



Crowding of Brownian spheres

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Abstract. We study two models consisting of reflecting one-dimensional Brownian “particles” of positive radius. We show that the stationary empirical distributions for the particle systems do not converge to the harmonic function for the generator of the individual particle process, unlike in the case when the particles are infinitely small.

1. Introduction

In this article we consider the dynamics of a collection of *hard Brownian spheres* with drifts or boundary conditions that includes instantaneous reflections upon collisions. The models are similar to existing ones in the literature that consider point masses instead of spheres of a positive radius. We will show that the (empirical) distribution of a family of Brownian spheres behaves differently from the (empirical) distribution of the point Brownian particles in some natural models. In particular, the distribution of Brownian spheres fails to satisfy the usual heat equation under circumstances that lead to the heat equation for the infinitely many infinitesimally small Brownian particles.

Various models of colliding Brownian particles have been considered in the statistical physics literature. One stream, pioneered by [Harris \(1965\)](#), considers a

Received by the editors February 4, 2010; accepted June 1st, 2010.

2000 Mathematics Subject Classification. 60K35.

Key words and phrases. hydrodynamic limit, Brownian motion, reflected Brownian motion.

K.B.: Research supported in part by NSF Grant DMS-0906743 and by grant N N201 397137, MNiSW, Poland.

J.S.: Research supported in part by NSA Grant H98230-09-1-0079.

countable collection of Brownian point masses on the line that collide and reflect instantaneously. Also see the follow-up work on tagged particle in the Harris model by Dürr, Goldstein, & Lebowitz, [Dürr et al. \(1985\)](#). A variation on the theme has been to replace the instantaneous reflection by a potential, and goes by the name of *gradient systems*. In these gradient systems, one studies the behavior of countably many particles under a repelling potential. Usually the potential is modeled as smooth with a singularity at zero; see the article by [Cépa and Lépingle \(1997\)](#). A particular example of this class includes the famous Dyson Brownian motion from Random Matrix theory; see [Dyson \(1962\)](#), and [Cépa and Lépingle \(2001\)](#). The other class of models, closer to our article, goes by the name of *hard-core* interactions, in which the Brownian particles are assumed to be hard balls of small radius, and consequently, there is instantaneous reflection whenever two such balls collide (plus possible additional interactions). This is the spirit taken in the articles by Dobrushin & Fritz in dimension one, [Dobrushin and Fritz \(1977\)](#), and Fritz & Dobrushin in dimension two, [Fritz and Dobrushin \(1977\)](#), [Lang \(1977a,b\)](#) (with a correction by [Shiga, 1979](#)). The main focus of these authors is the non-equilibrium dynamics of the gradient systems. Also see the articles by [Osada \(1996, 1998\)](#), and [Tanemura \(1996\)](#) all of which consider properties of a tagged particle in the infinite system.

In the discrete case, the various models of symmetric and asymmetric exclusion processes have been considered. Closest in spirit to the models discussed here is the totally asymmetric exclusion process (TASEP) considered by Baik, Deift, & Johansson, [Baik et al. \(1999\)](#), and [Johansson \(2000\)](#) in connection with random matrices and the longest increasing subsequence problem. Specifically, if the initial configuration in TASEP is \mathbb{Z}^- , then the probability that a particle initially at $-m$ moves at least n steps to the right in time t equals the probability distribution of the largest eigenvalue in a unitary Laguerre random matrix ensemble. In recent subsequent articles [Tracy and Widom \(2008, 2009\)](#), Tracy & Widom explicitly compute transition probabilities of individual particles in the asymmetric exclusion process, extending Johansson's work.

In this paper, we consider only one dimensional models, so our “spheres” are actually intervals. The title of this paper reflects our intention to study multidimensional models in future articles. We leave more detailed discussion to Section 4. That section also contains references to related research projects.

We consider two models, which have the following common features. Informally speaking, both models consist of families of Brownian “particles”. The k -th “particle” is represented by an interval $I_t^k = (X_t^k - \varepsilon/2, X_t^k + \varepsilon/2)$, where X_t^k is a Brownian-like process. The intervals I^k and I^j are always disjoint, for $k \neq j$. The processes X^k are driven by independent Brownian motions. When two intervals I^k and I^j collide, they reflect instantaneously. In the first model, the number of particles is constant and they are pushed by a barrier moving at a constant speed. In the second model, particles enter the interval $[0, 1]$ at the left, they reflect at 0, and they are killed when they hit the right endpoint. The second model is our primary focus because it is related to other models considered in mathematical physics literature—see Section 4.

We are grateful to Thierry Bodineau, Pablo Ferrari, Claudio Landim, Mario Primicerio and Jeremy Quastel for very helpful advice. We would like to thank the referee for the suggestions for improvement.

2. Extreme crowding

We start with an informal description of our first model, which consists of a fixed number n of “particles”. The k -th leftmost “particle” is represented by an interval $I_t^k = (X_t^k - \varepsilon/2, X_t^k + \varepsilon/2)$. The intervals I^k and I^j are always disjoint. The processes X^k are driven by independent Brownian motions. When two intervals I^k and I^j collide, they reflect instantaneously. The intervals are pushed from the left by a barrier with a constant velocity, that is, the leftmost interval reflects on the line $x = ct$.

Formally, we define $\{X^0, X^1, \dots, X^n\}$ to be continuous processes such that $X_t^0 = -\varepsilon/2 + ct$, $X_t^k - X_t^{k-1} \geq \varepsilon$ for all $k \geq 1$ and all $t \geq 0$, and for $k \geq 1$,

$$dX_t^k = dB_t^k + dL_t^k - dM_t^k,$$

where $\{B^1, \dots, B^n\}$ are iid Brownian motions, and L^k and M^k are nondecreasing processes such that

$$\int_0^\infty 1_{\{X_t^k - X_t^{k-1} > \varepsilon\}} dL_t^k = 0 \quad \text{and} \quad \int_0^\infty 1_{\{X_t^{k+1} - X_t^k > \varepsilon\}} dM_t^k = 0.$$

(Here, we may interpret $X^{n+1} \equiv \infty$.) The distributions of X_0^k for $1 \leq k \leq n$ will be specified later.

