Lemmas for the Skorohod Space

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The following lemmas were proved while working on a recent paper, but as of today, they do not appear in the final draft. Rather than having my hard work go to waste, I have decided to present them here in this small article. The context in which these lemmas occurred is described at the end of this article.

1 The Lemmas

A function is cadlag if it is right continuous and has left limits. If (E, r) is a metric space, then the Skorohod space, $D = D_E[0, \infty)$, is the space of cadlag functions from $[0, \infty)$ to E. A metric on D is given by

$$d(x, y) = \inf_{\lambda \in \Lambda} \left[\| \log \lambda' \|_\infty \vee \int_0^\infty e^{-u} \sup_{t \geq 0} \{ r(x(t \wedge u), y(\lambda(t) \wedge u)) \wedge 1 \} \, du \right], \tag{1}$$

where Λ is the collection of all strictly increasing, surjective, Lipschitz continuous functions $\lambda : [0, \infty) \to [0, \infty)$ such that $\| \log \lambda' \|_\infty < \infty$. If (E, r) is complete and separable, then (D, d) is complete and separable. This metric generates the Skorohod topology on D. See Chapter 3 of [1] for details.

Note that $D_E \times D_E$ is not the same space as $D_{E \times E}$. In particular, the map $(x, y) \to x + y$ is not continuous when viewed as a map from $D_{\mathbb{R}^d} \times D_{\mathbb{R}^d}$ to $D_{\mathbb{R}^d}$, but it is continuous as a map from $D_{\mathbb{R}^{2d}}$ to $D_{\mathbb{R}^d}$.

Lemma 1.1 *Suppose $x_n \to x$ in $D_{\mathbb{R}^d}[0, \infty)$ and $y_n \to y$ in $D_{\mathbb{R}^d}[0, \infty)$. If $\Delta x(t) \Delta y(t) = 0$ for all $t \geq 0$, then $x_n + y_n \to x + y$ in $D_{\mathbb{R}^d}[0, \infty)$.*

Proof. By Lemma 6.2 in [2], $v_n \to v$ in $D_{\mathbb{R}^d}[0, \infty)$ if and only if the following conditions hold.

(i) If $t_n \to t$, then $|v_n(t_n) - v(t)| \wedge |v_n(t_n) - v(t-)| \to 0$.

(ii) If $s_n \geq t_n$, $s_n, t_n \to t$, and $v_n(t_n) \to v(t)$, then $v_n(s_n) \to v(t)$.

Let $z_n = x_n + y_n$ and $z = x + y$. Suppose $t_n \to t$. Since $\Delta x(t) \Delta y(t) = 0$, either t is a continuity point of x or it is a continuity point of y. By symmetry, suppose it is a continuity point of x. In this case, choose strictly increasing, surjective $\lambda_n : [0,\infty) \to [0,\infty)$ such that $\lambda_n(t) \to t$ and $x_n(t) - x(\lambda_n(t)) \to 0$ uniformly on compacts. Then

$$|x_n(t_n) - x(t)| \le |x_n(t_n) - x(\lambda_n(t_n))| + |x(\lambda_n(t_n)) - x(t)|.$$

Since $\lambda_n(t_n) \to t$ and t is a continuity point of x, it follows that $x_n(t_n) \to x(t)$. Hence,

$$\begin{aligned}|z_n(t_n) - z(t)| \wedge |z_n(t_n) - z(t-)| &\le (|x_n(t_n) - x(t)| + |y_n(t_n) - y(t)|)\\ &\quad \wedge (|x_n(t_n) - x(t-)| + |y_n(t_n) - y(t-)|)\\ &= |x_n(t_n) - x(t)| + (|y_n(t_n) - y(t)| \wedge |y_n(t_n) - y(t-)|).\end{aligned}$$

Since (i) holds for $\{y_n\}$, this goes to zero, which verifies (i) for $\{z_n\}$.

