The Penny Game

by
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Notation

$\mathbb{N}$	The set of natural numbers, i.e. $\mathbb{N} = \{1, 2, 3, \ldots\}$
$\mathbb{Z}$	The set of integers, i.e. $\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$
$\in$	Is an element of
$\notin$	Is not an element of
$\exists$	There exists
$\forall$	For all
$A \cup B$	The union of A and B, i.e. $A \cup B = \{x : x \in A \text{ or } x \in B\}$
$A \cap B$	The intersection of A and B, i.e. $A \cap B = \{x : x \in A \text{ and } x \in B\}$
$A - B$	The relative complement of B in A, i.e. $A - B = \{x : x \in A \text{ and } x \notin B\}$
$\bigcup_{i=m}^{n} A_i$	The union as i goes from m to n of A_i, i.e. $\bigcup_{i=m}^{n} A_i = \{x : \exists\, i \in \mathbb{N}, m \leq i \leq n \text{ such that } x \in A_i\}$
$f : A \to B$	A function from A to B, i.e. a rule that assigns an element $f(a) \in B$ to every element $a \in A$
$\mathcal{P}(A)$	The power set of A, i.e. the set of all subsets of A
$\varnothing$	The null set, i.e. the set with no elements.
$\mathbf{x}$	A vector, i.e. $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ for some $n \in \mathbb{N}$
$\mathbf{0}$	The zero vector, i.e. $\mathbf{0} = (0, 0, 0, \ldots, 0)$
A^k	The set of k-tuples in A, i.e. $A^k = \{\mathbf{x} = (x_1, x_2, \ldots, x_k) : x_i \in A \text{ for all } i\}$
$\sum_{i=m}^{n} x_i$	The sum as i goes from m to n of x_i, i.e. $\sum_{i=m}^{n} x_i = x_m + x_{m+1} + \ldots + x_n$

Part I

The Game Space,
The Winners, and
The Fundamental Positions

Two players sit across from one another at a table. On the table are several rows of pennies. Turns alternate. The player with the turn selects a row, takes pennies from it, and removes them from the table. The player may take the whole row or part of the row, but must take at least one penny. The player who takes the last penny loses.

The question that led to the writing of this paper is the following: Adam and Brian are playing the penny game with four rows of pennies. They've decided to put 9 pennies in the first row, 18 pennies in the second, and 34 pennies in the third. They agree that if Adam can go first, Brian can decide how many pennies to put in the fourth row. How many pennies should Brian put in the fourth row to ensure his victory?

Questions that come to mind are: What does it mean to "ensure victory"? If this can be adequately defined, is there a solution? If there is a solution, is it unique? This paper will answer these questions and provide a mathematical framework for understanding the general game with any number of rows containing any number of pennies.

To begin with, let us define the set which will model The Penny Game.

Definition 1

Let N be the set of non-negative integers. If $G = N^k$ for some $k \in \mathbb{Z}$, $k \geq 2$, then G is called the **game space of order** k.

In the original question, Adam and Brian are playing in the game space of order 4, $G = N^4$. If Brian places 12 pennies in the fourth row, then the initial board setup would be (9, 18, 34, 12). If Adam begins the game by taking all 9 pennies from the first row, the resulting position would be (0, 18, 34, 12).

Note that the order of the game space depends only on the initial number of rows in the game and not on the number of rows remaining at any later time. The order of the game space remains constant from the start to the finish of the game.

In the penny game certain positions are related in ways that others are not. For example, (9, 18, 34, 12) and (9, 18, 18, 12) are related in that one can be created from the other in one turn of the game; (9, 18, 34, 12) and (0, 18, 34, 9) are not.

Definition 2

Let G be the game space of order k. Let $\mathbf{x}, \mathbf{y} \in G$. If $\exists\, j \in \mathbb{Z}$, $1 \leq j \leq k$ such that $x_j > y_j$ and $x_i = y_i$ for $i \neq j$, then $\mathbf{y}$ is a **reduction** of $\mathbf{x}$.

In our analysis of The Penny Game, it will be necessary to consider all possible moves available to our opponent at a given time. This is equivalent to considering the complete set of reductions of a given position.

Definition 3

Let G be the game space of order k. The **reduction function**, $s : G \rightarrow \mathcal{P}(G)$ is defined as: $s(\mathbf{x}) = \{\mathbf{y} \in G : \mathbf{y} \text{ is a reduction of } \mathbf{x}\}$.

