'(s) = V'(s)/(2\sigma(s))$ and

$$\partial_s r(s,t) = \frac{\partial_s \rho(s,t)}{\sigma(s)\sigma(t)} - \frac{\rho(s,t)}{\sigma(s)^2\sigma(t)}\sigma'(s) = \frac{\partial_s \rho(s,t)}{\sigma(s)\sigma(t)} - \frac{V'(s)r(s,t)}{2\sigma(s)^2}.$$

Thus,

$$\begin{aligned}\partial_s f(s,t) &= \tfrac{1}{2}V'(s)E[g''(X(s))h(X(t))] \\ &\quad + \tfrac{1}{2}V'(s)\sigma(s)^{-1}\sigma(t)r(s,t)E[g'(X(s))h'(X(t))] \\ &\quad + \partial_s \rho(s,t)E[g'(X(s))h'(X(t))] \\ &\quad - \tfrac{1}{2}V'(s)\sigma(s)^{-1}\sigma(t)r(s,t)E[g'(X(s))h'(X(t))] \\ &= \tfrac{1}{2}V'(s)E[g''(X(s))h(X(t))] + \partial_s \rho(s,t)E[g'(X(s))h'(X(t))]. \qquad \square\end{aligned}$$

THEOREM 5.4. *Let X, ρ and V be as in Lemma 5.2. Let $g, h \in C^3(\mathbb{R})$ have polynomial growth of order* 3 *with common constants K and r, and define $f(s,t) = E[g(X(s))h(X(t))]$. Then*

$$|\partial_{st} f(s,t)| \le C|\partial_{st}\rho(s,t)| + C\big(|V'(s)| + |\partial_s \rho(s,t)|\big)\big(|V'(t)| + |\partial_t \rho(s,t)|\big),$$

whenever $0 < s, t \le T$ and $s \ne t$, where C depends only on K, r and T.

PROOF. By (5.1),

$$\begin{aligned}\partial_{st} f(s,t) = {} & \tfrac{1}{2} V'(s)\partial_t\{E[g''(X(s))h(X(t))]\} + \partial_s\rho(s,t)\partial_t\{E[g'(X(s))h'(X(t))]\} \\ & + \partial_{st}\rho(s,t)E[g'(X(s))h'(X(t))].\end{aligned}$$

Applying (5.2), we have

$$\begin{aligned}\partial_{st} f(s,t) = {} & \tfrac{1}{4} V'(s)V'(t)E[g''(X(s))h''(X(t))] \\ & + \tfrac{1}{2} V'(s)\partial_t\rho(s,t)E[g'''(X(s))h'(X(t))] \\ & + \tfrac{1}{2} V'(t)\partial_s\rho(s,t)E[g'(X(s))h'''(X(t))] \\ & + \partial_s\rho(s,t)\partial_t\rho(s,t)E[g''(X(s))h''(X(t))] \\ & + \partial_{st}\rho(s,t)E[g'(X(s))h'(X(t))]\end{aligned}$$

and the theorem now follows. □

From Theorem 5.4, we immediately obtain the following corollary.

COROLLARY 5.5. *Let X, ρ and V be as in Lemma 5.2. Let $g, h \in C^3(\mathbb{R})$ have polynomial growth of order* 3 *with common constants K and r, and define $f(s,t) = E[g(X(s))h(X(t))]$. If*

$$|V'(t)| \le Ct^{-1/2},$$
$$|\partial_s\rho(s,t)| + |\partial_t\rho(s,t)| \le C\big(s^{-1/2} + (t-s)^{-1/2}\big)$$

and

$$|\partial_{st}\rho(s,t)| \le C\big(s^{-3/2} + (t-s)^{-3/2}\big)$$

for all $0 < s < t \le T$, where C depends on only T, then

$$|\partial_{st} f(s,t)| \le C\big(s^{-3/2} + (t-s)^{-3/2}\big)$$

for a (*possibly different*) *constant C that depends only on K, r and T.*

With this corollary in place, we can now begin proving Theorem 5.1.

LEMMA 5.6. *Suppose $g \in C_0^{1,1}(\mathbb{R} \times [0,\infty))$ has compact support. If $p > 0$, then*

$$E|g(F(t),t) - g(F(s),s)|^p \le C|t-s|^{p/4}$$

for all $0 \le s, t \le T$, where C depends only on g, p and T.

PROOF. We write

$$\begin{aligned} g(F(t),t) &- g(F(s),s) \\ &= \int_s^t \partial_t g(F(t),u)\,du \\ &\quad + \bigl(F(t) - F(s)\bigr) \int_0^1 g'\bigl(F(s) + u\bigl(F(t) - F(s)\bigr), s\bigr)\,du. \end{aligned}$$

Hence, $|g(F(t),t) - g(F(s),s)| \le C|t-s| + C|F(t) - F(s)|$. Since F is a Gaussian process, an application of (2.3) completes the proof. □

LEMMA 5.7. *Recall that $\sigma_j^2 = E\Delta F_j^2$. Under the hypotheses of Theorem 5.1,*

$$E\left|\sum_{j=c+1}^{d} \{g(F(t_{j-1}),t_{j-1}) - g(F(t_c),t_c)\}\sigma_j^2(-1)^j\right|^2 \le C|t_d - t_c|^{3/2},$$

where C depends only on g and T.

PROOF. By (2.4),

$$\sum_{j=c+1}^{d} \{g(F(t_{j-1}),t_{j-1}) - g(F(t_c),t_c)\}\sigma_j^2(-1)^j = S + \varepsilon,$$

where

$$S = \left(\frac{2}{\pi}\right)^{1/2} \Delta t^{1/2} \sum_{j=c+1}^{d} \{g(F(t_{j-1}),t_{j-1}) - g(F(t_c),t_c)\}(-1)^j$$

and, by Hölder's inequality,

$$\begin{aligned} |\varepsilon|^2 &\le C\Delta t\left(\sum_{j=c+1}^{d} |g(F(t_{j-1}),t_{j-1}) - g(F(t_c),t_c)|\,j^{-3/2}\right)^2 \\ &\le C\Delta t\left(\sum_{j=c+1}^{d} |g(F(t_{j-1}),t_{j-1}) - g(F(t_c),t_c)|^2\right)\left(\sum_{j=c+1}^{d} j^{-3}\right). \end{aligned}$$

Hence, by Lemma 5.6,

$$E|\varepsilon|^2 \le C\Delta t^{3/2} \sum_{j=c+1}^{d} |j-c|^{1/2} \le C\Delta t^{3/2}|d-c|^{3/2} = C|t_d - t_c|^{3/2}.$$

As for S, we assume, without loss of generality, that c and d are both even. In that case,

$$\begin{aligned} S &= \left(\frac{2}{\pi}\right)^{1/2} \Delta t^{1/2} \sum_{\substack{j=c+1\\ j \text{ even}}}^{d} \{g(F(t_{j-1}), t_{j-1}) - g(F(t_{j-2}), t_{j-2})\} \\ &= \left(\frac{2}{\pi}\right)^{1/2} \Delta t^{1/2} \sum_{\substack{j=c+1\\ j \text{ even}}}^{d} \left\{\int_{t_{j-2}}^{t_{j-1}} \partial_t g(F(t_{j-1}), s)\, ds \right. \\ &\qquad\qquad \left. + g(F(t_{j-1}), t_{j-2}) - g(F(t_{j-2}), t_{j-2})\right\}. \end{aligned}$$