To construct the solution to this Skorohod problem, consider first the processes $Y_t^k = X_t^k - (k-1)\varepsilon - \varepsilon/2 - ct$. These processes satisfy $Y^0 \equiv 0$, $Y_t^k - Y_t^{k-1} \geq 0$ for all $k \geq 1$ and all $t \geq 0$, and for $k \geq 1$, $dY_t^k = dB_t^k - c dt + dL_t^k - dM_t^k$, where

$$\int_0^\infty 1_{\{Y_t^k - Y_t^{k-1} > 0\}} dL_t^k = 0 \quad \text{and} \quad \int_0^\infty 1_{\{Y_t^{k+1} - Y_t^k > 0\}} dM_t^k = 0.$$

We may therefore construct the processes $\{Y^1, \dots, Y^n\}$ using order statistics. Namely, let $\{Z^1, \dots, Z^n\}$ be defined by $dZ_t^k = dB_t^k - c dt$, and reflected at 0. For every fixed $t \geq 0$, we let $Y_t^1, Y_t^2, \dots, Y_t^n$ be ordered Z_t^k 's, that is, $\{Y_t^1, \dots, Y_t^n\} = \{Z_t^1, \dots, Z_t^n\}$ and $Y_t^1 \leq Y_t^2 \leq \dots \leq Y_t^n$. Finally, we let $X_t^k = Y_t^k + (k-1)\varepsilon + \varepsilon/2 + ct$ and $I_t^k = (X_t^k - \varepsilon/2, X_t^k + \varepsilon/2)$.

Let $n\varepsilon = b$. We will fix $b > 0$ and analyze the behavior of the system of intervals $\{I^k\}$ as $n \rightarrow \infty$. In other words, we will keep the total length of all intervals I^k constant.

The stationary distribution for Z^k has the density $\varphi(z) = ce^{-cz}$ for $z \geq 0$, with $c = 2c_1$, because

$$\frac{1}{2} \frac{d^2}{dz^2} \varphi(z) + c_1 \frac{d}{dz} \varphi(z) = 0.$$

Consider any $0 \leq x_1 < x_2 < \infty$, let λ denote the Lebesgue measure, and let

$$\mathbf{d}([x_1, x_2]) = \mathbf{d}_t([x_1, x_2]) = \frac{\lambda\left([x_1 + ct, x_2 + ct] \cap \bigcup_{1 \leq k \leq n} I_t^k\right)}{x_2 - x_1}. \tag{2.1}$$

The quantity $\mathbf{d}([x_1, x_2])$ represents the average density of “particles” I^k on the interval $[x_1, x_2]$.

We will say that the intervals $\{I^k\}$ have the pseudo-stationary distribution if all Z_t^k 's are independent and have the stationary distribution φ for $t = 0$ and, therefore, for every $t \geq 0$.

Theorem 2.1. *Suppose that the intervals $\{I^k\}$ have the pseudo-stationary distribution. Fix arbitrary $p_1, d_1 < 1$, $d_2 > 0$, and $0 \leq x_1 < x_2 < b < x_3 < x_4 < \infty$. There exist $c_0, n_0 < \infty$ such that for $c \geq c_0$, $n \geq n_0$ and any $t \geq 0$, we have*

$$P(\mathbf{d}_t([x_1, x_2]) \geq d_1) \geq p_1, \quad (2.2)$$

$$P(\mathbf{d}_t([x_3, x_4]) \leq d_2) \geq p_1. \quad (2.3)$$

The theorem says that the ‘‘particles’’ I^k clump together and there is a sharp transition in density of ‘‘mass’’ around $x = b$. This is in contrast with infinitely small ‘‘particles’’ Z^k whose empirical distribution is close to the distribution with the density $\varphi(z) = ce^{-cz}$ that displays no sharp drop-off.

Proof of Theorem 2.1: Without loss of generality, we let $t = 0$. We define $y_k \in (0, \infty)$ in an implicit way by the following formula, for $k = 1, 2, 3, 4$,

$$x_k = b \int_0^{y_k} \varphi(z) dz + y_k.$$

Note that $y_1 < y_2$, and that for $\varepsilon > 0$ sufficiently small (that is, for $n = b\varepsilon^{-1}$ sufficiently large), it is possible to choose y_5, y_6 such that $y_1 < y_5 < y_6 < y_2$, and

$$\frac{b \int_{y_5}^{y_6} \varphi(z) dz - 2\varepsilon}{y_2 - y_1 + b \int_{y_1}^{y_2} \varphi(z) dz} \geq \frac{b \int_{y_1}^{y_2} \varphi(z) dz}{y_2 - y_1 + b \int_{y_1}^{y_2} \varphi(z) dz} - (1 - d_1)/2. \quad (2.4)$$

Since $b - x_2 > 0$, we can find c so large that,

$$\frac{c(b - x_2)}{1 + c(b - x_2)} \geq 1 - (1 - d_1)/2. \quad (2.5)$$

Let $\lceil a \rceil$ denote the smallest integer greater than or equal to a . By the law of large numbers, if n is sufficiently large, the number of Z_0^k 's in the interval $[0, y_1]$ is smaller than or equal to $n \int_0^{y_5} \varphi(z) dz$, with probability greater than $1 - (1 - p_1)/2$. If this event holds then there are exactly $\lceil n \int_0^{y_5} \varphi(z) dz \rceil$ processes Z_0^k in some (random) interval $[0, y_7]$ with $y_7 \geq y_1$. This implies that there are exactly $\lceil n \int_0^{y_5} \varphi(z) dz \rceil$ processes X_0^k in $[0, \varepsilon \lceil n \int_0^{y_5} \varphi(z) dz \rceil + y_7]$. Note that

$$\varepsilon \left\lceil n \int_0^{y_5} \varphi(z) dz \right\rceil + y_7 \geq b \int_0^{y_5} \varphi(z) dz + y_7 \geq b \int_0^{y_1} \varphi(z) dz + y_1 = x_1.$$

