Transfinite induction

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1 Preliminaries

If X and Y are sets, their *Cartesian product* is the set $X \times Y = \{(x, y) : x \in X, y \in Y\}$. A *relation* from X to Y is any subset of $X \times Y$. If R is a relation from X to Y, then we often write xRy to mean $(x, y) \in R$. A relation on X is a relation from X to X.

A *partial ordering* on a nonempty set X is a relation on X such that

(i) if xRy and yRz, then xRz,

(ii) if xRy and yRx, then $x = y$, and

(iii) xRx for all x.

A *total ordering* (or *linear ordering*) is a partial ordering R such that

(iv) if $x, y \in X$, then xRy or yRx.

We will often denote partial orderings by the symbol $\leq$, rather than the symbol R, in which case $x < y$ shall mean that $x \leq y$ and $x \neq y$. Two partially ordered sets are *order isomorphic* if there exists a bijection $f : X \to Y$ such that $a \leq b$ if and only if $f(a) \leq f(b)$.

If X is a partially ordered set and $E \subset X$, then $x \in E$ is a *maximal element* of E if, for all $y \in E$, $x \leq y$ implies $y = x$. Similarly, $x \in E$ is a *minimal element* of E if, for all $y \in E$, $y \leq x$ implies $y = x$. Maximal and minimal elements may or may not exist, and they may or may not be unique. If E is totally ordered, then maximal and minimal elements are unique, when they exist.

An *upper bound* for E is an element $x \in X$ such that $y \leq x$ for all $y \in E$. Similarly, a *lower bound* for E is an element $x \in X$ such that $x \leq y$ for all $y \in E$. Upper and lower bounds need not exist, and they need not be elements of the set E.

As an example, let $X = 2^{\mathbb{R}}$, the set of all subsets of $\mathbb{R}$. Then X is partially ordered by the inclusion relation, $\subseteq$. Let

$$E = \{A \in X : A \subseteq (-\infty, 0) \text{ or } A \subseteq (0, \infty)\}.$$

Note that $x = (-\infty, 0) \in E$ is a maximal element of E. To prove this, we simply observe that if $y \in E$ and $x \subseteq y$, then $y = x$. However, x is not an upper bound for E. The only upper bounds for E are $\mathbb{R}$ and $(-\infty, 0) \cup (0, \infty)$, neither of which are in E. This example shows that, in general, a maximal element of E need not be an upper bound for E. However, if E is totally ordered, then maximal and minimal elements of E, when they exist, are upper and lower bounds for E, respectively.

If X is a totally ordered set, and every nonempty subset of X has a minimal element, then X is *well ordered*, and the ordering is called a *well ordering*. For example, the natural numbers, $\mathbb{N}$, with their natural ordering, are well ordered, but the integers are not.

Theorem 1.1. (Zorn's lemma) *If X is a partially ordered set, and every totally ordered subset of X has an upper bound, then X has a maximal element.*

Zorn's lemma is logically equivalent to the axiom of choice.

Theorem 1.2. (Well ordering principle) *Every nonempty set can be well ordered.*

Proof. Let X be a nonempty set, and let

$$\mathcal{W} = \{(E, \leq) : E \subset X \text{ and } \leq \text{ is a well ordering on } E\}.$$

Define a partial ordering on $\mathcal{W}$ by $(E_1, \leq_1) \lesssim (E_2, \leq_2)$ if

(i) $E_1 \subset E_2$,

(ii) $\leq_1$ and $\leq_2$ agree on E_1, and

(iii) if $x \in E_2 \cap E_1^c$, then $y \leq_2 x$ for all $y \in E_1$.

It is left as an exercise for the reader to show that every totally ordered subset of $\mathcal{W}$ has an upper bound. Hence, by Zorn's lemma, $\mathcal{W}$ has a maximal element, $(E, \leq)$. Assume $E \neq X$. Choose $x_0 \in E^c$. Define $E' = E \cup \{x_0\}$ and $\leq' = \leq \cup \{(x, x_0) : x \in E\}$. Then $(E', \leq') \in \mathcal{W}$ and $(E, \leq) \lesssim (E', \leq')$. But this contradicts the fact that $(E, \leq)$ is a maximal element of $\mathcal{W}$. Therefore, $E = X$, and $\leq$ is a well ordering on X. □

2 Transfinite induction

The principle of mathematical induction can be stated as follows: If $A \subset \mathbb{N}$ satisfies

(i) $1 \in A$, and

(ii) if $\{1, \ldots, n-1\} \subset A$, then $n \in A$,

then $A = \mathbb{N}$. Note that (i) can be rephrased as

(i)$'$ if $\emptyset \subset A$, then $1 \in A$.