Note that for any game space, G, $s(\mathbf{x}) = \varnothing$ if and only if $\mathbf{x} = \mathbf{0}$.

The following positions play a critical role in the theory of the penny game.

Definition 4

Let G be the game space of order k. Let $\mathbf{x} \in$ G. If $\sum_{i=1}^{k} x_i = 1$, then $\mathbf{x}$ is a **unit**.

Units have the property that if one is on the table and it is your opponent's turn to play, you win. Other positions bear similar properties. Consider the position (2, 2, 0) $\in$ G = N^3. If this is on the table and it is your opponent's turn to play, then no matter what he does, you can make a unit, and thereby win. Mathematically speaking, if $\mathbf{x}$ = (2, 2, 0), then $\forall$ $\mathbf{y} \in s(\mathbf{x})$, $s(\mathbf{y})$ contains a unit. Let us call all such $\mathbf{x}$ with this property "first level winners". Now let us define "second level winners" as any $\mathbf{x}$ such that $\forall$ $\mathbf{y} \in s(\mathbf{x})$, $s(\mathbf{y})$ contains a unit or a first level winner. Continuing in this way, we can define n-th level winners for any $n \in \mathbb{N}$. Using this concept, if the position on the table is a unit or an n-th level winner for some $n \in \mathbb{N}$ and it is your opponent's turn to play, then victory is ensured.

Definition 5

Let G be the game space of order k. The **set of winners** of G is defined as:

$$\mathrm{W} = \bigcup_{n=0}^{\infty} \mathrm{W}_n, \text{ where}$$

$$\mathrm{W}_0 = \{\mathbf{x} \in \mathrm{G} : \mathbf{x} \text{ is a unit}\}, \text{ and}$$

$$\mathrm{W}_n = \left\{\mathbf{x} \in \mathrm{G}: s(\mathbf{x}) \neq \varnothing \text{ and } \forall \mathbf{y} \in s(\mathbf{x}), s(\mathbf{y}) \cap \left(\bigcup_{i=0}^{n-1} \mathrm{W}_i\right) \neq \varnothing\right\}, \text{ for } n \in \mathbb{N}.$$

Before we investigate the properties of this set, let us introduce an alternative concept. The Penny Game is sometimes played with the objective reversed, i.e. whoever takes the last penny wins. If this is the goal, units are no longer "winners", but **0** is. If we base our recursive construction of the set of "winners" on **0** instead of the units, we obtain a different, yet analogous set of winners.

Definition 6

Let G be the game space of order k. The **set of contrary winners** of G is defined as:

$$\mathrm{W}' = \bigcup_{n=0}^{\infty} \mathrm{W}'_n, \text{ where}$$

$$\mathrm{W}'_0 = \{\mathbf{0}\}, \text{ and}$$

$$\mathrm{W}'_n = \left\{\mathbf{x} \in \mathrm{G}: \forall \mathbf{y} \in s(\mathbf{x}), s(\mathbf{y}) \cap \left(\bigcup_{i=0}^{n-1} \mathrm{W}'_i\right) \neq \varnothing\right\}, \text{ for } n \in \mathbb{N}.$$

Note that the condition $s(\mathbf{x}) \neq \varnothing$ has been omitted. This is because $s(\mathbf{x}) = \varnothing$ is equivalent to $\mathbf{x} = \mathbf{0}$. In other words, we have allowed **0** to be an element of W'_n for all n.

Now, not only do W and W' share many of the same properties, but many of the same elements as well. In fact, only a finite number of elements are not common to both sets. Because these sets are so similar, it will be useful to discuss and prove properties of only one, which, for mathematical simplicity, will be W'. Therefore, in order to provide a complete understanding of W, let us specify the exact relationship between W and W'. Before we do this, we will need some preliminaries.

Lemma 1

Let G be the game space of order k and $\mathbf{x} \in$ G. Then

(i) if $s(\mathbf{x}) \neq \varnothing$ and $\forall\, \mathbf{y} \in s(\mathbf{x})$, $s(\mathbf{y}) \cap W \neq \varnothing$, then $\mathbf{x} \in W$; and
(ii) if $\forall\, \mathbf{y} \in s(\mathbf{x})$, $s(\mathbf{y}) \cap W' \neq \varnothing$, then $\mathbf{x} \in W'$.