Hence, the number of X_0^k 's in the interval $[0, x_1]$ is smaller than or equal to $n \int_0^{y_5} \varphi(z) dz + 1$, with probability greater than $1 - (1 - p_1)/2$. A completely analogous argument shows that, if n is sufficiently large, then the number of X_0^k 's in the interval $[x_2, \infty)$ is smaller than or equal to $n \int_{y_6}^{\infty} \varphi(z) dz + 1$, with probability greater than $1 - (1 - p_1)/2$. Both events hold with probability greater than $1 - 2(1 - p_1)/2 = p_1$, and then the number of X_0^k 's in $[x_1, x_2]$ is greater than or equal to $n \int_{y_5}^{y_6} \varphi(z) dz - 2$. This and (2.4) imply that

$$\begin{aligned} \mathbf{d}([x_1, x_2]) &\geq \frac{\varepsilon n \int_{y_5}^{y_6} \varphi(z) dz - 2\varepsilon}{x_2 - x_1} = \frac{b \int_{y_5}^{y_6} \varphi(z) dz - 2\varepsilon}{b \int_0^{y_2} \varphi(z) dz + y_2 - b \int_0^{y_1} \varphi(z) dz - y_1} \\ &= \frac{b \int_{y_5}^{y_6} \varphi(z) dz - 2\varepsilon}{y_2 - y_1 + b \int_{y_1}^{y_2} \varphi(z) dz} \geq \frac{b \int_{y_1}^{y_2} \varphi(z) dz}{y_2 - y_1 + b \int_{y_1}^{y_2} \varphi(z) dz} - (1 - d_1)/2. \end{aligned} \quad (2.6)$$

We have

$$x_2 = b \int_0^{y_2} \varphi(z) dz + y_2 = b \int_0^{y_2} ce^{-cz} dz + y_2 = y_2 + b - be^{-cy_2},$$

so $e^{-cy_2} = (y_2 - x_2 + b)/b$ and, therefore, for $z \leq y_2$,

$$\varphi(z) = ce^{-cz} \geq ce^{-cy_2} = (c/b)(y_2 - x_2 + b).$$

We combine this estimate with (2.6) and (2.5) to see that, with probability greater than p_1 ,

$$\begin{aligned} \mathbf{d}([x_1, x_2]) &\geq \frac{b \int_{y_1}^{y_2} \varphi(z) dz}{y_2 - y_1 + b \int_{y_1}^{y_2} \varphi(z) dz} - (1 - d_1)/2 \\ &\geq \frac{b \int_{y_1}^{y_2} (c/b)(y_2 - x_2 + b) dz}{y_2 - y_1 + b \int_{y_1}^{y_2} (c/b)(y_2 - x_2 + b) dz} - (1 - d_1)/2 \\ &= \frac{c(y_2 - y_1)(y_2 - x_2 + b)}{y_2 - y_1 + c(y_2 - y_1)(y_2 - x_2 + b)} - (1 - d_1)/2 \\ &= \frac{c(y_2 - x_2 + b)}{1 + c(y_2 - x_2 + b)} - (1 - d_1)/2 \\ &\geq \frac{c(b - x_2)}{1 + c(b - x_2)} - (1 - d_1)/2 \\ &\geq 1 - (1 - d_1)/2 - (1 - d_1)/2 = d_1. \end{aligned}$$

This completes the proof of (2.2). The proof of (2.3) is completely analogous. \square

3. Brownian gas under pressure

In this model, “particles” I^k are confined to the interval $[0, 1]$. More precisely, their centers are confined to this interval. The k -th leftmost “particle” is represented by an interval $I_t^k = (X_t^k - \varepsilon/2, X_t^k + \varepsilon/2)$. The intervals I^k and I^j are always disjoint. The processes X^k are driven by independent Brownian motions with the diffusion coefficient σ^2 . When two intervals I^k and I^j collide, they reflect instantaneously. The particles are added to the system at the left endpoint of $[0, 1]$ at a constant rate. In other words, they are pushed in at the speed a , so that a new particle enters the interval every ε/a units of time. As soon as X^k reaches 0, it starts moving as a Brownian motion reflected at 0. The k -th interval is removed from the system when X^k hits the right endpoint of $[0, 1]$.

Formally, we define $\{X^1, X^2, \dots\}$ to be a collection of right-continuous, $[0, \infty]$ -valued processes such that

$$X_0^k = -k\varepsilon + \varepsilon/2 \text{ for all } k, \quad (3.1)$$

$$\text{If } S_k = \inf\{t > 0 : X_{t-}^k = 1 - \varepsilon/2\},$$

$$\text{then } X_t^k \text{ is continuous on } [0, S_k) \text{ and } X_t^k = \infty \text{ for all } t > S_k, \quad (3.2)$$

$$X_t^k - X_t^{k+1} \geq \varepsilon \text{ for all } k \geq 1 \text{ and all } t \geq 0, \text{ and} \quad (3.3)$$

$$dX_t^k = \begin{cases} a dt & \text{if } t \in [0, k\varepsilon/a), \\ \sigma dB_t^k + dL_t^k - dM_t^k & \text{if } t \in [k\varepsilon/a, S_k), \end{cases} \quad (3.4)$$

where a and σ are positive constants, $\{B^1, B^2, \dots\}$ are iid Brownian motions, and L^k and M^k are nondecreasing processes such that

$$\int_{k\varepsilon/a}^{S_k} 1_{\{X_t^k - X_t^{k+1} > \varepsilon\}} dL_t^k = 0 \quad \text{and} \quad \int_{k\varepsilon/a}^{S_k} 1_{\{X_t^{k-1} - X_t^k > \varepsilon\}} dM_t^k = 0.$$

(Here, we may interpret $X^0 \equiv \infty$.)