Using (3.5) with $j = 0$, the integral is bounded by $C\Delta t$ and we have $E|S|^2 \le C\Delta t(|t_d - t_c|^2 + S_1 + S_2)$, where

$$S_1 = \sum_{\substack{j=c+1\\ j \text{ even}}}^{d} E|g(F(t_{j-1}), t_{j-2}) - g(F(t_{j-2}, t_{j-2}))|^2,$$

$$\begin{aligned} S_2 &= 2 \sum_{\substack{i=c+1\\ i \text{ even}}}^{d} \sum_{\substack{j=i+2\\ j \text{ even}}}^{d} |E[\{g(F(t_{i-1}), t_{i-2}) - g(F(t_{i-2}), t_{i-2})\} \\ &\qquad\qquad \times \{g(F(t_{j-1}), t_{j-2}) - g(F(t_{j-2}), t_{j-2})\}]| \\ &= 2 \sum_{\substack{i=c+1\\ i \text{ even}}}^{d} \sum_{\substack{j=i+2\\ j \text{ even}}}^{d} \left| \int_{t_{i-2}}^{t_{i-1}} \int_{t_{j-2}}^{t_{j-1}} \partial_{st} f_{ij}(s,t)\, dt\, ds \right| \end{aligned}$$

and $f_{ij}(s,t) = E[g(F(s), t_{i-2})g(F(t), t_{j-2})]$. Note that F is a Gaussian process satisfying the conditions of Corollary 5.5. Hence,

$$\begin{aligned} S_2 &\le C \sum_{\substack{i=c+1\\ i \text{ even}}}^{d} \sum_{\substack{j=i+2\\ j \text{ even}}}^{d} \int_{t_{i-2}}^{t_{i-1}} \int_{t_{j-2}}^{t_{j-1}} \big(s^{-3/2} + (t-s)^{-3/2}\big)\, dt\, ds \\ &\le C\Delta t^{1/2} \sum_{\substack{i=c+1\\ i \text{ even}}}^{d} \sum_{\substack{j=i+2\\ j \text{ even}}}^{d} \big((i-2)^{-3/2} + (j-i-1)^{-3/2}\big) \\ &\le C\Delta t^{1/2}(d-c) = C\Delta t^{-1/2}|t_d - t_c|. \end{aligned}$$

By Lemma 5.6, we also have $S_1 \le C\Delta t^{-1/2}|t_d - t_c|$. Hence,

$$E|S|^2 \le C\Delta t^{1/2}|t_d - t_c|.$$

Combined with the estimate on $E|\varepsilon|^2$, this completes the proof. □

LEMMA 5.8. *Let* $\widehat{\sigma}_{c,j} = E[(F(t_{j-1}) - F(t_c))\Delta F_j]$. *Under the hypotheses of Theorem* 5.1,

$$E\left|\sum_{j=c+1}^{d} g''(F(t_{j-1}), t_{j-1})\widehat{\sigma}_{c,j}^2(-1)^j\right|^2 \le C\Delta t|t_d - t_c|,$$

where C *depends only on* g *and* T.

PROOF. By Lemma 2.9(i) applied with $c = 0$, and (2.9), we have

$$\begin{aligned}|\widehat{\sigma}_{c,j} + (2\pi)^{-1/2}\Delta t^{1/2}| &= |\widehat{\sigma}_j - E[F(t_c)\Delta F_j] + (2\pi)^{-1/2}\Delta t^{1/2}|\\ &\le C\Delta t^{1/2}(j-c)^{-1/2}.\end{aligned}$$

Hence, by Lemma 2.9(i),

$$|\widehat{\sigma}_{c,j}^2 - (2\pi)^{-1}\Delta t| \le C\Delta t^{1/2}|\widehat{\sigma}_{c,j} + (2\pi)^{-1/2}\Delta t^{1/2}| \le C\Delta t(j-c)^{-1/2}.$$

Therefore,

$$\sum_{j=c+1}^{d} g''(F(t_{j-1}), t_{j-1})\widehat{\sigma}_{c,j}^2(-1)^j = S + \varepsilon,$$

where

$$S = (2\pi)^{-1}\Delta t \sum_{j=c+1}^{d} g''(F(t_{j-1}), t_{j-1})(-1)^j$$

and

$$|\varepsilon|^2 \le C\Delta t^2\left(\sum_{j=c+1}^{d}(j-c)^{-1/2}\right)^2 \le C\Delta t^2(d-c) = C\Delta t|t_d - t_c|.$$

The proof that $E|S|^2 \le C\Delta t|t_d - t_c|$ is similar to that in the proof of Lemma 5.7, except that we must use (3.5) with $j = 2$. □

LEMMA 5.9. *Under the hypotheses of Theorem* 5.1, *we have*

$$E\left|\sum_{j=c+1}^{d}\{g(F(t_c), t_{j-1}) - g(F(t_c), t_c)\}\Delta F_j^2(-1)^j\right|^2 \le C|t_d - t_c|^3,$$

where C *depends only on* g *and* T.

PROOF. Let $Y(t) = g(F(t_c), t) - g(F(t_c), t_c)$ and note that

$$E\left|\sum_{j=c+1}^{d} Y(t_{j-1})\Delta F_j^2(-1)^j\right|^2$$

$$= \sum_{i=c+1}^{d}\sum_{j=c+1}^{d} E[Y(t_{i-1})\Delta F_i^2 Y(t_{j-1})\Delta F_j^2](-1)^{i+j}.$$

For fixed i, j, define $f:\mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \left(\frac{g(x_1, t_{i-1}) - g(x_1, t_c)}{t_{i-1} - t_c}\right)\left(\frac{g(x_1, t_{j-1}) - g(x_1, t_c)}{t_{j-1} - t_c}\right)x_2^2.$$

By (3.5) with $j = 2$, f has polynomial growth of order 2 with constants K and r that do not depend on i or j.

Let $\xi_1 = F(t_c)$, $\xi_2 = \sigma_i^{-1}\Delta F_i$, $Y = \sigma_j^{-1}\Delta F_j$ and $h(y) = y^2$. By Corollary 4.2 with $k = 1$, $E[f(\xi)h(Y)] = E[f(\xi)] + R_1$, where $|R_1| \le C|\rho|^2$. Similarly, if $\widetilde{f}(x_1) = f(x_1, 1)$, then

$$E[f(\xi)] = E[\widetilde{f}(\xi_1)h(\xi_2)] = E[\widetilde{f}(\xi_1)] + R_2,$$

where $|R_2| \le C|E[\xi_1\xi_2]|^2$. Therefore,

$$E[Y(t_{i-1})\Delta F_i^2 Y(t_{j-1})\Delta F_j^2] = \sigma_i^2\sigma_j^2 E[Y(t_{i-1})Y(t_{j-1})] + R_3,$$

where

$$|R_3| = \sigma_i^2\sigma_j^2|t_{i-1} - t_c||t_{j-1} - t_c||R_1 + R_2| \le \Delta t^3|i - c||j - c||R_1 + R_2|.$$

Using Lemma 2.9(iii) and (2.8),

$$|\rho_1| = |E[\xi_1 Y]| \le C\Delta t^{1/4}|j - c|^{-1/2},$$

$$|\rho_2| = |E[\xi_2 Y]| \le C(|j - i| \vee 1)^{-3/2}, \qquad |E[\xi_1\xi_2]| \le C\Delta t^{1/4}|i - c|^{-1/2}.$$

This gives

$$|R_3| \le C\Delta t^{7/2}(|i - c| + |j - c|) + C\Delta t^3(|j - i| \vee 1)^{-3}(|i - c|^2 + |j - c|^2).$$

Observe that

$$\sum_{i=c+1}^{d}\sum_{j=c+1}^{d}|R_3(i, j)| \le C\Delta t^{7/2}(d - c)^3 + C\Delta t^3(d - c)^3 \le C|t_d - t_c|^3.$$

Hence, we are reduced to considering

$$\sum_{i=c+1}^{d}\sum_{j=c+1}^{d}\sigma_i^2\sigma_j^2 E[Y(t_{i-1})Y(t_{j-1})](-1)^{i+j} = E\left|\sum_{j=c+1}^{d} Y(t_{j-1})\sigma_j^2(-1)^j\right|^2.$$

Using (3.5) with $j=0$ and (2.4), we have

$$\begin{aligned}
&\left|\sum_{j=c+1}^{d} Y(t_{j-1})\sigma_j^2(-1)^j\right| \\
&\qquad = \left|\sum_{\substack{j=c+1 \\ j \text{ even}}}^{d} \bigl(Y(t_{j-1})\sigma_j^2 - Y(t_{j-2})\sigma_{j-1}^2\bigr)\right| \\
&\qquad \le \sum_{\substack{j=c+1 \\ j \text{ even}}}^{d} \bigl(|Y(t_{j-1})||\sigma_j^2 - \sigma_{j-1}^2| + |Y(t_{j-1}) - Y(t_{j-2})|\sigma_{j-1}^2\bigr) \\
&\qquad \le C \sum_{j=c+1}^{d} (|t_{j-1} - t_c| j^{-3/2} \Delta t^{1/2} + \Delta t^{3/2}) \\
&\qquad \le C \Delta t^{3/2} \sum_{j=c+1}^{d} |j-c|^{-1/2} \le C|t_d - t_c|^{3/2},
\end{aligned}$$

which completes the proof. $\square$

PROOF OF THEOREM 5.1. By Lemma 5.9, it will suffice to show that

$$E\left|\sum_{j=c+1}^{d} \{g(F(t_{j-1}), t_{j-1}) - g(F(t_c), t_{j-1})\}\Delta F_j^2(-1)^j\right|^2 \le C|t_d - t_c|^{3/2}.$$