Proof

Part (i): Let G be the game space of order k, $\mathbf{x} \in$ G, $s(\mathbf{x}) \neq \varnothing$, and $\forall\, \mathbf{y} \in s(\mathbf{x})$, $s(\mathbf{y}) \cap W \neq \varnothing$. Let $h = |s(\mathbf{x})| = \sum_{i=1}^{k} x_i$ and $s(\mathbf{x}) = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_h\}$. For each $\mathbf{a}_i$, find $\mathbf{b}_i \in s(\mathbf{a}_i) \cap W$. Then, for each $\mathbf{b}_i \in W = \bigcup_{n=0}^{\infty} W_n$, find $N_i \in \mathbb{Z}$, $N_i \geq 0$ such that $\mathbf{b}_i \in W_{N_i}$. Let $N = \max \{N_i\}_{i=1}^{h}$. It follows then that for all i, $\mathbf{b}_i \in \bigcup_{i=0}^{N} W_i$, i.e. $\forall\, \mathbf{y} \in s(\mathbf{x})$, $s(\mathbf{y}) \cap \left(\bigcup_{i=0}^{N} W_i\right) \neq \varnothing$. Thus, by definition, $\mathbf{x} \in W_{N+1} \subset W$.

Part (ii): Let G be the game space of order k, $\mathbf{x} \in$ G, and $\forall\, \mathbf{y} \in s(\mathbf{x})$, $s(\mathbf{y}) \cap W' \neq \varnothing$. If $s(\mathbf{x}) = \varnothing$, then $\mathbf{x} = \mathbf{0} \in W_0' \subset W'$. Assume $s(\mathbf{x}) \neq \varnothing$. Let $h = |s(\mathbf{x})| = \sum_{i=1}^{k} x_i$ and $s(\mathbf{x}) = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_h\}$. For each $\mathbf{a}_i$, find $\mathbf{b}_i \in s(\mathbf{a}_i) \cap W'$. Then, for each $\mathbf{b}_i \in W' = \bigcup_{n=0}^{\infty} W'_n$, find $N_i \in \mathbb{Z}$, $N_i \geq 0$ such that $\mathbf{b}_i \in W'_{N_i}$. Let $N = \max \{N_i\}_{i=1}^{h}$. It follows then that for all i, $\mathbf{b}_i \in \bigcup_{i=0}^{N} W'_i$, i.e. $\forall\, \mathbf{y} \in s(\mathbf{x})$, $s(\mathbf{y}) \cap \left(\bigcup_{i=0}^{N} W'_i\right) \neq \varnothing$. Thus, by definition, $\mathbf{x} \in W'_{N+1} \subset W'$. ■

The above lemma is simply a reformulation of the definition of the set of (contrary) winners, but with the cumbersome usage of W_n and W'_n eliminated.

Definition 7

Let G be the game space of order k. The **set of fundamental positions** is $F = \{\mathbf{x} \in G : \forall\, i \in \mathbb{Z}, 1 \leq i \leq k, x_i \leq 1\}$. The **odd positions** are $F_o = \{\mathbf{x} \in F : \sum_{i=1}^{k} x_i$ is odd$\}$. The **even positions** are $F_e = F - F_o$.

Examples of fundamental positions are (1, 0, 0, 1, 1) which is odd, (1, 1) which is even, and (1, 1, 1, 0) which is odd.

Lemma 2
Let G be the game space of order k, F the set of fundamental positions, and $\mathbf{x} \in$ G. Then
(i) if $\mathbf{x} \in F_o$, then $s(\mathbf{x}) \subset F_e$; and
(ii) if $\mathbf{x} \in F_e$, then $s(\mathbf{x}) \subset F_o$.

Proof
Part (i): Let G be the game space of order k and $\mathbf{x} \in F_o \subset G$. Since $\mathbf{0} \notin F_o$, $s(\mathbf{x}) \neq \varnothing$. Let $\mathbf{y} \in s(\mathbf{x})$ be arbitrary. Find $j \in \mathbb{Z}$, $1 \leq j \leq k$ such that $x_j > y_j$ and $x_i = y_i$ for $i \neq j$. Since $\mathbf{x} \in F_o \subset F$, $x_j \leq 1$, i.e. $x_j = 1$ and $y_j = 0$. Thus,

$$\sum_{i=1}^{k} y_i = \left(\sum_{i=1}^{k} x_i \right) - 1$$

is even and $\mathbf{y} \in F_e$. Since $\mathbf{y}$ was arbitrary, $s(\mathbf{x}) \subset F_e$.