To construct the solution to this Skorohod problem, consider first the processes $Y_t^k = X_t^k + k\varepsilon - \varepsilon/2 - at$. These processes satisfy

$$Y_0^k = 0 \text{ for all } k, \tag{3.5}$$

If $S_k = \inf\{t > 0 : Y_{t-}^k = 1 - at + (k-1)\varepsilon\}$,

$$\text{then } Y_t^k \text{ is continuous on } [0, S_k) \text{ and } Y_t^k = \infty \text{ for all } t > S_k, \tag{3.6}$$

$$Y_t^k - Y_t^{k+1} \geq 0 \text{ for all } k \geq 1 \text{ and all } t \geq 0, \text{ and} \tag{3.7}$$

$$dY_t^k = \begin{cases} 0 & \text{if } t \in [0, k\varepsilon/a), \\ \sigma dB_t^k - a dt + dL_t^k - dM_t^k & \text{if } t \in [k\varepsilon/a, S_k), \end{cases} \tag{3.8}$$

where

$$\int_{k\varepsilon/a}^{S_k} 1_{\{Y_t^k - Y_t^{k+1} > 0\}} dL_t^k = 0 \quad \text{and} \quad \int_{k\varepsilon/a}^{S_k} 1_{\{Y_t^{k-1} - Y_t^k > 0\}} dM_t^k = 0.$$

Again, we shall construct the processes $\{Y^1, Y^2, \dots\}$ using order statistics.

Let Z^k be a $[0, \infty)$ -valued process, satisfying the SDE $dZ_t^k = \sigma dB_t^k - a dt$, and reflected at 0. The process Z_t^k is defined on the time interval $t \in [k\varepsilon/a, \infty)$, and starts at $Z_{k\varepsilon/a}^k = 0$. At any time $t \in [k\varepsilon/a, (k+1)\varepsilon/a)$, only processes Z^j , $1 \leq j \leq k$, are defined. Let $\lfloor a \rfloor$ denote the greatest integer less than or equal to a , and

$$S_0 = 0,$$

$$\mathcal{A}_t^1 = \{j \in \mathbb{Z} : 1 \leq j \leq \lfloor ta/\varepsilon \rfloor\}, \quad t \geq 0,$$

$$S_1 = \inf\{t > 0 : \sup_{j \in \mathcal{A}_t^1} Z_t^j \geq 1 - at\},$$

$$\mathcal{A}_t^k = \mathcal{A}_t^{k-1} \setminus \{m \in \mathbb{Z} : Z_{S_{k-1}}^m = 1 - at + (k-2)\varepsilon\}, \quad t \geq S_{k-1}, k \geq 2,$$

$$S_k = \inf\{t > S_{k-1} : \sup_{j \in \mathcal{A}_t^k} Z_t^j \geq 1 - at + (k-1)\varepsilon\}, \quad k \geq 2.$$

Note that it is possible that $\mathcal{A}_t^k = \emptyset$ for some random k and $t > 0$.

Convention (C). For the sake of future reference, it is convenient to say that the process Z^m is killed at the time S_{k-1} , where $\{m\} = \mathcal{A}_{S_{k-1}}^{k-1} \setminus \mathcal{A}_{S_{k-1}}^k$. In other words, Z^m is killed when the corresponding interval I^k , defined below, hits the right endpoint of the interval $[0, 1]$.

For every $t \in [S_{k-1}, S_k)$, note that there are $\lfloor ta/\varepsilon \rfloor - (k-1)$ elements in \mathcal{A}_t^k . Let $Y_t^k, Y_t^{k+1}, \dots, Y_t^{\lfloor ta/\varepsilon \rfloor}$ be reverse-ordered Z_t^j 's, $j \in \mathcal{A}_t^k$, that is, $\{Y_t^k, \dots, Y_t^{\lfloor ta/\varepsilon \rfloor}\} = \{Z_t^j, j \in \mathcal{A}_t^k\}$ and $Y_t^k \geq Y_t^{k+1} \geq \dots \geq Y_t^{\lfloor ta/\varepsilon \rfloor}$. Let $Y_t^j = \infty$ for $j < k$ and $Y_t^j = 0$ for $j > \lfloor ta/\varepsilon \rfloor$.

It is elementary to check that $\{Y^1, Y^2, \dots\}$ satisfy (3.5)-(3.8). We may therefore define

$$\begin{aligned} X_t^k &= Y_t^k - k\varepsilon + \varepsilon/2 + at, \\ I_t^k &= (X_t^k - \varepsilon/2, X_t^k + \varepsilon/2). \end{aligned}$$

We have to modify slightly the definition (2.1) of density to match the current model. For $t \in [S_{k-1}, S_k)$, let

$$\mathbf{d}([x_1, x_2]) = \mathbf{d}_t([x_1, x_2]) = \frac{\lambda \left([x_1, x_2] \cap \bigcup_{k \leq j \leq \lfloor ta/\varepsilon \rfloor} I_t^j \right)}{x_2 - x_1}. \quad (3.9)$$

Theorem 3.1. *Fix arbitrary $0 < x_1 < x_2 < 1$, $p_1 < 1$ and $a, \sigma, c_0 > 0$. There exist $t_0 < \infty$ and $\varepsilon_0 > 0$ such that for $t \geq t_0$ and $\varepsilon \in (0, \varepsilon_0)$,*

$$P \left(\frac{1 - x_2}{1 - x_2 + \sigma^2/(2a)} - c_0 \leq \mathbf{d}_t([x_1, x_2]) \leq \frac{1 - x_1}{1 - x_1 + \sigma^2/(2a)} + c_0 \right) \geq p_1. \quad (3.10)$$

Intuitively speaking, the theorem says that the mass density at $x \in (0, 1)$ is close to $(1 - x)/(1 - x + \sigma^2/(2a))$, for large t and small ε .