This allows us to combine both conditions into a single condition and state the following.

Theorem 2.1. (Principle of mathematical induction) *Let $A \subset \mathbb{N}$. If $I_n \subset A$ implies $n \in A$, where $I_n = \{j \in \mathbb{N} : j < n\}$, then $A = \mathbb{N}$.*

We now extend this principle so that it applies not only to the set $\mathbb{N}$, but to any well ordered set X. Let X be a well ordered set. If $A \subset X$ is nonempty, then A has a minimal element, which we call the *infimum* of A, and denote by $\inf A$. If A has an upper bound, then

$$B = \{x \in X : x \text{ is an upper bound for } A\}$$

is nonempty, and we define the *supremum* of A by $\sup A = \inf B$. For $x \in X$, we define the *initial segment* of x by

$$I_x = \{y \in X : y < x\}.$$

The elements of I_x are called the *predecessors* of x.

Theorem 2.2. (Principle of transfinite induction) *Let X be a well ordered set. Let $A \subset X$. If $I_x \subset A$ implies $x \in A$, then $A = X$.*

Proof. Suppose $A \neq X$ and let $x = \inf(A^c)$. Then $I_x \subset A$, but $x \notin A$. □

Theorem 2.3. (Set of countable ordinals) *There is an uncountable well ordered set Ω, called the* set of countable ordinals*, such that I_x is countable for all $x \in \Omega$. If Ω' is another set with the same properties, then Ω and Ω' are order isomorphic. Every countable subset of Ω has an upper bound.*

Proof. Let X be an uncountable well ordered set, whose existence is guaranteed by the well ordering principle. Let $A = \{x \in X : I_x \text{ is uncountable}\}$. If A is empty, then let $\Omega = X$. If A is nonempty, then let $\Omega = I_{x_0}$, where $x_0 = \inf A$. It follows, then, that I_x is countable for all $x \in \Omega$.

Suppose Ω' is another set with this property. Let $\mathcal{I} = \{I_x : x \in \Omega\} \cup \{\Omega\}$ and $\mathcal{I}' = \{I_x : x \in \Omega'\} \cup \{\Omega'\}$. Let $\mathcal{F}$ be the collection of order isomorphisms $f : X \to Y$, where $X \in \mathcal{I}$ and $Y \in \mathcal{I}'$. The set $\mathcal{F}$ is nonempty, since the unique map $f : \{\inf \Omega\} \to \{\inf \Omega'\}$ belongs to $\mathcal{F}$. Also, the set $\mathcal{F}$ is partially ordered by inclusion, where we regard each $f \in \mathcal{F}$ as a subset of $\Omega \times \Omega'$. It is left as an exercise for the reader to verify that the hypotheses of Zorn's lemma are satisfied. Hence, $\mathcal{F}$ has a maximal element, $f : A \to B$.

Suppose $A = I_x$ for some $x \in \Omega$. Then A is countable and, since f is a bijection, B is also countable. But Ω' is uncountable. Hence, $B = I_y$ for some $y \in \Omega'$. This means, however, that f can be extended to an order isomorphism $f : A \cup \{x\} \to B \cup \{y\}$ by setting $f(x) = y$, and this contradicts the maximality of f. Therefore, $A = \Omega$. Since Ω is uncountable and I_y is countable for all $y \in \Omega'$, it follows that $B = \Omega'$, and f is an order isomorphism from Ω to Ω'.

