The expectation of a product of Gaussian random variables

Jason Swanson

October 16, 2007

Let $X_1, X_2, \ldots, X_{2n}$ be a collection of random variables which are jointly Gaussian. Assume that each X_j has mean zero and variance one. In this note, we will derive a formula for the expectation of their product in terms of their pairwise covariances. To state the formula, we introduce the following notation. A *partition* of a set S is a collection of pairwise disjoint, nonempty subsets of S whose union is S. Let $\mathcal{A}$ be a partition of $\{1, 2, \ldots, 2n\}$. We will call $\mathcal{A}$ a *pair-partition* if each set $A \in \mathcal{A}$ has exactly two elements. (In particular, this means $\mathcal{A}$ consists of exactly n sets.) Let $\mathbb{A} = \mathbb{A}^n$ denote the set of all pair-partitions of $\{1, 2, \ldots, 2n\}$. We shall show that

$$E\left[\prod_{j=1}^{2n} X_j\right] = \sum_{\mathcal{A} \in \mathbb{A}} \prod_{\{i,j\} \in \mathcal{A}} \rho_{ij},$$

where $\rho_{ij} = E[X_i X_j]$. For example,

$$E[X_1 X_2 X_3 X_4] = \rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{14}\rho_{23},$$

and

$$\begin{aligned} E[X_1 X_2 X_3 X_4 X_5 X_6] = {} & \rho_{12}\rho_{34}\rho_{56} + \rho_{12}\rho_{35}\rho_{46} + \rho_{12}\rho_{36}\rho_{45} \\ & + \rho_{13}\rho_{24}\rho_{56} + \rho_{13}\rho_{25}\rho_{46} + \rho_{13}\rho_{26}\rho_{45} \\ & + \rho_{14}\rho_{23}\rho_{56} + \rho_{14}\rho_{25}\rho_{36} + \rho_{14}\rho_{26}\rho_{35} \\ & + \rho_{15}\rho_{23}\rho_{46} + \rho_{15}\rho_{24}\rho_{36} + \rho_{15}\rho_{26}\rho_{34} \\ & + \rho_{16}\rho_{23}\rho_{45} + \rho_{16}\rho_{24}\rho_{35} + \rho_{16}\rho_{25}\rho_{34}. \end{aligned}$$

To prove this formula, we begin with the following lemma.

Lemma 1 *If $X_1, \ldots, X_{2n}$ are jointly Gaussian with $EX_j = 0$ and $EX_j^2 = 1$, then*

$$E\left[\prod_{j=1}^{2n} X_j\right] = \frac{1}{n} \sum_{i<j} \rho_{ij} E\left[\prod_{k \notin \{i,j\}} X_k\right],$$

where $\rho_{ij} = E[X_i X_j]$.

Proof. Let $W_1, \dots, W_{2n}$ be dependent Brownian motions with covariation $[W_i, W_j]_t = \rho_{ij} t$. Using Itô's formula, we have

$$\begin{aligned} E[X_1 \cdots X_{2n}] &= E[W_1(1) \cdots W_{2n}(1)] \\ &= \sum_{i<j} \int_0^1 E\Bigg[\prod_{k \notin \{i,j\}} W_k(s)\Bigg] \rho_{ij}\, ds \\ &= \sum_{i<j} \int_0^1 (\sqrt{s})^{2n-2} E\Bigg[\prod_{k \notin \{i,j\}} X_k\Bigg] \rho_{ij}\, ds \\ &= \sum_{i<j} \frac{1}{n} E\Bigg[\prod_{k \notin \{i,j\}} X_k\Bigg] \rho_{ij}, \end{aligned}$$

completing the proof. □

We can now prove the general formula.

Theorem 2 *If $X_1, \dots, X_{2n}$ are jointly Gaussian with $EX_j = 0$ and $EX_j^2 = 1$, then*

$$E\Bigg[\prod_{j=1}^{2n} X_j\Bigg] = \sum_{\mathcal{A} \in \mathbb{A}} \prod_{\{i,j\} \in \mathcal{A}} \rho_{ij}, \tag{1}$$

where $\rho_{ij} = E[X_i X_j]$.