Now suppose $s_n \ge t_n$, $s_n, t_n \to t$, and $z_n(t_n) \to z(t)$. Again, by symmetry, assume t is a continuity point of x. We then have that $y_n(t_n) = z_n(t_n) - x_n(t_n) \to z(t) - x(t) = y(t)$. Hence, by (ii), we must have $y_n(s_n) \to y(t)$. But this implies $z_n(s_n) = x_n(s_n) + y_n(s_n) \to x(t) + y(t) = z(t)$ and this verifies (ii) for $\{z_n\}$. □

Lemma 1.2 *If $2 \le d < \infty$, then $\{(X_n^1, \dots, X_n^d)\}$ is relatively compact in $D_{\mathbb{R}^d}[0,\infty)$ if and only if $\{X_n^k\}$ and $\{X_n^k + X_n^\ell\}$ are relatively compact in $D_{\mathbb{R}}[0,\infty)$.*

Proof. Problem 3.22(c) in [1]. □

Lemma 1.3 *For each n, let X_n and Y_n be independent random variables taking values in $D_{\mathbb{R}^k}[0,\infty)$ and $D_{\mathbb{R}^m}[0,\infty)$, respectively. Suppose that $(X_n, Y_n) \Rightarrow (X, Y)$ in $D_{\mathbb{R}^k}[0,\infty) \times D_{\mathbb{R}^m}[0,\infty)$. If*

$$P(\Delta X(t)\Delta Y(t) = 0 \text{ for all } t \ge 0) = 1,$$

then $(X_n, Y_n) \Rightarrow (X, Y)$ in $D_{\mathbb{R}^k \times \mathbb{R}^m}[0,\infty)$.

Proof. By the Skorohod Representation Theorem, we can assume that $X_n \to X$ and $Y_n \to Y$ a.s. By Lemma 1.1, $X_n + Y_n \to X + Y$ a.s. Hence, by Lemma 1.2, $\{(X_n, Y_n)\}$ is relatively compact in $D_{\mathbb{R}^{k+m}}[0,\infty)$. If (U, V) is a subsequential limit, then $U \overset{d}{=} X$, $V \overset{d}{=} Y$, and U and V are independent. Hence, $(U, V) \overset{d}{=} (X, Y)$, so $(X_n, Y_n) \Rightarrow (X, Y)$. □

Lemma 1.4 *Let (E, r) be a complete and separable metric space. Let X_n be a sequence of E-valued random variables and suppose, for each k, there exists a sequence $\{X_{n,k}\}_{n=1}^\infty$ such that*

$$\limsup_{n\to\infty} E[r(X_n, X_{n,k})] \le \delta_k,$$

where $\delta_k \to 0$ as $k \to \infty$. Suppose also that for each k, there exists Y_k such that $X_{n,k} \Rightarrow Y_k$ as $n \to \infty$. Then there exists X such that $X_n \Rightarrow X$ and $Y_k \Rightarrow X$.

Proof. Let $\mathcal{P}(E)$ be the family of all probability measures on E, endowed with the Prohorov metric,

$$\rho(P, Q) = \inf\{\varepsilon > 0 : P(F) \le Q(F^\varepsilon) + \varepsilon \text{ for all } F \in \mathcal{C}\},$$

where $\mathcal{C}$ is the collection of closed sets in E and $F^\varepsilon = \{x \in E : r(x, F) < \varepsilon\}$. Under this metric, $(\mathcal{P}(E), \rho)$ is complete and separable, and $Z_n \Rightarrow Z$ if and only if $\rho(PZ_n^{-1}, PZ^{-1}) \to 0$.

Let $\varepsilon > 0$ be given and choose k_0 such that $\delta_k < \varepsilon^2$ whenever $k \geq k_0$. For each fixed $k \geq k_0$, choose $N(k)$ and $M(k)$ such that $E[r(X_n, X_{n,k})] < \varepsilon^2$ whenever $n \geq N(k)$ and $\rho(PX_{n,k}^{-1}, PY_k^{-1}) < \varepsilon$ whenever $n \geq M(k)$. Let $n \geq N(k)$ be arbitrary. Then for all $F \in \mathcal{C}$,

$$P(X_n \in F) \leq P(X_n \in F, r(X_n, X_{n,k}) < \varepsilon) + P(r(X_n, X_{n,k}) \geq \varepsilon) \leq P(X_{n,k} \in F^\varepsilon) + \varepsilon.$$

It follows then that $\rho(PX_n^{-1}, PX_{n,k}^{-1}) \leq \varepsilon$ whenever $n \geq N(k)$.