For brevity, let $\mathcal{X}(t) = g(F(t), t) - g(F(t_c), t)$ and write

$$\begin{aligned}
&E\left|\sum_{j=c+1}^{d} \mathcal{X}(t_{j-1})\Delta F_j^2(-1)^j\right|^2 \\
&\qquad = \sum_{i=c+1}^{d} \sum_{j=c+1}^{d} E[\mathcal{X}(t_{i-1})\Delta F_i^2 \mathcal{X}(t_{j-1})\Delta F_j^2](-1)^{i+j}.
\end{aligned}$$

Recall that $\sigma_j^2 = E\Delta F_j^2$. Let $\delta_c(t) = F(t) - F(t_c)$. Let $\sigma_{c,j}^2 = E\delta_c(t_j)^2$. Let $\xi_1 = F(t_c)$, $\xi_2 = \sigma_{c,i-1}^{-1}\delta_c(t_{i-1})$, $\xi_3 = \sigma_{c,j-1}^{-1}\delta_c(t_{j-1})$, $\xi_4 = \sigma_i^{-1}\Delta F_i$ and $\xi = (\xi_1, \dots, \xi_4)$. For $x \in \mathbb{R}^4$, define $f = f_{ij}$ by

$$\begin{aligned}
f(x) &= \left(\frac{g(x_1 + \sigma_{c,i-1}x_2, t_{i-1}) - g(x_1, t_{i-1})}{\sigma_{c,i-1}}\right) \\
&\qquad \times \left(\frac{g(x_1 + \sigma_{c,j-1}x_3, t_{j-1}) - g(x_1, t_{j-1})}{\sigma_{c,j-1}}\right) x_4^2.
\end{aligned}$$

Let $Y = \sigma_j^{-1} \Delta F_j$ and $h(y) = y^2$.

Note that for $\theta \in (0, 1]$ and $t_j \in [0, T]$, $x \mapsto \theta^{-1}(g(x_1 + \theta x_2, t_j) - g(x_1, t_j))$ has polynomial growth of order 6 with constants K and r that do not depend on θ or j. Hence, f has polynomial growth of order 6 with constants K and r. Thus, by Corollary 4.2 with $k = 5$, if $\sigma = \sigma_{c,i-1}\sigma_{c,j-1}\sigma_i^2\sigma_j^2$, then

$$\begin{aligned} &E[\mathcal{X}(t_{i-1})\Delta F_i^2 \mathcal{X}(t_{j-1})\Delta F_j^2] \\ &\quad = \sigma E[f(\xi)h(Y)] \\ &\quad = \sigma \biggl(\sum_{|\alpha| \le 5} \frac{1}{\alpha!} \rho^\alpha E[\partial^\alpha f(\xi)] E\bigl[h_{|\alpha|}(Y)Y^2\bigr] + R_1 \biggr), \end{aligned}$$

where $\rho_j = E[\xi_j Y]$ and $|R_1| \le C|\rho|^6$. If p is a positive integer, then by (2.19), (2.20) and (2.22) with $r = 1$,

$$E\bigl[h_{|\alpha|}(Y)Y^p\bigr] = 0 \qquad \text{if } p - |\alpha| \text{ is odd or } |\alpha| > p. \tag{5.3}$$

Hence, since $E[h_2(Y)Y^2] = E[Y^4 - Y^2] = 2$,

$$E[\mathcal{X}(t_{i-1})\Delta F_i^2 \mathcal{X}(t_{j-1})\Delta F_j^2] = \sigma E[f(\xi)] + \sigma \rho_3^2 E[\partial_3^2 f(\xi)] + \sigma R_2,$$

where R_2 incorporates all terms of the form $\rho^\alpha E[\partial^\alpha f(\xi)]$ with $|\alpha| = 2$, except $\alpha = (0, 0, 2, 0)$. It follows that

$$\begin{aligned} &E[\mathcal{X}(t_{i-1})\Delta F_i^2 \mathcal{X}(t_{j-1})\Delta F_j^2] \\ &\quad = \sigma_j^2 E[\mathcal{X}(t_{i-1})\Delta F_i^2 \mathcal{X}(t_{j-1})] \\ &\qquad + \widehat{\sigma}_{c,j}^2 E[\mathcal{X}(t_{i-1})\Delta F_i^2 g''(F(t_{j-1}), t_{j-1})] + \sigma R_2, \end{aligned} \tag{5.4}$$

where $\widehat{\sigma}_{c,j} = E[\delta_c(t_{j-1})\Delta F_j]$ and

$$|R_2| \le C(|\rho_1|^2 + |\rho_2|^2 + |\rho_4| + |\rho_1\rho_2| + |\rho_1\rho_3| + |\rho_2\rho_3| + |\rho_3|^6).$$

The terms $|\rho_1\rho_4|$, $|\rho_2\rho_4|$, $|\rho_3\rho_4|$ and $|\rho_4|^2$ are not listed on the right-hand side of the above estimate because $|\rho_1\rho_4| + |\rho_2\rho_4| + |\rho_3\rho_4| + |\rho_4|^2 \le C|\rho_4|$. Using (2.3) and Lemma 2.9, we have

$$\begin{aligned} |\sigma| &\le C\Delta t^{3/2}|i - c|^{1/4}|j - c|^{1/4}, \\ |\rho_2| &\le C|i - c|^{-1/4}(|j - i|^{-1/2} + |j - c|^{-1/2}), \\ |\rho_1| &\le C\Delta t^{1/4}|j - c|^{-1/2} \\ &\le C|i - c|^{-1/4}(|j - i|^{-1/2} + |j - c|^{-1/2}), \\ |\rho_3| &\le C|j - c|^{-1/4}, \\ |\rho_4| &\le C|j - i|^{-3/2}. \end{aligned}$$

Note that the above factors of $|j-i|$ are actually $(|j-i| \vee 1)$, although we have omitted this to simplify the notation. These estimates now yield

$$\begin{aligned}|\sigma R_2| \le C\Delta t^{3/2}(&|i-c|^{-1/4}|j-c|^{1/4}|j-i|^{-1}\\ &+|i-c|^{-1/4}|j-c|^{-3/4}+|i-c|^{1/4}|j-c|^{-5/4}\\ &+|i-c|^{-1/4}|j-c|^{1/4}|j-i|^{-3/2}+|j-i|^{-1/2}+|j-c|^{-1/2}).\end{aligned}$$

Using $|j-c| \le |j-i|+|i-c|$ and $|i-c| \le |j-i|+|j-c|$, we can show that

$$|\sigma R_2| \le C\Delta t^{3/2}(|i-c|^{1/4}|j-c|^{-5/4}+|j-i|^{-1/2}+|j-c|^{-1/2})$$

and, therefore, that

$$\sum_{i=c+1}^{d}\sum_{j=c+1}^{d}|\sigma R_2| \le C\Delta t^{3/2}\sum_{i=c+1}^{d}(d-c)^{1/2} \le C\Delta t^{3/2}(d-c)^{3/2} = C|t_d-t_c|^{3/2}.$$

By (5.4), we are now reduced to considering the sums

$$\begin{aligned}&\sum_{i=c+1}^{d}\sum_{j=c+1}^{d}\sigma_j^2 E[\mathcal{X}(t_{i-1})\Delta F_i^2\mathcal{X}(t_{j-1})](-1)^{i+j}\\ &\qquad+\sum_{i=c+1}^{d}\sum_{j=c+1}^{d}\widehat{\sigma}_{c,j}^2 E[\mathcal{X}(t_{i-1})\Delta F_i^2 g''(F(t_{j-1}),t_{j-1})](-1)^{i+j},\end{aligned} \tag{5.5}$$

which will require two more applications of Corollary 4.2. We will be brief in our presentation because the following estimates can be obtained in a way very similar to the one presented above.