Proof of Theorem 3.1: We will use the coupling technique. Recall processes Z^1, Z^2, \dots used in the definition of Y^k 's—we will use the same Z^k 's to construct auxiliary processes. Fix some $v_1 > 0$, let $\hat{S}_k = \inf\{t \geq 0 : Z_t^k = v_1\}$, and let \hat{Z}_t^k be the process Z^k killed at the time \hat{S}_k . Let n_t be the number of processes \hat{Z}^k alive at time t . Let $\hat{Y}_t^1, \hat{Y}_t^2, \dots, \hat{Y}_t^{n_t}$ be ordered \hat{Z}_t^j 's, that is, $\{\hat{Y}_t^1, \dots, \hat{Y}_t^{n_t}\} = \{\hat{Z}_t^j, \hat{S}_j > t\}$ and $\hat{Y}_t^1 \leq \hat{Y}_t^2 \leq \dots \leq \hat{Y}_t^{n_t}$. For $t \in [\hat{S}_{k-1}, \hat{S}_k)$ and $j = k, \dots, k + n_t - 1$, we let

$$\begin{aligned} \hat{X}_t^j &= \hat{Y}_t^{n_t+k-j} + (n_t + k - j - 1)\varepsilon + \varepsilon/2 + (t - \lfloor ta/\varepsilon \rfloor)\varepsilon/a, \\ \hat{I}_t^j &= (\hat{X}_t^j - \varepsilon/2, \hat{X}_t^j + \varepsilon/2). \end{aligned}$$

Every process \hat{X}_t^j is defined on the interval $[j\varepsilon/a, \hat{S}_j)$ and it is continuous on this interval. Although it may not be apparent from the above formulas, the processes \hat{Y}^j, \hat{X}^j and \hat{I}^j are constructed from \hat{Z}^j 's in the same way as Y^j, X^j and I^j were constructed from Z^j 's. We leave the verification of this claim to the reader.

We will find the Green function $G_{v_1}(v)$ of \hat{Z}^k , i.e., the density of its occupation measure. Consider a process V with values in $[-v_1, v_1]$, satisfying the SDE $dV_t = dB_t - a \operatorname{sign}(V_t)dt$, where B is Brownian motion, $V_0 = 0$, and such that V is killed when it hits $-v_1$ or v_1 . Note that the Green function $G_{v_1}^V(v)$ of V is one half of $G_{v_1}(v)$ for $v > 0$. The scale function $S(v)$ and the speed measure $m(v)$ for V can be calculated as follows (see Karlin and Taylor, 1981, pp. 194-195),

$$\begin{aligned} s(v) &= \exp \left(\int_0^v -(-2a \operatorname{sign}(x)/\sigma^2)dx \right) = \exp(2av \operatorname{sign}(v)/\sigma^2), \\ S(v) &= \int_0^v s(x)dx = \frac{\operatorname{sign}(v)\sigma^2}{2a} (\exp(2av \operatorname{sign}(v)/\sigma^2) - 1), \\ m(v) &= 1/(\sigma^2 s(v)) = (1/\sigma^2) \exp(-2av \operatorname{sign}(v)/\sigma^2). \end{aligned}$$

We will use formula (3.11) on page 197 of Karlin and Taylor (1981). In that formula, we take $x = 0$, so $u(0) = 1/2$, by symmetry. We apply the formula to functions $g(v)$

of the form $g(v) = \mathbf{1}_{[v_3, v_4]}(v)$, to conclude that for $v \in (0, v_1)$, the Green function $G_{v_1}^V(v)$ is given by

$$\begin{aligned} G_{v_1}^V(v) &= (S(v_1) - S(v))m(v) \\ &= \frac{1}{2a}(\exp(2av_1/\sigma^2) - \exp(2av/\sigma^2)) \exp(-2av/\sigma^2) \\ &= \frac{1}{2a}(\exp(2a(v_1 - v)/\sigma^2) - 1). \end{aligned}$$

It follows that

$$G_{v_1}(v) = 2G_{v_1}^V(v) = \frac{1}{a}(\exp(2a(v_1 - v)/\sigma^2) - 1).$$

Define $v_0 \in (0, \infty)$ by setting

$$\varphi(v) = aG_{v_0}(v) = \frac{1}{a}(\exp(2a(v_0 - v)/\sigma^2) - 1),$$

and the following condition,

$$\begin{aligned} 1 &= \int_0^{v_0} \varphi(v)dv + v_0 = \int_0^{v_0} (\exp(2a(v_0 - v)/\sigma^2) - 1)dv + v_0 \\ &= (\sigma^2/2a)(\exp(2av_0/\sigma^2) - 1). \end{aligned} \quad (3.11)$$

We define $y_k \in (0, \infty)$, $k = 1, 2$, by the following formula,

$$\begin{aligned} x_k &= \int_0^{y_k} \varphi(v)dv + y_k \\ &= \int_0^{y_k} (\exp(2a(v_0 - v)/\sigma^2) - 1)dv + y_k \\ &= (-(\sigma^2/2a) \exp(2a(v_0 - v)/\sigma^2) - v) \Big|_{v=0}^{v=y_k} + y_k \\ &= (\sigma^2/2a)(\exp(2av_0/\sigma^2) - \exp(2a(v_0 - y_k)/\sigma^2)). \end{aligned} \quad (3.12)$$

Choose $y_1 < y_3 < y_4 < y_2$ and $v_1 < v_0$ such that,

$$\frac{a \int_{y_3}^{y_4} G_{v_1}(v)dv}{y_2 - y_1 + \int_{y_1}^{y_2} \varphi(z)dz} \geq \frac{\int_{y_1}^{y_2} \varphi(z)dz}{y_2 - y_1 + \int_{y_1}^{y_2} \varphi(z)dz} - c_0. \quad (3.13)$$

Recall that $\lceil a \rceil$ denotes the smallest integer greater than or equal to a . Let $\lfloor a \rfloor$ denote the largest integer smaller than or equal to a .