Now let $n, m \geq N(k_0) \vee M(k_0)$. Then

$$\begin{aligned}\rho(PX_n^{-1}, PX_m^{-1}) &\leq \rho(PX_n^{-1}, PX_{n,k_0}^{-1}) + \rho(PX_{n,k_0}^{-1}, PY_{k_0}^{-1}) \\ &\qquad + \rho(PY_{k_0}^{-1}, PX_{m,k_0}^{-1}) + \rho(PX_{m,k_0}^{-1}, PX_m^{-1}) \\ &< 4\varepsilon.\end{aligned}$$

Hence, $\{PX_n^{-1}\}$ is Cauchy in $\mathcal{P}(E)$, so there exists X such that $X_n \Rightarrow X$.

Now let $k \geq k_0$ and choose $n \geq N(k) \vee M(k)$ such that $\rho(PX_n^{-1}, PX^{-1}) < \varepsilon$. Then

$$\rho(PY_k^{-1}, PX^{-1}) \leq \rho(PY_k^{-1}, PX_{n,k}^{-1}) + \rho(PX_{n,k}^{-1}, PX_n^{-1}) + \rho(PX_n^{-1}, PX^{-1}) < 3\varepsilon.$$

Hence, $Y_k \Rightarrow X$. □

Lemma 1.5 *Suppose $x, y \in D$ and $x(t) = y(t)$ for all $t < T$. Then $d(x, y) \leq e^{-T}$.*

Proof. Taking $\lambda(t) = t$ in (1) gives

$$d(x, y) \leq \int_0^\infty e^{-u} \sup_{t\in[0,u]} \{r(x(t), y(t)) \wedge 1\}\, du.$$

If $x(t) = y(t)$ for all $t < T$, then $d(x, y) \leq \int_T^\infty e^{-u}\, du = e^{-T}$. □

Lemma 1.6 *For $x \in D = D_{\mathbb{R}^d}[0, \infty)$ and $\varepsilon > 0$, let*

$$h_\varepsilon(x) = \inf\{t \geq 0 : |x(t)| \wedge |x(t-)| \leq \varepsilon\}. \tag{2}$$

If $(x_n, h_\varepsilon(x_n)) \to (x, T) \in D \times [0, \infty]$, then $h_\varepsilon(x) \leq T$.

Proof. Let $s < t < h_\varepsilon(x)$, so that $\inf_{[0,t]} |x(r)| > \varepsilon$. Since $x(\cdot) \mapsto \inf_{[0,\cdot]} |x(r)|$ is continuous in the Skorohod topology, for sufficiently large n, $\inf_{[0,s]} |x_n(r)| > \varepsilon$, which implies that $s \leq h_\varepsilon(x_n)$. Letting $n \to \infty$ gives $s \leq T$. Letting $s \uparrow h_\varepsilon(x)$ gives $h_\varepsilon(x) \leq T$. □

2 The Context

These lemmas were proved while working on a paper in which we applied the theorems in [2]. We did not need the full power of these theorems. Rather, we simply used the following "watered down" versions.

This first theorem is a special case of Theorem 2.2 in [2].

Theorem 2.1 *For each* n*, let* Y_n *be a cadlag,* $\mathbb{R}^m$*-valued semimartingale with respect to a filtration* $\{\mathcal{F}_t^n\}$*. Suppose that* $Y_n = M_n + A_n$*, where* M_n *is an* $\{\mathcal{F}_t^n\}$*-local martingale and* A_n *is a finite variation process, and that*

$$\sup_n E[[M_n]_t + V_t(A_n)] < \infty \tag{3}$$

for each $t \geq 0$*, where* $V_t(A_n)$ *is the total variation of* A_n *on* $[0, t]$*. Let* X_n *be a cadlag,* $\{\mathcal{F}_t^n\}$*-adapted,* $\mathbb{R}^{k\times m}$*-valued process and define*

$$Z_n(t) = \int_0^t X_n(s-)\, dY_n(s).$$

Suppose that $(X_n, Y_n) \Rightarrow (X, Y)$ *in* $D_{\mathbb{R}^{k\times m}\times\mathbb{R}^m}[0, \infty)$*. Then* Y *is a semimartingale with respect to a filtration to which* X *and* Y *are adapted, and* $(X_n, Y_n, Z_n) \Rightarrow (X, Y, Z)$ *in* $D_{\mathbb{R}^{k\times m}\times\mathbb{R}^m\times\mathbb{R}^k}[0, \infty)$*, where*