For $x \in \mathbb{R}^3$, define $\widetilde{f}_1(x) = f(x_1,x_2,x_3,1)$. Let $\widetilde{Y} = \xi_4$, $\widetilde{\xi} = (\xi_1,\xi_2,\xi_3)$ and $\widetilde{\rho}_j = E[\xi_j\widetilde{Y}]$. Note that $\widetilde{f}_1$ and $\widetilde{f}_2 = \sigma_{c,j-1}^{-1}\partial_3^2\widetilde{f}$ both have polynomial growth of order 5 with constants K and r. Applying Corollary 4.2 with $k=4$ and using (5.3), we have

$$\begin{aligned}&\sigma_j^2 E[\mathcal{X}(t_{i-1})\Delta F_i^2\mathcal{X}(t_{j-1})]\\ &\quad=\sigma E[\widetilde{f}_1(\widetilde{\xi})h(\widetilde{Y})]\\ &\quad=\sigma E[\widetilde{f}_1(\widetilde{\xi})]+\sigma\widetilde{\rho}_2^2E[\partial_2^2\widetilde{f}_1(\widetilde{\xi})]+\sigma R_3\\ &\quad=\sigma_i^2\sigma_j^2E[\mathcal{X}(t_{i-1})\mathcal{X}(t_{j-1})]\\ &\qquad+\widehat{\sigma}_{c,i}^2\sigma_j^2E[g''(F(t_{i-1}),t_{i-1})\mathcal{X}(t_{j-1})]+\sigma R_3,\end{aligned} \tag{5.6}$$

where

$$|R_3| \le C(|\widetilde{\rho}_1|^2+|\widetilde{\rho}_3|^2+|\widetilde{\rho}_1\widetilde{\rho}_2|+|\widetilde{\rho}_1\widetilde{\rho}_3|+|\widetilde{\rho}_2\widetilde{\rho}_3|+|\widetilde{\rho}_2|^5). \tag{5.7}$$

As before,

$$
\begin{aligned}
|\widetilde{\rho}_3| &\le C|j-c|^{-1/4}(|i-j|^{-1/2}+|i-c|^{-1/2}),\\
|\widetilde{\rho}_1| &\le C\Delta t^{1/4}|i-c|^{-1/2}\\
&\le C|j-c|^{-1/4}(|i-j|^{-1/2}+|i-c|^{-1/2}),\\
|\widetilde{\rho}_2| &\le C|i-c|^{-1/4},
\end{aligned}
$$

which gives

$$|\sigma R_3| \le C\Delta t^{3/2}(|i-c|^{1/4}|i-j|^{-5/4}+|i-j|^{-1/2}+|i-c|^{-1/2})$$

and shows that

$$\sum_{i=c+1}^{d}\sum_{j=c+1}^{d}|\sigma R_3| \le C|t_d-t_c|^{3/2}. \tag{5.8}$$

Similarly, if $\widetilde{\sigma}=\widehat{\sigma}_{c,j}^2\sigma_i^2\sigma_{c,i-1}$, then

$$
\begin{aligned}
&\widehat{\sigma}_{c,j}^2 E[\mathcal{X}(t_{i-1})\Delta F_i^2 g''(F(t_{j-1}),t_{j-1})]\\
&\qquad = \widetilde{\sigma}E[\widetilde{f}_2(\widetilde{\xi})h(\widetilde{Y})]\\
&\qquad = \widetilde{\sigma}E[\widetilde{f}_2(\widetilde{\xi})]+\widetilde{\sigma}\widetilde{\rho}_2^2E[\partial_2^2\widetilde{f}_2(\widetilde{\xi})]+\widetilde{\sigma}R_4 \qquad (5.9)\\
&\qquad = \sigma_i^2\widehat{\sigma}_{c,j}^2E[\mathcal{X}(t_{i-1})g''(F(t_{j-1}),t_{j-1})]\\
&\qquad\quad + \widehat{\sigma}_{c,i}^2\widehat{\sigma}_{c,j}^2E[g''(F(t_{i-1}),t_{i-1})g''(F(t_{j-1}),t_{i-1})]+\widetilde{\sigma}R_4,
\end{aligned}
$$

where R_4 also satisfies (5.7). Note that $|\widetilde{\sigma}|\le C\Delta t^{7/4}|i-c|^{1/4}$. Since this is a better estimate than the one we use for $|\sigma|$, the estimates above also give

$$\sum_{i=c+1}^{d}\sum_{j=c+1}^{d}|\widetilde{\sigma} R_4| \le C|t_d-t_c|^{3/2}. \tag{5.10}$$

By (5.5), (5.6), (5.8), (5.9) and (5.10), we are reduced to considering the sums

$$
\begin{aligned}
&\sum_{i=c+1}^{d}\sum_{j=c+1}^{d}\sigma_i^2\sigma_j^2E[\mathcal{X}(t_{i-1})\mathcal{X}(t_{j-1})](-1)^{i+j}\\
&\quad+\sum_{i=c+1}^{d}\sum_{j=c+1}^{d}\widehat{\sigma}_{c,i}^2\sigma_j^2E[g''(F(t_{i-1}),t_{i-1})\mathcal{X}(t_{j-1})](-1)^{i+j}\\
&\quad+\sum_{i=c+1}^{d}\sum_{j=c+1}^{d}\sigma_i^2\widehat{\sigma}_{c,j}^2E[\mathcal{X}(t_{i-1})g''(F(t_{j-1}),t_{j-1})](-1)^{i+j}\\
&\quad+\sum_{i=c+1}^{d}\sum_{j=c+1}^{d}\widehat{\sigma}_{c,i}^2\widehat{\sigma}_{c,j}^2E[g''(F(t_{i-1}),t_{i-1})g''(F(t_{j-1}),t_{j-1})](-1)^{i+j}.
\end{aligned}
$$

Note that this can be simplified to

$$E\left|\sum_{j=c+1}^{d}\sigma_j^2\mathcal{X}(t_{j-1})(-1)^j+\sum_{j=c+1}^{d}\widehat{\sigma}_{c,j}^2 g''(F(t_{j-1}),t_{j-1})(-1)^j\right|^2$$
$$\le C\left(E\left|\sum_{j=c+1}^{d}\sigma_j^2\mathcal{X}(t_{j-1})(-1)^j\right|^2 + E\left|\sum_{j=c+1}^{d}\widehat{\sigma}_{c,j}^2 g''(F(t_{j-1}),t_{j-1})(-1)^j\right|^2\right).$$

By Lemmas 5.7 and 5.8, this completes the proof. □

COROLLARY 5.10. *Recall $J_n(g,t)$ from* (4.10). *If $g \in C_2^{7,1}(\mathbb{R}\times[0,\infty))$ has compact support, then $\{J_n(g,\cdot)\}$ is relatively compact in $D_{\mathbb{R}}[0,\infty)$.*

PROOF. We shall apply Corollary 2.2 with $\beta = 4$. First, note that $q(x+y)^4 \le C(|x|^2+|y|^4)$. Fix $0 \le s \le t \le T$. Let $c = 2\lfloor ns/2\rfloor$ and $d = 2\lfloor nt/2\rfloor$. Then,

$$E\big[q\big(J_n(t)-J_n(s)\big)^4\big]$$
$$\le CE\left|\sum_{j=c+1}^{d}\{g(F(t_{j-1}),t_{j-1})-g(F(t_c),t_c)\}\Delta F_j^2(-1)^j\right|^2$$
$$+CE\left|g(F(t_c),t_c)\sum_{j=c+1}^{d}\Delta F_j^2(-1)^j\right|^4.$$

By Theorem 5.1 and (2.12),

$$E\big[q\big(J_n(t)-J_n(s)\big)^4\big] \le C|t_d-t_c|^{3/2}+C|t_d-t_c|^2 \le C\left(\frac{2\lfloor nt/2\rfloor-2\lfloor ns/2\rfloor}{n}\right)^{3/2}.$$

This shows that one of the assumptions of Corollary 2.2 holds. The other assumption follows from the same estimate applied with $s = 0$. By Corollary 2.2, $\{J_n\}$ is relatively compact. □

6. Convergence to a Brownian integral. Recall that $J_n(g,t)$ is given by (4.10) and $B_n(t)$ is given by (2.11). Note that

$$J_n(g,t)=\kappa\int_0^t g(F_n(s-),N(s-))\,dB_n(s),$$

where $N(t) = \lfloor nt\rfloor/n$ and $F_n(t) = F(N(t))$. In light of Theorem 2.10, we would like to apply Theorem 2.7. Unfortunately, though, $\{B_n\}$ cannot be decomposed in a way that satisfies (2.2). This is essentially due to the numerous local oscillations of B_n. To overcome this difficulty, we consider a modified version of B_n.