Part (ii): Let G be the game space of order k and $\mathbf{x} \in F_e \subset G$. If $s(\mathbf{x}) = \varnothing$, then $s(\mathbf{x}) \subset F_o$ trivially. Assume $s(\mathbf{x}) \neq \varnothing$. Let $\mathbf{y} \in s(\mathbf{x})$ be arbitrary. Find $j \in \mathbb{Z}$, $1 \leq j \leq k$ such that $x_j > y_j$ and $x_i = y_i$ for $i \neq j$. Since $\mathbf{x} \in F_e \subset F$, $x_j \leq 1$, i.e. $x_j = 1$ and $y_j = 0$. Thus,

$$\sum_{i=1}^{k} y_i = \left(\sum_{i=1}^{k} x_i \right) - 1$$

is odd and $\mathbf{y} \in F_o$. Since $\mathbf{y}$ was arbitrary, $s(\mathbf{x}) \subset F_o$. ■

The above lemma states that the reduction of an odd position is even and the reduction of an even position is odd.

Lemma 3
Let G be the game space of order k, W the set of winners, W' the set of contrary winners, and F the set of fundamental positions. Then

(i) $W \cap F = F_o$, and
(ii) $W' \cap F = F_e$.

Proof
Part (i): First, note that since W_0 is the set of units, $W_0 \cap F \subset F_o$. Now let N be an arbitrary non-negative integer and assume that $W_n \cap F \subset F_o$ for all $n \leq N$. Let $\mathbf{x} \in W_{N+1} \cap F$ be arbitrary. Suppose $\mathbf{x} \notin F_o$, i.e. $\mathbf{x} \in F_e$. Since $\mathbf{0} \notin W$, $s(\mathbf{x}) \neq \varnothing$. Choose $\mathbf{y} \in s(\mathbf{x})$. By lemma 2, $\mathbf{y} \in F_o$. Since $\mathbf{x} \in W_{N+1}$, $\exists\, \mathbf{z} \in s(\mathbf{y}) \cap \left(\bigcup_{i=0}^{N} W_i \right)$, i.e. $\mathbf{z} \in W_n$ for some $n \leq N$. By lemma 2, $\mathbf{z} \in F_e \subset F$. Thus, by hypothesis, $\mathbf{z} \in W_n \cap F \subset F_o$, i.e. $\mathbf{z} \in F_e \cap F_o = \varnothing$. Hence, by contradiction, $\mathbf{x} \in F_o$. Since $\mathbf{x} \in W_{N+1} \cap F$ was arbitrary, $W_{N+1} \cap F \subset F_o$. Therefore, by induction, $W_n \cap F \subset F_o$ for all $n \geq 0$, i.e.

$$W \cap F = \left(\bigcup_{i=0}^{\infty} W_i \right) \cap F = \bigcup_{i=0}^{\infty} \left(W_i \cap F \right) \subset F_o .$$

Now define $F_o(q) = \{\mathbf{x} \in F_o : \sum_{i=1}^{k} x_i = q\}$. Note that $F_o(1) = W_0 = W_0 \cap F \subset W \cap F$. Now let $N \in \mathbb{N}$ be arbitrary and assume that $F_o(n) \subset W \cap F$ for all $n \leq N$. If $F_o(N + 1) = \varnothing$, then, trivially, $F_o(N + 1) \subset W \cap F$. Let $F_o(N + 1) \neq \varnothing$ and let $\mathbf{x} \in F_o(N + 1) \subset F_o \subset F$ be arbitrary. Since $\mathbf{0} \notin F_o$, $s(\mathbf{x}) \neq \varnothing$. Let $\mathbf{y} \in s(\mathbf{x})$ be arbitrary. By lemma 2, $\mathbf{y} \in F_e$. Since $\mathbf{x} \in F$ and $\mathbf{x}$ is not a unit, $\mathbf{0} \notin s(\mathbf{x})$, i.e. $s(\mathbf{y}) \neq \varnothing$. Choose $\mathbf{z} \in s(\mathbf{y})$. Let $n = \sum_{i=1}^{k} z_i < \sum_{i=1}^{k} x_i = N + 1$. Since, by lemma 2, $\mathbf{z} \in F_o$, it follows that $\mathbf{z} \in F_o(n)$. Since $n \leq N$, then by hypothesis, $\mathbf{z} \in W \cap F \subset W$. Since $\mathbf{y}$ was arbitrary, then by lemma 1, $\mathbf{x} \in W$. Hence, $\mathbf{x} \in W \cap F$ and, since $\mathbf{x}$ was arbitrary, $F_o(N + 1) \subset W \cap F$.