Finally, suppose $A \subset \Omega$ is countable. Let $J = \bigcup_{x \in A} I_x$. Since J is countable, $J \neq \Omega$. Choose any $b \in J^c$. Let $x \in A$ be arbitrary. Since $b \in J^c$, we have $b \notin I_x$. Hence, $b \not< x$. Since Ω is totally ordered, this means $x \leq b$, so b is an upper bound for A. □

Remark 2.4. **(Regarding Ω as an extension of $\mathbb{N}$.)** Let $A = \{x \in \Omega : I_x \text{ is infinite}\}$ and let $\omega = \inf A$. Then $\mathbb{N}$ and I_ω are order isomorphic, which allows us to identify $\mathbb{N}$ with the subset $I_\omega \subset \Omega$. An explicit order isomorphism can be constructed as follows: let $f(1) = \inf \Omega$ and, for $n > 1$, let $f(n) = \inf(\{f(1), \ldots, f(n-1)\}^c)$.

Remark 2.5. **(Successors)** Let $x \in \Omega$. Then $\{y \in X : y \leq x\} = I_x \cup \{x\}$ is countable. Hence, $S_x = \{y \in X : x < y\} = (I_x \cup \{x\})^c$ is nonempty. The *successor* of x is defined as $x + 1 = \inf S_x$. We say that x is the *immediate predecessor* of $x + 1$.

Theorem 2.6. *If $\alpha, \beta \in \Omega$, then α is the successor of β if and only if β is the maximal element of I_α. In particular, if α is the successor of β, then $x < \alpha$ if and only if $x \le \beta$, so that $I_\alpha = I_\beta \cup \{\beta\}$.*

If $\omega = \inf A$, where $A = \{x \in \Omega : I_x \text{ is infinite}\}$, then ω does not have an immediate predecessor.

Proof. Suppose α is the successor of β, so that $\alpha = \inf S_\beta$. In particular, $\alpha \in S_\beta$, so that $\beta < \alpha$, which implies $\beta \in I_\alpha$. Suppose $x \in I_\alpha$ and $\beta \le x$. If $\beta \ne x$, then $x \in S_\beta$. Since $\alpha = \inf S_\beta$, this implies $\alpha \le x$. But this contradicts $x \in I_\alpha$. Hence, $\beta = x$, so β is the maximal element of I_α.

Conversely, suppose β is the maximal element of I_α. Then $\beta < \alpha$, which implies $\alpha \in S_\beta$. Suppose $x \in S_\beta$ and $x \le \alpha$. Suppose $x \ne \alpha$. Then $x \in I_\alpha$, and since β is the maximal element of I_α, we must have $x \le \beta$. Therefore, $x \in I_\beta \cup \{\beta\} = S_\beta^c$, a contradiction. Hence, $x = \alpha$, so $\alpha = \inf S_\beta$, which means α is the successor of β.

Suppose α is the successor of β. If $x \le \beta$, then since $\beta < \alpha$, we have $x < \alpha$. Conversely, if $x < \alpha$, then $x \in I_\alpha$, so $x \le \beta$, since β is the maximal element of I_α.

Finally, suppose ω has an immediate predecessor, β. Since $\beta < \omega$, it follows that I_β is finite. But this implies $I_\omega = I_\beta \cup \{\beta\}$ is finite, which contradicts $\omega \in A$. □

3 An example

Let X be a set and 2^X the collection of all subsets of X. A nonempty collection $\mathcal{F} \subset 2^X$ is called a σ-algebra if

(i) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, and

(ii) if $\{A_j\}_{j=1}^\infty \subset \mathcal{F}$, then $\bigcup_{j=1}^\infty A_j \in \mathcal{F}$.

If $\mathcal{E} \subset 2^X$, then the σ-algebra generated by $\mathcal{E}$, denoted by $\sigma(\mathcal{E})$, is the smallest σ-algebra on X that contains $\mathcal{E}$. More precisely,

$$\sigma(\mathcal{E}) = \bigcap_{\mathcal{F} \in \Gamma} \mathcal{F},$$

where $\Gamma = \{\mathcal{F} \subset 2^X : \mathcal{F} \text{ is a } \sigma\text{-algebra and } \mathcal{E} \subset \mathcal{F}\}$. That is, $A \in \sigma(\mathcal{E})$ if and only if $A \in \mathcal{F}$ for all $\mathcal{F} \in \Gamma$. It is left as an exercise for the reader to verify that

(i) $\sigma(\mathcal{E})$ is a σ-algebra,

(ii) $\mathcal{E} \subset \sigma(\mathcal{E})$, and

(iii) if $\mathcal{F}$ is any σ-algebra on X such that $\mathcal{E} \subset \mathcal{F}$, then $\sigma(\mathcal{E}) \subset \mathcal{F}$.