$$Z(t) = \int_0^t X(s-)\, dY(s).$$

If $(X_n, Y_n) \to (X, Y)$ *in probability, then* $Z_n \to Z$ *in probability.*

Remark 2.2 *In the setting of Theorem 2.1, if* $\{V_n\}$ *is another sequence of cadlag adapted processes and* $(V_n, X_n, Y_n) \Rightarrow (V, X, Y)$*, then* $(V_n, X_n, Y_n, Z_n) \Rightarrow (V, X, Y, Z)$*. This can be seen by applying Theorem 2.1 to* $(\bar{X}_n, \bar{Y}_n)$*, where* $\bar{X}_n = (V_n, X_n)$ *and* $\bar{Y}_n = (0, Y_n)^T$*.*

This next theorem is a special case of Theorem 5.4 and Corollary 5.6 in [2].

Theorem 2.3 *For each* n*, let* Y_n *be a cadlag,* $\mathbb{R}^m$*-valued semimartingale with respect to a filtration* $\{\mathcal{F}_t^n\}$*. Suppose that* $\{Y_n\}$ *satisfies* (3)*. Let* U_n *be a cadlag,* $\{\mathcal{F}_t^n\}$*-adapted,* $\mathbb{R}^k$*-valued process and suppose that* $(U_n, Y_n) \Rightarrow (U, Y)$ *in* $D_{\mathbb{R}^k\times\mathbb{R}^m}[0, \infty)$*. Let* G_n *and* G *be continuous functions from* $\mathbb{R}^k$ *to* $\mathbb{R}^{k\times m}$ *such that* $G_n \to G$ *uniformly on compacts, and suppose that* X_n *satisfies*

$$X_n(t) = U_n(t) + \int_0^t G_n(X_n(s-))\, dY_n(s).$$

Consider the integral equation

$$X(t) = U(t) + \int_0^t G(X(s-))\, dY(s).$$

Suppose that for every version of (U, Y)*, this equation has a unique strong solution for all time* $t \geq 0$*. Then* $(U_n, X_n, Y_n) \Rightarrow (U, X, Y)$ *in* $D_{\mathbb{R}^k\times\mathbb{R}^k\times\mathbb{R}^m}[0, \infty)$*. If* $(U_n, Y_n) \to (U, Y)$ *in probability, then* $X_n \to X$ *in probability.*

Remark 2.4 *As in Remark 2.2, if* $\{V_n\}$ *is another sequence of cadlag adapted processes and* $(V_n, U_n, Y_n) \Rightarrow (V, U, Y)$*, then* $(V_n, U_n, X_n, Y_n) \Rightarrow (V, U, X, Y)$*. This can be seen by applying Theorem 2.3 to* $(\bar{U}_n, Y_n)$ *and* $\bar{G}_n$*, where* $\bar{U}_n = (V_n, U_n)$ *and* $\bar{G}_n = (0, G_n)$*.*

The final theorem in this section is a generalization of these two, which follows from Remark 2.5 in [2].

Theorem 2.5 *Suppose all of the hypotheses of Theorem 2.1 (or Theorem 2.3) hold, except for* (3)*. If* $\{(\bar{X}_n, \bar{Y}_n)\}$ *is relatively compact in* $D_{\mathbb{R}^{m\times\ell}\times\mathbb{R}^\ell}[0,\infty)$, $\{\bar{Y}_n\}$ *satisfies* (3)*, and*

$$Y_n(t) = \int_0^t \bar{X}_n(s-)\, d\bar{Y}_n(s),$$

then the conclusions of Theorem 2.1 (or Theorem 2.3) hold.

References

[1] Stewart N. Ethier and Thomas G. Kurtz, *Markov Processes: Characterization and Convergence*. Wiley-Interscience, 1986.

[2] Thomas G. Kurtz and Philip Protter, Weak Limit Theorems for Stochastic Integrals and Stochastic Differential Equations. *The Annals of Probability*, **19(3)** (1991), 1035–1070.