Thus, by induction, $F_o(q) \subset W \cap F$ for all $q \in \mathbb{N}$. Since $F_o = \bigcup_{q=1}^{\infty} F_o(q)$, $F_o \subset W \cap F$, i.e. $W \cap F = F_o$.

Part (ii): First, note that since $W'_0 = \{\mathbf{0}\}$, $W'_0 \cap F \subset F_e$. Now let N be an arbitrary non-negative integer and assume that $W'_n \cap F \subset F_e$ for all $n \leq N$. Let $\mathbf{x} \in W'_{N+1} \cap F$ be arbitrary. Suppose $\mathbf{x} \notin F_e$, i.e. $\mathbf{x} \in F_o$. Since $\mathbf{0} \notin F_o$, $s(\mathbf{x}) \neq \varnothing$. Choose $\mathbf{y} \in s(\mathbf{x})$. By lemma 2, $\mathbf{y} \in F_e$. Since $\mathbf{x} \in W'_{N+1}$, $\exists\, \mathbf{z} \in s(\mathbf{y}) \cap \left(\bigcup_{i=0}^{N} W'_i\right)$, i.e. $\mathbf{z} \in W'_n$ for some $n \leq N$. Now by lemma 2, $\mathbf{z} \in F_o \subset F$. Thus, by hypothesis, $\mathbf{z} \in W'_n \cap F \subset F_e$, i.e. $\mathbf{z} \in F_o \cap F_e = \varnothing$. Hence, by contradiction, $\mathbf{x} \in F_e$. Since $\mathbf{x} \in W'_{N+1} \cap F$ was arbitrary, $W'_{N+1} \cap F \subset F_e$. Therefore, by induction, $W_n \cap F \subset F_e$ for all $n \geq 0$, i.e.

$$W' \cap F = \left(\bigcup_{i=0}^{\infty} W'_i\right) \cap F = \bigcup_{i=0}^{\infty} (W'_i \cap F) \subset F_e .$$

Now define $F_e(q) = \{\mathbf{x} \in F_e : \sum_{i=1}^{k} x_i = q\}$. Note that $F_e(0) = W'_0 = W'_0 \cap F \subset W' \cap F$. Now let N be an arbitrary non-negative integer and assume that $F_e(n) \subset W' \cap F$ for all $n \leq N$. If $F_e(N + 1) = \varnothing$, then, trivially, $F_e(N + 1) \subset W' \cap F$. Let $F_e(N + 1) \neq \varnothing$ and let $\mathbf{x} \in F_e(N + 1) \subset F_e \subset F$ be arbitrary. Since $N + 1 \geq 1$, $\mathbf{x} \neq \mathbf{0}$, i.e. $s(\mathbf{x}) \neq \varnothing$. Let $\mathbf{y} \in s(\mathbf{x})$ be arbitrary. By lemma 2, $\mathbf{y} \in F_o$. Since $\mathbf{x} \in F$ and $\mathbf{x}$ is not a unit, $\mathbf{0} \notin s(\mathbf{x})$, i.e. $s(\mathbf{y}) \neq \varnothing$. Choose $\mathbf{z} \in s(\mathbf{y})$. Let $n = \sum_{i=1}^{k} z_i < \sum_{i=1}^{k} x_i = N + 1$. Since, by lemma 2, $\mathbf{z} \in F_e$, it follows that $\mathbf{z} \in F_e(n)$. Since $n \leq N$, then by hypothesis, $\mathbf{z} \in W' \cap F \subset W'$. Since $\mathbf{y}$ was arbitrary, then by lemma 1, $\mathbf{x} \in W'$. Hence, $\mathbf{x} \in W' \cap F$ and, since $\mathbf{x}$ was arbitrary, $F_e(N + 1) \subset W' \cap F$.