For example, the Borel σ-algebra on $\mathbb{R}$, denoted by $\mathcal{B}(\mathbb{R})$, is the σ-algebra generated by the open sets. That is, $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{E})$, where $\mathcal{E} = \{U \subset \mathbb{R} : U \text{ is open}\}$.

The above definition of $\sigma(\mathcal{E})$ may not be very intuitive. What does a typical set in $\sigma(\mathcal{E})$ look like? Informally, we construct $\sigma(\mathcal{E})$ by starting with $\mathcal{E}$ and then throwing in whatever else we need so that $\sigma(\mathcal{E})$ is closed under complements and countable unions. We could start by defining $\mathcal{E}_1 = \{A \subset X : A \in \mathcal{E} \text{ or } A^c \in \mathcal{E}\}$. Of course, this may not be closed under countable unions, so we define $\mathcal{D}_2 = \{\bigcup_{j=1}^\infty A_j : \{A_j\}_{j=1}^\infty \subset \mathcal{E}_1\}$. Unfortunately, this new set may no longer be closed under complements. So we repeat, defining $\mathcal{E}_2 = \{A \subset X : A \in \mathcal{D}_2 \text{ or } A^c \in \mathcal{D}_2\}$. But now this new set may no longer be closed under countable unions. We could go on like this, defining for $n > 1$,

$$\mathcal{D}_n = \left\{ \bigcup_{j=1}^\infty A_j : \{A_j\}_{j=1}^\infty \subset \mathcal{E}_{n-1} \right\},$$
$$\mathcal{E}_n = \{A \subset X : A \in \mathcal{D}_n \text{ or } A^c \in \mathcal{D}_n\}.$$

We now have a nested sequence of families of sets, $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots$, none of which is equal to $\sigma(\mathcal{E})$. We might try taking the union, $\bigcup_{j=1}^\infty \mathcal{E}_j$, but this is not in general equal to $\sigma(\mathcal{E})$ either.

In order to make a scheme like this work, we need transfinite induction. Let Ω be the set of countable ordinals. Let $\alpha \in \Omega$. If $\alpha = 1$, then $\mathcal{E}_\alpha$ is defined as above. If α has an immediate predecessor, β, then define

$$\mathcal{D}_\alpha = \left\{ \bigcup_{j=1}^\infty A_j : \{A_j\}_{j=1}^\infty \subset \mathcal{E}_\beta \right\},$$
$$\mathcal{E}_\alpha = \{A \subset X : A \in \mathcal{D}_\alpha \text{ or } A^c \in \mathcal{D}_\alpha\}.$$

Otherwise, define $\mathcal{E}_\alpha = \bigcup_{\beta<\alpha} \mathcal{E}_\beta$. Does this actually define $\mathcal{E}_\alpha$ for every $\alpha \in \Omega$? To see that it does, we use the principle of transfinite induction. Let $A \subset \Omega$ be the set of all α for which $\mathcal{E}_\alpha$ is defined. We see from the above that if $I_\alpha \subset A$, then $\alpha \in A$. Hence, $A = \Omega$.

Lemma 3.1. *If $\beta < \alpha$, then $\mathcal{E}_\beta \subset \mathcal{E}_\alpha$.*