The process B_n has a jump after every Δt units of time. To “smooth out” this process, we shall restrict it so that it jumps only after every $\Delta t^{1/4}$ units of time. Define

$$\overline{B}_n(t) = \kappa^{-1} \sum_{j=1}^{2m^3\lfloor mt/2\rfloor} \Delta F_j^2(-1)^j, \tag{6.1}$$

where $m = \lfloor n^{1/4}\rfloor$.

LEMMA 6.1. *The sequence* $\{\overline{B}_n\}$ *given by* (6.1) *satisfies* (2.2) *and* $B_n - \overline{B}_n \to 0$ *ucp*.

PROOF. Given k, let $d = d(k) = 2m^3k$ and $c = c(k) = 2m^3(k-1)$. Write $\overline{B}_n(t) = \kappa^{-1}\sum_{k=1}^{\lfloor mt/2\rfloor} \xi_k$, where

$$\xi_k = \sum_{j=c+1}^{d} \Delta F_j^2(-1)^j.$$

For $c < j \le d$, let $\Delta\overline{F}_j = \Delta F_j - E[\Delta F_j|\mathcal{F}_{t_c}]$, where $\mathcal{F}_t$ is given by (2.13). Let

$$\overline{\xi}_k = \sum_{j=c+1}^{d} \Delta\overline{F}_j^2(-1)^j$$

so that $\{\overline{\xi}_k\}$ is an i.i.d. sequence, by the remarks following (2.13). In particular, $M_n(t) = \kappa^{-1}\sum_{k=1}^{\lfloor mt/2\rfloor} \overline{\xi}_k$ is a martingale. Let $A_n = \overline{B}_n - M_n$. We must now verify (2.2).

Since $\{\Delta\overline{F}_j\}_{j=c+1}^{\infty}$ has the same law as $\{\Delta F_j\}_{j=1}^{\infty}$, (2.12) implies that

$$E|\overline{\xi}_k|^2 = E\left|\sum_{j=1}^{2m^3} \Delta F_j^2(-1)^j\right|^2 = E|\kappa B_n(2m^3/n)|^2 \le Cn^{-1/4}.$$

It follows that $E[M_n]_t = \kappa^{-1}\sum_{k=1}^{\lfloor mt/2\rfloor} E|\overline{\xi}_k|^2 \le Ct$ for all n. Also, by (2.15),

$$E|\xi_k - \overline{\xi}_k| \le C\Delta t^{1/2} \sum_{j=c+1}^{d} (j-c)^{-3/4} \le C\Delta t^{1/2}(2m^3)^{1/4} \le Cn^{-5/16}.$$

It follows that $EV_t(A_n) = \kappa^{-1}\sum_{k=1}^{\lfloor mt/2\rfloor} E|\xi_k - \overline{\xi}_k| \le Ctn^{-1/16}$ and $\{\overline{B}_n\}$ satisfies (2.2).

By (2.12),

$$E|\overline{B}_n(t) - \overline{B}_n(s)|^4 \le C\left(\frac{2m^3\lfloor mt/2\rfloor - 2m^3\lfloor ms/2\rfloor}{n}\right)^2.$$

By Corollary 2.2, $\{\overline{B}_n\}$ is relatively compact. By Corollary 2.4 and Theorem 2.10, $\{B_n - \overline{B}_n\}$ is relatively compact. Hence, by Lemma 2.6, in order to show that $B_n - \overline{B}_n \to 0$ ucp, it will suffice to show that $B_n(t) - \overline{B}_n(t) \to 0$ in probability for each fixed t.

For this, note that $n^{1/4} - 1 < m \le n^{1/4}$. Hence, $m^3\lfloor mt/2\rfloor \le nt/2$. Since $m^3\lfloor mt/2\rfloor$ is an integer, $m^3\lfloor mt/2\rfloor \le \lfloor nt/2\rfloor$. By (2.12),

$$
\begin{aligned}
E|B_n(t) - \overline{B}_n(t)|^4 &= E\left|\kappa^{-1}\sum_{j=2m^3\lfloor mt/2\rfloor+1}^{2\lfloor nt/2\rfloor} \Delta F_j^2(-1)^j\right|^4 \\
&\le C\left(\frac{2\lfloor nt/2\rfloor - 2m^3\lfloor mt/2\rfloor}{n}\right)^2 \\
&\le C\left(\frac{nt - m^4 t + 2m^3}{n}\right)^2 \\
&\le C\left(\frac{nt - (n^{1/4}-1)^4 t + 2n^{3/4}}{n}\right)^2.
\end{aligned}
\tag{6.2}
$$

Letting $n \to \infty$ completes the proof. □

With this lemma in place, we are finally ready to prove our main result.

THEOREM 6.2. *Let $I_n(g,t)$ be given by (1.4) and κ, B_n by (2.10) and (2.11), respectively. Let B be a standard Brownian motion, independent of F. If $g \in C_4^{9,1}(\mathbb{R} \times [0,\infty))$, then $(F, B_n, I_n(g', \cdot)) \to (F, B, I^{F,B}(g', \cdot))$ in law in $D_{\mathbb{R}^3}[0,\infty)$, where $I^{F,B}(g', \cdot)$ is given by (1.6).*

REMARK 6.3. Suppose $\{W_n\}$ is another sequence of cadlag, $\mathbb{R}^\ell$-valued processes, adapted to a filtration of the form $\{\mathcal{F}_t \vee \mathcal{G}_t^n\}$, where $\{\mathcal{F}_t\}$ and $\{\mathcal{G}_t^n\}$ are independent. If $(W_n, F, B_n) \to (W, F, B)$ in law in $D_{\mathbb{R}^{\ell+2}}[0,\infty)$, then $(W_n, F, B_n, I_n(g', \cdot)) \to (W, F, B, I^{F,B}(g', \cdot))$ in law in $D_{\mathbb{R}^{\ell+3}}[0,\infty)$. This can be seen by applying Remark 2.8 to (6.3) below.

PROOF OF THEOREM 6.2. By Lemma 6.1 and Theorem 2.10, $\overline{B}_n \to B$ in law. Define $N(t) = 2m^3\lfloor mt/2\rfloor/n$ and $\overline{F}_n(t) = F(N(t))$. By continuity, $g''(\overline{F}_n(\cdot), N(\cdot))$ converges to $g''(F(\cdot), \cdot)$ a.s. Hence, by Corollary 2.4 and Lemma 2.5,

$$(F, g''(\overline{F}_n(\cdot), N(\cdot)), \overline{B}_n) \to (F, g''(F(\cdot), \cdot), B)$$

in law in $D_{\mathbb{R}^3}[0,\infty)$. Therefore, by Lemma 6.1, Theorem 2.7 and Remark 2.8,

$$
\begin{aligned}
&\left(F, g''(\overline{F}_n(\cdot), N(\cdot)), \overline{B}_n, \kappa\int_0^\cdot g''(\overline{F}_n(s-), N(s-))\, d\overline{B}_n(s)\right) \\
&\quad \to \left(F, g''(F(\cdot), \cdot), B, \kappa\int_0^\cdot g''(F(s), s)\, dB(s)\right)
\end{aligned}
\tag{6.3}
$$

in law in $D_{\mathbb{R}^4}[0,\infty)$. By Corollary 4.5 and Lemma 6.1,

$$(F, B_n, I_n(g', t)) \approx \biggl(F, \overline{B}_n, g(F(\cdot), \cdot) - g(F(0), 0) - \int_0^t \partial_t g(F(s), s)\, ds - \frac{\kappa}{2} \int_0^{\cdot} g''(\overline{F}_n(s-), N(s-))\, d\overline{B}_n(s)\biggr) - \frac{1}{2}(0, 0, \zeta_n(t)),$$

where

$$\zeta_n(t) = J_n(g'', t) - \kappa \int_0^t g''(\overline{F}_n(s-), N(s-))\, d\overline{B}_n(s).$$

Hence, it will suffice to show that $\zeta_n \to 0$ ucp.