Thus, by induction, $F_e(q) \subset W' \cap F$ for all non-negative integers q. Since $F_e = \bigcup_{q=0}^{\infty} F_e(q)$, $F_e \subset W' \cap F$, i.e. $W' \cap F = F_e$. ■

The above lemma states that an odd position is a winner and an even position is a contrary winner. It also states that an odd position is not a contrary winner and an even position is not a winner.

Lemma 4
Let G be the game space of order k and F the set of fundamental positions. If $\mathbf{x} \in$ G – F, then $s(\mathbf{x}) \cap F_o = \varnothing$ if and only if $s(\mathbf{x}) \cap F_e = \varnothing$.

Proof
Let G be the game space of order k, F the set of fundamental positions, and $\mathbf{x} \in$ G – F. Assume $s(\mathbf{x}) \cap F_o = \varnothing$.

Suppose $s(\mathbf{x}) \cap F_e \neq \varnothing$, i.e. $\exists\, \mathbf{y} \in s(\mathbf{x}) \cap F_e$. Since $\mathbf{y} \in s(\mathbf{x})$, find $j \in \mathbb{N}$, $1 \le j \le k$ such that $x_j > y_j$ and $x_i = y_i$ for $i \neq j$. Since $\mathbf{x} \notin$ F and $\mathbf{y} \in F_e \subset$ F, $x_j \ge 2$. Define $\mathbf{z} = \{z_1, z_2, \ldots, z_k\} \in$ F such that $z_j = 1 - y_j$ and $z_i = y_i$ for $i \neq j$. Then $x_j > z_j$ and $x_i = y_i$ for $i \neq j$, i.e. $\mathbf{z} \in s(\mathbf{x})$. Also,

$$\sum_{i=1}^{k} z_i = \left(\sum_{i=1}^{k} y_i\right) - y_j + (1 - y_j) = \left(\sum_{i=1}^{k} y_i\right) - 2y_j + 1.$$

Since $\sum_{i=1}^{k} y_i$ is even, $\sum_{i=1}^{k} z_i$ is odd, i.e. $\mathbf{z} \in F_o$. Hence, $\mathbf{z} \in s(\mathbf{x}) \cap F_o = \varnothing$; so by contradiction, $s(\mathbf{x}) \cap F_e = \varnothing$.

A similar argument proves the converse. Assume $s(\mathbf{x}) \cap F_e = \varnothing$. Suppose $s(\mathbf{x}) \cap F_o \neq \varnothing$, i.e. $\exists\, \mathbf{y} \in s(\mathbf{x}) \cap F_o$. Since $\mathbf{y} \in s(\mathbf{x})$, find $j \in \mathbb{N}$, $1 \le j \le k$ such that $x_j > y_j$ and $x_i = y_i$ for $i \neq j$. Since $\mathbf{x} \notin$ F and $\mathbf{y} \in F_o \subset$ F, $x_j \ge 2$. Define $\mathbf{z} = \{z_1, z_2, \ldots, z_k\} \in$ F such that $z_j = 1 - y_j$ and $z_i = y_i$ for $i \neq j$. Then $x_j > z_j$ and $x_i = y_i$ for $i \neq j$, i.e. $\mathbf{z} \in s(\mathbf{x})$. Also,

$$\sum_{i=1}^{k} z_i = \left(\sum_{i=1}^{k} y_i\right) - y_j + (1 - y_j) = \left(\sum_{i=1}^{k} y_i\right) - 2y_j + 1.$$

Since $\sum_{i=1}^{k} y_i$ is odd, $\sum_{i=1}^{k} z_i$ is even, i.e. $\mathbf{z} \in F_e$. Hence, $\mathbf{z} \in s(\mathbf{x}) \cap F_e = \varnothing$; so by contradiction, $s(\mathbf{x}) \cap F_o = \varnothing$. ■

The above lemma states that the reductions of a non-fundamental position either contain both odd and even positions or neither.

With these preliminaries, we can now state the exact relationship between W and W'.