Proof. Let $A = \{\alpha \in \Omega : \bigcup_{\beta<\alpha} \mathcal{E}_\beta \subset \mathcal{E}_\alpha\}$. Let $\alpha \in \Omega$ and assume $I_\alpha \subset A$. If $\alpha = 1$ or α has no immediate predecessor, then it is obvious from the definition of $\mathcal{E}_\alpha$ that $\alpha \in A$. Assume that α has an immediate predecessor, β. By the definition of $\mathcal{E}_\alpha$, we have $\mathcal{E}_\beta \subset \mathcal{E}_\alpha$. Let $x \in I_\alpha = I_\beta \cup \{\beta\}$ be arbitrary. If $x = \beta$, then as noted above, $\mathcal{E}_x \subset \mathcal{E}_\alpha$. Assume $x \in I_\beta$. Since $\beta \in I_\alpha \subset A$, we have $\mathcal{E}_x \subset \bigcup_{y<\beta} \mathcal{E}_y \subset \mathcal{E}_\beta \subset \mathcal{E}_\alpha$. We have thus shown that $\mathcal{E}_x \subset \mathcal{E}_\alpha$ for all $x \in I_\alpha$. This implies $\bigcup_{x<\alpha} \mathcal{E}_x \subset \mathcal{E}_\alpha$, and hence, $\alpha \in A$. By the principle of transfinite induction, $A = \Omega$. □

Theorem 3.2. $\sigma(\mathcal{E}) = \bigcup_{\alpha\in\Omega} \mathcal{E}_\alpha$.

Proof. Let $A = \{\alpha \in \Omega : \mathcal{E}_\alpha \subset \sigma(\mathcal{E})\}$. By the definition of $\mathcal{E}_\alpha$, if $I_\alpha \subset A$, then $\alpha \in A$. Hence, by transfinite induction, $A = \Omega$. Therefore, $\mathcal{E}_\alpha \subset \sigma(\mathcal{E})$ for all $\alpha \in \Omega$, which implies $\bigcup_{\alpha\in\Omega} \mathcal{E}_\alpha \subset \sigma(\mathcal{E})$.

Since $\mathcal{E} \subset \bigcup_{\alpha\in\Omega} \mathcal{E}_\alpha$, the reverse inclusion will follow once we show that $\bigcup_{\alpha\in\Omega} \mathcal{E}_\alpha$ is a σ-algebra. We first show that $\bigcup_{\alpha\in\Omega} \mathcal{E}_\alpha$ is closed under complements, for which it will suffice to show that $\mathcal{E}_\alpha$ is closed under complements for all $\alpha \in \Omega$. Let

$$A = \{\alpha \in \Omega : \mathcal{E}_\alpha \text{ is closed under complements}\}.$$

Suppose $I_\alpha \subset A$. If $\alpha = 1$ or α has an immediate predecessor, then it is clear from the definition of $\mathcal{E}_\alpha$ that $\alpha \in A$. Suppose $\alpha > 1$ does not have an immediate predecessor. Let $E \in \mathcal{E}_\alpha = \bigcup_{\beta<\alpha} \mathcal{E}_\beta$. Then $E \in \mathcal{E}_\beta$ for some $\beta \in I_\alpha$. Since $I_\alpha \subset A$, it follows that $\mathcal{E}_\beta$ is closed under complements. Thus, $E^c \in \mathcal{E}_\beta \subset \mathcal{E}_\alpha$, so $\mathcal{E}_\alpha$ is closed under complements, and $\alpha \in A$. We have shown that $I_\alpha \subset A$ implies $\alpha \in A$. By transfinite induction, $A = \Omega$.

Finally, we must show that $\bigcup_{\alpha\in\Omega} \mathcal{E}_\alpha$ is closed under countable unions. Let $\{E_j\}_{j=1}^\infty \subset \bigcup_{\alpha\in\Omega} \mathcal{E}_\alpha$, so that $E_j \in \mathcal{E}_{\alpha_j}$. By Theorem 2.3, $A = \{\alpha_j : j \in \mathbb{N}\}$ has an upper bound. Let $\beta = \sup A$ and let α be the successor of β. Since $\alpha_j \leq \beta$ implies $\mathcal{E}_{\alpha_j} \subset \mathcal{E}_\beta$, we have $\{E_j\}_{j=1}^\infty \subset \mathcal{E}_\beta$. Thus, by definition, $\bigcup_{j=1}^\infty E_j \in \mathcal{D}_\alpha \subset \mathcal{E}_\alpha \subset \bigcup_{\alpha\in\Omega} \mathcal{E}_\alpha$. □

References

[1] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications.* Wiley-Interscience, 1999.