By (6.1), $\overline{B}_n$ jumps only at times of the form $s = 2k/m$, where k is an integer. At such a time, $N(s-) = 2m^3(k-1)/n$ and $\overline{F}_n(s-) = F(N(s-))$. Using the notation in the proof of Lemma 6.1, this gives

$$\begin{aligned}
\kappa &\int_0^t g''(\overline{F}_n(s-), N(s-))\, d\overline{B}_n(s) \\
&= \kappa \sum_{0<s\le t} g''(\overline{F}_n(s-), N(s-)) \Delta \overline{B}_n(s) \\
&= \kappa \sum_{k=1}^{\lfloor mt/2 \rfloor} g''\bigl(F\bigl(t_{2m^3(k-1)}\bigr), t_{2m^3(k-1)}\bigr)\kappa^{-1} \sum_{j=2m^3(k-1)+1}^{2m^3k} \Delta F_j^2 (-1)^j \\
&= \sum_{k=1}^{\lfloor mt/2 \rfloor} \sum_{j=c+1}^{d} g''(F(t_c), t_c) \Delta F_j^2 (-1)^j.
\end{aligned}$$

Hence, by (4.10), $\zeta_n(t) = \sum_{k=1}^{\lfloor mt/2 \rfloor} S_k + \varepsilon_n$, where

$$S_k = \sum_{j=c+1}^{d} \{g''(F(t_{j-1}), t_{j-1}) - g''(F(t_c), t_c)\} \Delta F_j^2 (-1)^j \tag{6.4}$$

and

$$\varepsilon_n = \sum_{j=2m^3\lfloor mt/2 \rfloor+1}^{2\lfloor nt/2 \rfloor} g''(F(t_{j-1}), t_{j-1}) \Delta F_j^2 (-1)^j.$$

By the truncation argument in the proof of Theorem 3.2, we may assume that g has compact support. Hence, by Corollary 5.10, $\{J_n(g'', \cdot)\}$ is relatively compact, so by Corollary 2.4 and (6.3), $\{\zeta_n\}$ is relatively compact. Therefore, by Lemma 2.6, it will suffice to show that $\zeta_n(t) \to 0$ in probability for fixed t.

If $M = 2m^3\lfloor mt/2\rfloor$ and $N = 2\lfloor nt/2\rfloor$, then

$$\varepsilon_n = \sum_{j=M+1}^{N} \{g''(F(t_{j-1}), t_{j-1}) - g''(F(t_M), t_M)\}\Delta F_j^2(-1)^j$$
$$+ g''(F(t_M), t_M) \sum_{j=M+1}^{N} \Delta F_j^2(-1)^j.$$

Note that g'' is bounded and, by (2.11) and (2.12),

$$E\left|\sum_{j=M+1}^{N} \Delta F_j^2(-1)^j\right|^4 = E|B_n(N/n) - B_n(M/n)|^4 \le C|t_N - t_M|^2.$$

As in (6.2), this goes to zero as $n \to \infty$. Also, by Theorem 5.1,

$$E\left|\sum_{j=M+1}^{N} \{g''(F(t_{j-1}), t_{j-1}) - g''(F(t_M), t_M)\}\Delta F_j^2(-1)^j\right|^2 \le C|t_N - t_M|^{3/2}.$$

Hence, $\varepsilon_n \to 0$ in probability and it remains only to check that $\sum_{k=1}^{\lfloor mt/2\rfloor} S_k \to 0$ in probability.

Still using the notation from the proof of Lemma 6.1, let

$$\overline{S}_k = \sum_{j=c+1}^{d} \{g''(F(t_{j-1}), t_{j-1}) - g''(F(t_c), t_c)\}\Delta \overline{F}_j^2(-1)^j, \tag{6.5}$$

$\overline{m}_k = E[\overline{S}_k|\mathcal{F}_{t_c}]$ and $\overline{N}_k = \overline{S}_k - \overline{m}_k$. We claim that

$$E|S_k - \overline{N}_k|^2 \le C\Delta t^{5/8}. \tag{6.6}$$

For the moment, let us grant that this claim is true. In that case,

$$E\left|\sum_{k=1}^{\lfloor mt/2\rfloor} S_k\right| \le \sum_{k=1}^{\lfloor mt/2\rfloor} E|S_k - \overline{N}_k| + \left(E\left|\sum_{k=1}^{\lfloor mt/2\rfloor} \overline{N}_k\right|^2\right)^{1/2}.$$

Since $m \le n^{1/4} = \Delta t^{-1/4}$, (6.6) gives $\sum_{k=1}^{\lfloor mt/2\rfloor} E|S_k - \overline{N}_k| \le C\Delta t^{1/16} \to 0$. Also, if $k < \ell$, then $E[\overline{N}_k\overline{N}_\ell] = E[\overline{N}_k E[\overline{N}_\ell \mid \mathcal{F}_{t_{c(\ell)}}]] = 0$. Hence,

$$E\left|\sum_{k=1}^{\lfloor mt/2\rfloor} \overline{N}_k\right|^2 = \sum_{k=1}^{\lfloor mt/2\rfloor} E\overline{N}_k^2 \le C\sum_{k=1}^{\lfloor mt/2\rfloor} E|\overline{N}_k - S_k|^2 + C\sum_{k=1}^{\lfloor mt/2\rfloor} ES_k^2.$$

As above, the first summation goes to zero. For the second summation, note that $g'' \in C_2^{7,1}(\mathbb{R} \times [0, \infty))$ has compact support. Thus, by (6.4), Theorem 5.1 and the fact that $d - c = 2m^3 \le 2\Delta t^{-3/4}$, we have

$$ES_k^2 = E\left|\sum_{j=c+1}^{d} \{g''(F(t_{j-1}), t_{j-1}) - g''(F(t_c), t_c)\}\Delta F_j^2 (-1)^j\right|^2$$
$$\le C|t_d - t_c|^{3/2} = C\Delta t^{3/2}(d-c)^{3/2} \le C\Delta t^{3/8}.$$

Hence, $\sum_{k=1}^{\lfloor mt/2 \rfloor} ES_k^2 \le C\Delta t^{1/8} \to 0$, which completes the proof of the theorem.

It remains only to prove (6.6). By (6.4) and (6.5),

$$E|S_k - \overline{S}_k|^2$$
$$= E\left|\sum_{j=c+1}^{d} \{g''(F(t_{j-1}), t_{j-1}) - g''(F(t_c), t_c)\}(\Delta F_j^2 - \Delta \overline{F}_j^2)(-1)^j\right|^2$$
$$\le (d-c) \sum_{j=c+1}^{d} E[|g''(F(t_{j-1}), t_{j-1}) - g''(F(t_c), t_c)|^2 (\Delta F_j^2 - \Delta \overline{F}_j^2)^2].$$

By Hölder's inequality, Lemma 5.6 and (2.15),

$$E|S_k - \overline{S}_k|^2 \le C(d-c) \sum_{j=c+1}^{d} (t_j - t_c)^{1/2} \Delta t (j-c)^{-3/2}$$
$$= C\Delta t^{3/2}(d-c) \sum_{j=c+1}^{d} (j-c)^{-1}$$
$$\le C\Delta t^{3/2}(d-c)^{7/6} \le C\Delta t^{5/8}.$$

Hence, it will suffice to show that $E|\overline{m}_k|^2 \le C\Delta t^{5/8}$.