Let $c_1 = 1 - p_1$ and $p_2 = 1 - c_1/8$. By the law of large numbers, we can find a large t_0 and make $\varepsilon_0 > 0$ smaller, if necessary, such that if $t \geq t_0$ and $\varepsilon \in (0, \varepsilon_0)$ then with probability greater than p_2 , the number of processes \widehat{Z}_t^k in the interval $[0, y_1]$ is smaller than or equal to $(a/\varepsilon) \int_0^{y_3} G_{v_1}(v)dv$. If this event holds then there are exactly $\lceil (a/\varepsilon) \int_0^{y_3} G_{v_1}(v)dv \rceil$ processes \widehat{Z}_0^k in some (random) interval $[0, y_5]$ with $y_5 \geq y_1$. This implies that there are exactly $\lceil (a/\varepsilon) \int_0^{y_3} G_{v_1}(v)dv \rceil$ processes \widehat{X}_0^k in $[0, \varepsilon \lceil (a/\varepsilon) \int_0^{y_3} G_{v_1}(v)dv \rceil + y_5]$. For fixed y_1 and y_3 , we make $v_1 < v_0$ larger, if necessary, so that

$$\begin{aligned} \varepsilon \left[(a/\varepsilon) \int_0^{y_3} G_{v_1}(v)dv \right] + y_5 &\geq a \int_0^{y_3} G_{v_1}(v)dv + y_5 \geq a \int_0^{y_3} G_{v_1}(v)dv + y_1 \\ &\geq a \int_0^{y_1} G_{v_0}(v)dv + y_1 = \int_0^{y_1} \varphi(v)dv + y_1 = x_1. \end{aligned}$$

Hence, the number of \widehat{X}_0^k 's in the interval $[0, x_1]$ is smaller than or equal to $(a/\varepsilon) \int_0^{y_3} G_{v_1}(v)dv$, with probability greater than p_2 .

We can make t_0 larger and $\varepsilon_0 > 0$ smaller, if necessary, so that by the law of large numbers, if $t \geq t_0$ and $\varepsilon \in (0, \varepsilon_0)$ then with probability greater than p_2 , the number of processes $\widehat{Z}_{t_3}^k$ in the interval $[0, y_2]$ is greater than or equal to $(a/\varepsilon) \int_0^{y_4} G_{v_1}(v)dv$. If this event holds then there are exactly $\lfloor (a/\varepsilon) \int_0^{y_4} G_{v_1}(v)dv \rfloor$ processes \widehat{Z}_0^k in some (random) interval $[0, y_6]$ with $y_6 \leq y_2$. This implies that there are exactly $\lfloor (a/\varepsilon) \int_0^{y_4} G_{v_1}(v)dv \rfloor$ processes \widehat{X}_0^k in $[0, \varepsilon \lfloor (a/\varepsilon) \int_0^{y_4} G_{v_1}(v)dv \rfloor + y_6]$. Note that,

$$\varepsilon \left[(a/\varepsilon) \int_0^{y_4} G_{v_1}(v)dv \right] + y_6 \leq a \int_0^{y_4} G_{v_1}(v)dv + y_2 \leq a \int_0^{y_2} G_{v_0}(v)dv + y_2 = x_2.$$

Hence, the number of \widehat{X}_0^k 's in the interval $[0, x_2]$ is greater than or equal to $(a/\varepsilon) \int_0^{y_4} G_{v_1}(v)dv$, with probability greater than p_2 .

Let $\widehat{\mathbf{d}}$ be defined as in (3.9) but relative to \widehat{I}^k in place of I^k . The two events described in the last two paragraphs hold simultaneously with probability greater than $1 - c_1/4$. Then the number of X_0^k 's in $[x_1, x_2]$ is greater than or equal to $(a/\varepsilon) \int_{y_3}^{y_4} G_{v_1}(v)dv$. This and (3.13) imply that

$$\begin{aligned} \widehat{\mathbf{d}}_t([x_1, x_2]) &\geq \frac{\varepsilon \left((a/\varepsilon) \int_{y_5}^{y_6} G_{v_1}(v)dv \right)}{x_2 - x_1} = \frac{a \int_{y_5}^{y_6} G_{v_1}(v)dv}{\int_0^{y_2} \varphi(z)dz + y_2 - \int_0^{y_1} \varphi(z)dz - y_1} \\ &= \frac{a \int_{y_5}^{y_6} G_{v_1}(v)dv}{y_2 - y_1 + \int_{y_1}^{y_2} \varphi(z)dz} \geq \frac{\int_{y_1}^{y_2} \varphi(z)dz}{y_2 - y_1 + \int_{y_1}^{y_2} \varphi(z)dz} - c_0. \end{aligned} \quad (3.14)$$

It follows from (3.12) that

$$\exp(2a(v_0 - y_2)/\sigma^2) = \exp(2av_0/\sigma^2) - 2ax_2/\sigma^2$$

and, therefore, for $v \leq y_2$,

$$\begin{aligned} \varphi(v) &= \exp(2a(v_0 - v)/\sigma^2) - 1 \geq \exp(2a(v_0 - y_2)/\sigma^2) - 1 \\ &= \exp(2av_0/\sigma^2) - 2ax_2/\sigma^2 - 1. \end{aligned}$$

We combine this estimate with (3.14) to see that, with probability greater than $1 - c_1/4$,

$$\begin{aligned} \widehat{\mathbf{d}}_t([x_1, x_2]) &\geq \frac{\int_{y_1}^{y_2} \varphi(v)dv}{y_2 - y_1 + \int_{y_1}^{y_2} \varphi(v)dv} - c_0 \\ &\geq \frac{\int_{y_1}^{y_2} (\exp(2av_0/\sigma^2) - 2ax_2/\sigma^2 - 1)dv}{y_2 - y_1 + \int_{y_1}^{y_2} (\exp(2av_0/\sigma^2) - 2ax_2/\sigma^2 - 1)dv} - c_0 \\ &= \frac{(y_2 - y_1)(\exp(2av_0/\sigma^2) - 2ax_2/\sigma^2 - 1)}{y_2 - y_1 + (y_2 - y_1)(\exp(2av_0/\sigma^2) - 2ax_2/\sigma^2 - 1)} - c_0 \\ &= \frac{\exp(2av_0/\sigma^2) - 2ax_2/\sigma^2 - 1}{\exp(2av_0/\sigma^2) - 2ax_2/\sigma^2} - c_0 \\ &= \frac{(\sigma^2/2a)(\exp(2av_0/\sigma^2) - 1) - x_2}{(\sigma^2/2a)(\exp(2av_0/\sigma^2) - 1) - x_2 + \sigma^2/2a} - c_0 \\ &= \frac{1 - x_2}{1 - x_2 + \sigma^2/2a} - c_0. \end{aligned} \quad (3.15)$$

The last equality follows from (3.11).