Proof. If $n = 1$, the claim is trivial. Suppose the formula holds for $n - 1$. For each i and j, define $\mathbb{A}_{ij} = \{\mathcal{A} \in \mathbb{A}^n : \{i,j\} \in \mathcal{A}\}$ and $\mathbb{B}_{ij} = \{\mathcal{A} \setminus \{\{i,j\}\} : \mathcal{A} \in \mathbb{A}_{ij}\}$. Note that $\mathbb{B}_{ij}$ is simply the set of pair-partitions of $\{1, \dots, 2n\} \setminus \{i,j\}$. By Lemma 1 and the inductive hypothesis,

$$\begin{aligned} E\Bigg[\prod_{j=1}^{2n} X_j\Bigg] &= \frac{1}{n} \sum_{i<j} \rho_{ij} E\Bigg[\prod_{k \notin \{i,j\}} X_k\Bigg] \\ &= \frac{1}{n} \sum_{i<j} \rho_{ij} \sum_{\mathcal{A} \in \mathbb{B}_{ij}} \prod_{\{m,n\} \in \mathcal{A}} \rho_{mn} \\ &= \frac{1}{n} \sum_{i<j} \sum_{\mathcal{A} \in \mathbb{A}_{ij}} \prod_{\{m,n\} \in \mathcal{A}} \rho_{mn}. \end{aligned}$$

Since each partition in $\mathbb{A}^n$ is counted exactly n times in the above double sum, this completes the proof. □

Corollary 3 *If $X_1, \dots, X_{2n}$ are jointly Gaussian with $EX_j = 0$ and $EX_j^2 = 1$, then*

$$E\Bigg[\prod_{j=1}^{2n} X_j\Bigg] = \sum_{k=2}^{2n} \rho_{1k} E\Bigg[\prod_{\ell \notin \{1,k\}} X_\ell\Bigg],$$

where $\rho_{1k} = E[X_1 X_k]$.

Proof. By Theorem 2,

$$E\left[\prod_{j=1}^{2n} X_j\right] = \sum_{\mathcal{A}\in\mathbb{A}} \prod_{\{i,j\}\in\mathcal{A}} \rho_{ij}$$
$$= \sum_{k=2}^{2n} \sum_{\mathcal{A}\in\mathbb{A}_{1k}} \prod_{\{i,j\}\in\mathcal{A}} \rho_{ij}$$
$$= \sum_{k=2}^{2n} \rho_{1k} \sum_{\mathcal{A}\in\mathbb{B}_{1k}} \prod_{\{i,j\}\in\mathcal{A}} \rho_{ij}.$$

Since $\mathbb{B}_{1k}$ is the set of pair-partitions of $\{1,\ldots,2n\}\setminus\{1,k\}$, a second application of Theorem 2 completes the proof. □

As an example of the use of Corollary 3, we compute the following:

$$\begin{aligned} E[X_1X_2^2X_3X_4^2] &= 2\rho_{12}E[X_2X_3X_4^2] + \rho_{13}E[X_2^2X_4^2] + 2\rho_{14}E[X_2^2X_3X_4] \\ &= 2\rho_{12}(\rho_{23} + 2\rho_{24}\rho_{34}) + \rho_{13}(1 + 2\rho_{24}^2) + 2\rho_{14}(\rho_{34} + 2\rho_{23}\rho_{24}) \\ &= 2\rho_{12}\rho_{23} + 4\rho_{12}\rho_{24}\rho_{34} + \rho_{13} + 2\rho_{13}\rho_{24}^2 + 2\rho_{14}\rho_{34} + 4\rho_{14}\rho_{23}\rho_{24}. \end{aligned}$$

As can be seen, Corollary 3 provides an iterative method for computing this expectation.

The assumption that each X_j has variance 1 is unnecessary.

Corollary 4 *If $Y_1,\ldots,Y_{2n}$ are jointly Gaussian, then*

$$E\left[\prod_{j=1}^{2n} Y_j\right] = \sum_{k=2}^{2n} E[Y_1Y_k]E\left[\prod_{\ell\notin\{1,k\}} Y_\ell\right],$$

provided $EY_j = 0$ for all j.

Proof. Let $\sigma_j^2 = EY_j^2$, $X_j = \sigma_j^{-1}Y_j$, and $\rho_{ij} = E[X_iX_j]$. Apply Corollary 3. □

Acknowledgments

Thanks go to Wenbo Li for bringing Equation (1) to my attention.