By (6.5),

$$\overline{m}_k = \sum_{j=c+1}^{d} E[g''(F(t_{j-1}), t_{j-1})\Delta \overline{F}_j^2 \mid \mathcal{F}_{t_c}](-1)^j$$
$$- \sum_{j=c+1}^{d} g''(F(t_c), t_c) E[\Delta \overline{F}_j^2](-1)^j$$
$$= \sum_{j=c+1}^{d} E\big[g''\big(G(t_{j-c-1}) + X_{j-c}, t_{j-1}\big)\Delta G_{j-c}^2 \mid \mathcal{F}_{t_c}\big](-1)^j$$
$$- \sum_{j=c+1}^{d} g''(F(t_c), t_c) E[\Delta G_{j-c}^2](-1)^j,$$

where $G(t) = F(t + t_c) - E[F(t + t_c) \mid \mathcal{F}_{t_c}]$ and $X_j = E[F(t_{j+c-1}) \mid \mathcal{F}_{t_c}]$. As noted in the discussion following (2.13), G is independent of $\mathcal{F}_{t_c}$ and has the same law as F. Thus,

$$\overline{m}_k = \sum_{\substack{j=1 \\ j \text{ even}}}^{d-c} \bigl(\varphi_j(X_j) - \varphi_{j-1}(X_{j-1})\bigr) - g''(F(t_c), t_c) \sum_{\substack{j=1 \\ j \text{ even}}}^{d-c} (\sigma_j^2 - \sigma_{j-1}^2),$$

where

$$\varphi_j(x) = E\bigl[g''\bigl(F(t_{j-1}) + x, t_{j+c-1}\bigr)\Delta F_j^2\bigr].$$

Using (2.25), if $\sigma^2 = EF(t_{j-1})^2$, then we have

$$\begin{aligned}
\varphi_j(x) &= \sigma_j^2 E\bigl[g''\bigl(F(t_{j-1}) + x, t_{j+c-1}\bigr)\bigr] \\
&\quad + \sigma_j^2 E\bigl[g''\bigl(\sigma(\sigma^{-1}F(t_{j-1})) + x, t_{j+c-1}\bigr)h_2(\sigma_j^{-1}\Delta F_j)\bigr] \\
&= \sigma_j^2 E\bigl[g''\bigl(F(t_{j-1}) + x, t_{j+c-1}\bigr)\bigr] + \sigma_j^2 (E[\sigma^{-1}F(t_{j-1})\sigma_j^{-1}\Delta F_j])^2 \\
&\quad \times E\bigl[\sigma^2 g^{(4)}\bigl(\sigma(\sigma^{-1}F(t_{j-1})) + x, t_{j+c-1}\bigr)h_0(\sigma_j^{-1}\Delta F_j)\bigr] \\
&= \sigma_j^2 E\bigl[g''\bigl(F(t_{j-1}) + x, t_{j+c-1}\bigr)\bigr] \\
&\quad + (E[F(t_{j-1})\Delta F_j])^2 E\bigl[g^{(4)}\bigl(F(t_{j-1}) + x, t_{j+c-1}\bigr)\bigr] \\
&= \sigma_j^2 b_j(x) + \widehat{\sigma}_j^2 c_j(x),
\end{aligned}$$

where

$$\begin{aligned}
b_j(x) &= E\bigl[g''\bigl(F(t_{j-1}) + x, t_{j+c-1}\bigr)\bigr], \\
c_j(x) &= E\bigl[g^{(4)}\bigl(F(t_{j-1}) + x, t_{j+c-1}\bigr)\bigr].
\end{aligned}$$

We may therefore write

$$\overline{m}_k = \sum_{\substack{j=1 \\ j \text{ even}}}^{d-c} \sum_{i=1}^{5} \mathcal{E}_i, \tag{6.7}$$

where

$$\begin{aligned}
\mathcal{E}_1 &= (\sigma_j^2 - \sigma_{j-1}^2) b_j(X_j), \\
\mathcal{E}_2 &= \sigma_{j-1}^2 \bigl(b_j(X_j) - b_{j-1}(X_{j-1})\bigr), \\
\mathcal{E}_3 &= (\widehat{\sigma}_j^2 - \widehat{\sigma}_{j-1}^2) c_j(X_j), \\
\mathcal{E}_4 &= \widehat{\sigma}_{j-1}^2 \bigl(c_j(X_j) - c_{j-1}(X_{j-1})\bigr), \\
\mathcal{E}_5 &= -g''(F(t_c), t_c)(\sigma_j^2 - \sigma_{j-1}^2).
\end{aligned}$$

For $\mathcal{E}_1$, (2.4) gives $|\sigma_j^2 - \sigma_{j-1}^2| \le Cj^{-3/2}\Delta t^{1/2}$. Hence,

$$E\left|\sum_{\substack{j=1\\ j \text{ even}}}^{d-c} \mathcal{E}_1\right|^2 \le C\Delta t.$$

The same estimate also applies to $\mathcal{E}_5$. For $\mathcal{E}_2$, let us write

$$\begin{aligned}
&|b_j(x_j) - b_{j-1}(x_{j-1})| \\
&\quad \le |b_j(x_j) - b_j(x_{j-1})| + |b_j(x_{j-1}) - b_{j-1}(x_{j-1})| \\
&\quad \le C|x_j - x_{j-1}| \\
&\qquad + \big|E\big[g''\big(F(t_{j-1}) + x_{j-1}, t_{j+c-1}\big) - g''\big(F(t_{j-2}) + x_{j-1}, t_{j+c-2}\big)\big]\big| \\
&\quad \le C|x_j - x_{j-1}| \\
&\qquad + \big|E\big[g''\big(F(t_{j-1}) + x_{j-1}, t_{j+c-1}\big) - g''\big(F(t_{j-2}) + x_{j-1}, t_{j+c-1}\big)\big]\big| \\
&\qquad + \big|E\big[g''\big(F(t_{j-2}) + x_{j-1}, t_{j+c-1}\big) - g''\big(F(t_{j-2}) + x_{j-1}, t_{j+c-2}\big)\big]\big| \\
&\quad \le C|x_j - x_{j-1}| + |\beta_2'(t^*)|\Delta t + C\Delta t,
\end{aligned}$$

where $\beta_2(t) = E[g''(F(t) + x_{j-1}, t_{j+c-1})]$ and $t^* \in (t_{j-2}, t_{j-1})$, and where we have used (3.5) with $j = 2$. By Lemma 5.2, $|\beta_2'(t)| \le Ct^{-1/2}$. Also, note that $X_j - X_{j-1} = E[\Delta F_{j+c-1} \mid \mathcal{F}_{t_c}]$ so that by (2.14), $E|X_j - X_{j-1}|^2 \le Cj^{-3/2}\Delta t^{1/2}$. Thus,

$$\begin{aligned}
E\left|\sum_{\substack{j=1\\ j \text{ even}}}^{d-c} \mathcal{E}_2\right|^2 &\le \left(\sum_{j=1}^{d-c} \sigma_{j-1}^4\right)\left(\sum_{j=1}^{d-c} E|b_j(X_j) - b_{j-1}(X_{j-1})|^2\right) \\
&\le C\Delta t^{1/4} \sum_{j=1}^{d-c} (j^{-3/2}\Delta t^{1/2} + j^{-1}\Delta t) \qquad (6.8)\\
&\le C\big(\Delta t^{3/4} + \Delta t^{5/4}(d-c)^{2/3}\big) \le C\Delta t^{3/4}.
\end{aligned}$$

For $\mathcal{E}_3$, (2.9) gives $|\widehat{\sigma}_j^2 - \widehat{\sigma}_{j-1}^2| \le Cj^{-1/2}\Delta t$. Hence,

$$E\left|\sum_{\substack{j=1\\ j \text{ even}}}^{d-c} \mathcal{E}_3\right|^2 \le C\Delta t^2(d-c) \le C\Delta t^{5/4}.$$

For $\mathcal{E}_4$, as above, we have

$$|c_j(x_j) - c_{j-1}(x_{j-1})| \le C|x_j - x_{j-1}| + |\beta_4'(t^*)|\Delta t + C\Delta t,$$

where $\beta_4(t) = E[g^{(4)}(F(t) + x_{j-1}, t_{j+c-1})]$ and $t^* \in (t_{j-2}, t_{j-1})$, and where we have used (3.5) with $j = 4$. It therefore follows, as in (6.8), that

$$E\left|\sum_{\substack{j=1\\ j \text{ even}}}^{d-c} \mathcal{E}_4\right|^2 \le C\Delta t^{3/4}.$$

Applying these five estimates to (6.7) shows that $E|\overline{m}_k|^2 \le C\Delta t^{3/4} \le C\Delta t^{5/8}$ and completes the proof. □

COROLLARY 6.4. *Let ξ be a continuous stochastic process, independent of F, such that* (1.3) *holds. Let $X = cF + \xi$, where $c \in \mathbb{R}$. Let $I_n^X(g, t)$ be given by* (1.4) *and κ, B_n by* (2.10) *and* (2.11), *respectively. Let B be a standard Brownian motion, independent of (F, ξ). If $g \in C_4^{9,1}(\mathbb{R} \times [0, \infty))$, then $(F, \xi, B_n, I_n^X(g', \cdot)) \to (F, \xi, B, I^{X,c^2B}(g', \cdot))$ in law in $D_{\mathbb{R}^4}[0, \infty)$, where $I^{X,Y}$ is given by* (1.6).