Recall that n_t is the number of processes \widehat{Z}^k alive at time t . Note that for any $0 \leq t_1 < t_2 < \infty$ with $t_2 - t_1 \geq \varepsilon/a$, we have,

$$n_{t_2} - n_{t_1} \leq (a/\varepsilon)(t_2 - t_1). \tag{3.16}$$

Fix arbitrary $t_1 \geq t_0$ and choose $\delta > 0$ such that

$$\int_0^{v_1} G_{v_1}(v)dv + \delta \leq \int_0^{v_0} G_{v_0}(v)dv. \tag{3.17}$$

We make $\varepsilon_0 > 0$ smaller, if necessary, so that, by the law of large numbers, if $\varepsilon \in (0, \varepsilon_0)$ then with probability greater than $1 - c_1/4$, for all s_k of the form $s_k = k\delta/2$, $k = 0, \dots, \lfloor t_1/\delta \rfloor + 1$, we have $n_{s_k} < (a/\varepsilon) (\int_0^{v_1} G_{v_1}(v)dv + \delta/2)$. It follows from (3.16) and (3.17) that for $\varepsilon < a\delta/2$,

$$\sup_{0 \leq t \leq t_1} n_t < (a/\varepsilon) \left(\int_0^{v_1} G_{v_1}(v)dv + \delta \right) \leq (a/\varepsilon) \int_0^{v_0} G_{v_0}(v)dv.$$

Suppose that this event holds. Then, for every $t \leq t_1$, the right edge of the rightmost interval \widehat{I}_t^k is to the left of

$$\varepsilon(a/\varepsilon) \int_0^{v_0} G_{v_0}(v)dv + v_1 = \int_0^{v_0} G_{v_0}(v)dv + v_0 + (v_1 - v_0) = 1 + (v_1 - v_0) < 1. \tag{3.18}$$

The second equality in the above formula follows from (3.11).

Recall the definitions given before the statement of the theorem. A process Z^k is killed when the right end of the rightmost interval I^k hits 1. Since the processes \widehat{Z}^k are driven by the same Brownian motions as Z^k , (3.18) implies that every Z^k has a longer lifetime than \widehat{Z}^k . This implies that $\mathbf{d}_{t_1}([x_1, x_2]) \geq \widehat{\mathbf{d}}_{t_1}([x_1, x_2])$. We combine this with (3.15) to conclude that, with probability greater than $1 - c_1/2$,

$$\mathbf{d}_{t_1}([x_1, x_2]) \geq \widehat{\mathbf{d}}_{t_1}([x_1, x_2]) \geq \frac{1 - x_2}{1 - x_2 + \sigma^2/2a} - c_0.$$

A completely analogous argument shows that, with probability greater than $1 - c_1/2$,

$$\mathbf{d}_{t_1}([x_1, x_2]) \leq \frac{1 - x_1}{1 - x_1 + \sigma^2/2a} + c_0.$$

This completes the proof of the theorem. □

4. Discussion

Remark 4.1. In the following remarks we will refer to the model analyzed in Section 3 as model (C). We will present another model, which we will call (R). Here, C represents the “constant” rate of influx of new particles, and R stands for the “random” rate of influx. Model (R) consists of a constant number n of “particles” I^k which are confined to the interval $[0, 1]$. The k -th leftmost “particle” is represented by an interval $I_t^k = (X_t^k - \varepsilon/2, X_t^k + \varepsilon/2)$. The intervals I^k and I^j are always disjoint. The processes X^k are driven by independent Brownian motions with the diffusion coefficient σ^2 . When two intervals I^k and I^j collide, they reflect instantaneously. The number of particles n is such that $n\varepsilon = b$, a constant. When X^k hits 1, it jumps to 0. We conjecture that as $\varepsilon \rightarrow 0$, the mass density \mathbf{d} in the

stationary regime for this process has the density $(1-x)/(1-x+\sigma^2/(2a))$, just like in model (C), where a, σ and b are related by the following formula,

$$\int_0^1 \frac{1-x}{1-x+\sigma^2/(2a)} = b.$$

Heuristically, we expect processes X^k in model (R) to jump at a more or less constant rate in the stationary regime, so this is why we believe that models (R) and (C) have the same hydrodynamic limit. We chose not to analyze model (R) in this paper as it appears to be harder from the technical point of view while it seems to illustrate the same phenomenon as model (C).

Remark 4.2. Model (R) is closely related to a model studied by T. Bodineau, B. Derrida and J. Lebowitz ([Bodineau et al., 2010](#)). In their model, one considers a periodic system of L sites with N particles. The particles perform random walks but cannot cross each other—it is the symmetric simple exclusion process. At some fixed edge, the jump rates are no longer symmetric but jumps occur with rates p in one direction and $1-p$ in the other direction. The case $p = 1$ corresponds to model (R) described in the previous remark. In the stationary state, the rescaled density varies linearly on the unit line segment, with a discontinuity located where the jump rates are biased. Hence, away from the singularity, the stationary empirical distribution is harmonic for the generator of the single particle process, i.e., Laplacian. This does not apply to the density of mass \mathbf{d} in our models (C) and (R).

Remark 4.3. Since the density \mathbf{d} of the intervals I^k has the form $(1-x)/(1-x+\sigma^2/(2a))$, it is elementary to check that the typical gap size between I^k 's is $\varepsilon\sigma^2/(2a(1-x))$. In a model with infinitely small particles X^k , the gap size is also $c/(1-x)$ but we do not have any heuristic explanation why the two functions representing the typical gap size should have the same form in both models.