REMARK 6.5. Recall Q_n^X from Section 1 and note that $Q_n^F = \kappa B_n$. Note that $Q_n^X(t) \approx c^2 Q_n^F(t)$ because $\Delta X^2 = c^2 \Delta F^2 + o(\Delta t)$. This, together with Corollary 6.4, implies that $(X, Q_n^X, I_n^X(g', \cdot)) \to (X, \kappa c^2 B, I^{X,c^2B}(g', \cdot))$ in law in $D_{\mathbb{R}^3}[0, \infty)$.

REMARK 6.6. Suppose $\{W_n\}$ is another sequence of cadlag, $\mathbb{R}^\ell$-valued processes, adapted to a filtration of the form $\{\mathcal{F}_t \vee \mathcal{G}_t^n\}$, where $\{\mathcal{F}_t\}$ and $\{\mathcal{G}_t^n\}$ are independent. As in Remark 6.3, if $(W_n, F, B_n) \to (W, F, B)$ in law in $D_{\mathbb{R}^{\ell+2}}[0, \infty)$, then $(W_n, F, \xi, B_n, I_n^X(g', \cdot)) \to (W, F, \xi, B, I^{X,c^2B}(g', \cdot))$ in law in $D_{\mathbb{R}^{\ell+4}}[0, \infty)$.

PROOF OF COROLLARY 6.4. The claim is trivial when $c = 0$. Suppose $c \ne 0$. We first assume ξ is deterministic. Let $h = h_\xi$ be given by $h(x, t) = g(cx + \xi(t), t)$. We claim that $h \in C_4^{9,1}(\mathbb{R} \times [0, \infty))$. Note that $h^{(j)}(F(t), t) = c^j g^{(j)}(X(t), t)$ for all $j \le 9$. It is straightforward to verify (3.2) and (3.3). Conditions (3.4) and (3.5) follow from the fact that

$$\partial_t h^{(j)}(x, t) = c^j g^{(j+1)}\big(cx + \xi(t), t\big)\xi'(t) + c^j \partial_t g^{(j)}\big(cx + \xi(t), t\big) \tag{6.9}$$

for all $j \le 4$.

Observe that

$$I_n^X(g', t) = I_n(h', t) + c^{-1} \sum_{j=1}^{\lfloor nt/2 \rfloor} h'(F(t_{2j-1}), t_{2j-1})\big(\xi(t_{2j}) - \xi(t_{2j-2})\big).$$

By our hypotheses on ξ, and the continuity of h' and F, the above sum converges uniformly on compacts, with probability one, to $\int_0^t h'(F(s), s)\xi'(s)\, ds$. Thus, by

Theorem 6.2 and Remark 6.3, $(F, \xi, B_n, I_n^X(g', \cdot)) \to (F, \xi, B, \mathcal{I})$, where

$$\mathcal{I} = h(F(t), t) - h(F(0), 0) - \int_0^t \partial_t h(F(s), s)\, ds - \frac{\kappa}{2} \int_0^t h''(F(s), s)\, dB(s) \\ + c^{-1} \int_0^t h'(F(s), s)\xi'(s)\, ds.$$

Using (6.9) with $j = 0$, this gives

$$\mathcal{I} = g(X(t), t) - g(X(0), 0) - \int_0^t \partial_t g(X(s), s)\, ds - \frac{\kappa c^2}{2} \int_0^t g''(X(s), s)\, dB(s),$$

completing the proof.

Now, suppose ξ is random and independent of F. Let $H : D_{\mathbb{R}^4}[0, \infty) \to \mathbb{R}$ be bounded and continuous. Since we have proven the result for deterministic ξ, it follows that

$$E[H(F, \xi, B_n, I_n^X(g', \cdot)) \mid \xi] \to E[H(F, \xi, B, I^{X,c^2B}(g', \cdot)) \mid \xi] \qquad \text{a.s.}$$

Applying the dominated convergence theorem completes the proof. □

We now give two examples of processes X satisfying the conditions of Corollary 6.4.

EXAMPLE 6.7. Consider the stochastic heat equation $\partial_t u = \frac{1}{2}\partial_x^2 u + \dot{W}(x, t)$ with initial conditions $u(x, 0) = f(x)$. Under suitable conditions on f, the unique solution is

$$u(x, t) = \int_{\mathbb{R}\times[0,t]} p(x - y, t - r) W(dy \times dr) + v(t, x),$$

where

$$v(x, t) = \int_{\mathbb{R}} p(x - y, t) f(y)\, dy.$$

For example, if f has polynomial growth, then this is the unique solution and, moreover, $\partial_t v$ is continuous on $\mathbb{R} \times [0, \infty)$. This implies that $t \mapsto v(x, t)$ satisfies (1.3). Hence, $X(t) = u(x, t) = F(t) + v(x, t)$ satisfies the conditions of Corollary 6.4. This remains true when f is allowed to be a stochastic process, independent of W.

EXAMPLE 6.8. This example is based on a decomposition of bifractional Brownian motion due to Lei and Nualart [8]. Let W be a standard Brownian motion, independent of F. Define

$$\xi(t) = (16\pi)^{-1/4} \int_0^\infty (1 - e^{-st}) s^{-3/4}\, dW(s).$$

By Proposition 1 and Theorem 1 in [8], we have $\xi \in C^1((0,\infty))$ a.s. Moreover, if $c = (\pi/2)^{1/4}$, then $X = cF + \xi$ has the same law as $B^{1/4}$, fractional Brownian motion with Hurst parameter $H = 1/4$. If $\varphi \in C^\infty[0,\infty)$ with $\varphi = 0$ on $[0, \varepsilon/4]$ and $\varphi = 1$ on $[\varepsilon/2, \infty)$, then $\varphi\xi$ satisfies (1.3) and we may apply Corollary 6.4 to $X_\varepsilon = cF + \varphi\xi$ to obtain that

$$\begin{aligned}(X(t), Q_n^X(t), I_n^X(g', t)) &- (X(\varepsilon), Q_n^X(\varepsilon), I_n^X(g', \varepsilon))\\ &= \Bigg(X(t) - X(\varepsilon), \sum_{j=\lfloor n\varepsilon/2\rfloor+1}^{\lfloor nt/2\rfloor} (\Delta X_{2j}^2 - \Delta X_{2j-1}^2),\\ &\qquad \sum_{j=\lfloor n\varepsilon/2\rfloor+1}^{\lfloor nt/2\rfloor} g(X(t_{2j-1}), t_{2j-1})\big(X(t_{2j}) - X(t_{2j-2})\big)\Bigg)\end{aligned}$$

converges in law in $D_{\mathbb{R}^3}[\varepsilon, \infty)$ as $n \to \infty$ to

$$\begin{aligned}(X(t), \kappa c^2 B(t), I^{X,c^2B}(g', t)) &- (X(\varepsilon), \kappa c^2 B(\varepsilon), I^{X,c^2B}(g', \varepsilon))\\ = \bigg(X(t) - X(\varepsilon), \kappa c^2\big(B(t) - B(\varepsilon)\big), &\, g(X(t), t) - g(X(\varepsilon), \varepsilon)\\ &- \int_\varepsilon^t \partial_t g(X(s), s)\, ds - \frac{\kappa c^2}{2}\int_\varepsilon^t \partial_x^2 g(X(s), s)\, dB(s)\bigg).\end{aligned}$$

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