Remark 4.4. We conjecture that the motion of an individual tagged particle I^k in model (C) converges, as $\varepsilon \rightarrow 0$, to a deterministic motion with fluctuations having the “fractional Brownian motion” structure. In other words, we conjecture that the fluctuations are Gaussian with the local scaling of space and time given by $\Delta x = (\Delta t)^{1/4}$. Our conjecture is inspired by the results in [Dürr et al. \(1985\)](#); [Harris \(1965\)](#); [Swanson \(2007, 2008\)](#) on families of one dimensional Brownian motions reflecting from each other.

Remark 4.5. A d -dimensional counterpart of model (C) can be represented as follows. Let I^k be balls with radius ε and center X^k . Our d -dimensional model consists of a constant number n of I^k 's which are confined to the cube $[0, 1]^d$. The balls I^k and I^j are always disjoint. The processes X^k are driven by independent d -dimensional Brownian motions with the diffusion coefficient σ^2 . When two balls I^k and I^j collide, they reflect instantaneously. Let S_ℓ and S_r be two opposite $(d-1)$ -dimensional sides on the boundary of $[0, 1]^d$. Balls I^k are pushed into the cube through S_ℓ at a constant rate a , i.e., $\varepsilon^{-(d-1)}$ balls are pushed into the cube every ε/a units of time, uniformly over S_ℓ . Once inside the cube, the balls reflect from all sides except S_r . When a ball hits S_r , it is removed from the cube. We conjecture that in the stationary regime, when ε is small, the density of the mass analogous to \mathbf{d} will be a function of the distance x from S_ℓ , i.e., a function of depending only on one coordinate. We do not see any obvious reason why the density should have the form $(1-x)/(1-x+\sigma^2/(2a))$. In relation to Remark 4.4, we conjecture that

the motion of a tagged particle in the present model is diffusive, with the diffusion coefficient depending on x . If this is true, it means that the “pressure” applied to particles in one direction can have a dampening effect on the size of oscillations of an individual particle in orthogonal directions.

References

- J. Baik, P. Deift and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* **12** (4), 1119–1178 (1999). [MR1682248](#).
- T. Bodineau, B. Derrida and J. Lebowitz. Forthcoming paper (2010).
- E. Cépa and D. Lépingle. Diffusing particles with electrostatic repulsion. *Probab. Theory Related Fields* **107** (4), 429–449 (1997). [MR1440140](#).
- E. Cépa and D. Lépingle. Brownian particles with electrostatic repulsion on the circle: Dyson’s model for unitary random matrices revisited. *ESAIM Probab. Statist.* **5**, 203–224 (electronic) (2001). [MR1875671](#).
- R. L. Dobrushin and J. Fritz. Non-equilibrium dynamics of one-dimensional infinite particle systems with a hard-core interaction. *Comm. Math. Phys.* **55** (3), 275–292 (1977). [MR0468978](#).
- D. Dürr, S. Goldstein and J. L. Lebowitz. Asymptotics of particle trajectories in infinite one-dimensional systems with collisions. *Comm. Pure Appl. Math.* **38** (5), 573–597 (1985). [MR803248](#).
- F. J. Dyson. A Brownian-motion model for the eigenvalues of a random matrix. *J. Mathematical Phys.* **3**, 1191–1198 (1962). [MR0148397](#).
- J. Fritz and R. L. Dobrushin. Non-equilibrium dynamics of two-dimensional infinite particle systems with a singular interaction. *Comm. Math. Phys.* **57** (1), 67–81 (1977). [MR0469025](#).
- T. E. Harris. Diffusion with “collisions” between particles. *J. Appl. Probability* **2**, 323–338 (1965). [MR0184277](#).
- K. Johansson. Shape fluctuations and random matrices. *Comm. Math. Phys.* **209** (2), 437–476 (2000). [MR1737991](#).
- S. Karlin and H. M. Taylor. *A second course in stochastic processes*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York (1981). ISBN 0-12-398650-8. [MR611513](#).
- R. Lang. Unendlich-dimensionale Wienerprozesse mit Wechselwirkung. I. Existenz. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **38** (1), 55–72 (1977a). [MR0431435](#).
- R. Lang. Unendlich-dimensionale Wienerprozesse mit Wechselwirkung. II. Die reversiblen Masse sind kanonische Gibbs-Masse. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **39** (4), 277–299 (1977b). [MR0455161](#).
- H. Osada. Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions. *Comm. Math. Phys.* **176** (1), 117–131 (1996). [MR1372820](#).
- H. Osada. Positivity of the self-diffusion matrix of interacting Brownian particles with hard core. *Probab. Theory Related Fields* **112** (1), 53–90 (1998). [MR1646432](#).
- T. Shiga. A remark on infinite-dimensional Wiener processes with interactions. *Z. Wahrsch. Verw. Gebiete* **47** (3), 299–304 (1979). [MR525311](#).
- J. Swanson. Weak convergence of the scaled median of independent Brownian motions. *Probab. Theory Related Fields* **138** (1-2), 269–304 (2007). [MR2288071](#).

- J. Swanson. Fluctuations of the empirical quantiles of independent Brownian motions. *ArXiv Mathematics e-prints* (2008). [arXiv:math/0812.4102v2](https://arxiv.org/abs/math/0812.4102v2).
- H. Tanemura. A system of infinitely many mutually reflecting Brownian balls in \mathbf{R}^d . *Probab. Theory Related Fields* **104** (3), 399–426 (1996). [MR1376344](https://doi.org/10.1007/BF01206844).
- C. A. Tracy and H. Widom. Integral formulas for the asymmetric simple exclusion process. *Comm. Math. Phys.* **279** (3), 815–844 (2008). [MR2386729](https://doi.org/10.1007/s00220-008-0529-1).
- C. A. Tracy and H. Widom. Asymptotics in ASEP with step initial condition. *Comm. Math. Phys.* **290** (1), 129–154 (2009). [MR2520510](https://doi.org/10.1007/s00220-009-0883-1).