Theorem 1
Let G be the game space of order k, W the set of winners, W' the set of contrary winners, and F the set of fundamental positions. Then
(i) W – W' = F_o
(ii) W' – W = F_e

Proof
Part (i): Let $\mathbf{x} \in W_0$ – W' be arbitrary. Since $\mathbf{x} \in W_0$, $\mathbf{x}$ is a unit, i.e. $\mathbf{x} \in F_o$. Since $\mathbf{x}$ was arbitrary, W_0 – W' $\subset F_o$. Now let N be an arbitrary non-negative integer and assume that W_n – W' $\subset F_o$ for all $n \le N$.

Let $\mathbf{x} \in W_{N+1} - W'$ be arbitrary. Suppose $\mathbf{x} \notin F_o$. Since $\mathbf{x} \in W_{N+1} \subset W$, $\mathbf{x} \neq \mathbf{0}$, i.e. $s(\mathbf{x}) \neq \varnothing$. Let $\mathbf{y} \in s(\mathbf{x})$ be arbitrary.

Assume $\mathbf{y} \in F_o$. Then, since $\mathbf{x} \in W_{N+1}$ and $\mathbf{y} \in s(\mathbf{x})$, $\exists\, \mathbf{z} \in s(\mathbf{y}) \cap \left(\bigcup_{i=0}^{N} W_i\right)$. But since $\mathbf{y} \in F_o$, by lemma 2, $s(\mathbf{y}) \subset F_e$. Thus, $\mathbf{z} \in F_e \subset F$. Also, $\mathbf{z} \in \left(\bigcup_{i=0}^{N} W_i\right) \subset W$, i.e. $\mathbf{z} \in W \cap F = F_o$, by lemma 3. Hence, $\mathbf{z} \in F_e \cap F_o = \varnothing$. So, by contradiction, $\mathbf{y} \notin F_o$.

Assume $\mathbf{y} \in F_e$. By supposition, $\mathbf{x} \notin F_o$. Since $\mathbf{x} \in W$, by lemma 3, $\mathbf{x} \notin F_e$. Thus, $\mathbf{x} \notin F$. Therefore, by lemma 4, since $\mathbf{y} \in s(\mathbf{x}) \cap F_e$, $s(\mathbf{x}) \cap F_o \neq \varnothing$. Find $\mathbf{y}' \in s(\mathbf{x}) \cap F_o$. Then, since $\mathbf{x} \in W_{N+1}$ and $\mathbf{y}' \in s(\mathbf{x})$, $\exists\, \mathbf{z} \in s(\mathbf{y}') \cap \left(\bigcup_{i=0}^{N} W_i\right)$. But since $\mathbf{y}' \in F_o$, by lemma 2, $s(\mathbf{y}) \subset F_e$, Thus, $\mathbf{z} \in F_e \subset F$. Also, $\mathbf{z} \in \left(\bigcup_{i=0}^{N} W_i\right) \subset W$, i.e. $\mathbf{z} \in W \cap F = F_o$, by lemma 3. Hence, $\mathbf{z} \in F_e \cap F_o = \varnothing$. So, by contradiction, $\mathbf{y} \notin F_e$.

Therefore, since $F_o \cup F_e = F$, $\mathbf{y} \notin F$. Now, since $\mathbf{x} \in W_{N+1}$, find $\mathbf{z} \in s(\mathbf{y}) \cap \left(\bigcup_{i=0}^{N} W_i\right)$.

Assume $\mathbf{z} \in F_o$. Then, since $\mathbf{y} \notin F$, by lemma 4, $\exists\, \mathbf{z}' \in s(\mathbf{y}) \cap F_e$. By lemma 3, part (ii), $F_e \subset W'$, i.e. $\mathbf{z}' \in W'$. Thus, $s(\mathbf{y}) \cap W' \neq \varnothing$.

Assume $\mathbf{z} \notin F_o$. Then, since $\mathbf{z} \in W_n$ for some $n \leq N$, then by hypothesis, $\mathbf{z} \in W'$. Hence, $s(\mathbf{y}) \cap W' \neq \varnothing$.

Since either $\mathbf{z} \in F_o$ or $\mathbf{z} \notin F_o$, it follows that $s(\mathbf{y}) \cap W' \neq \varnothing$. Since $\mathbf{y}$ was arbitrary, then by lemma 1, part (ii), $\mathbf{x} \in W'$. But we have defined $\mathbf{x} \in W_{N+1} - W'$, i.e. $\mathbf{x} \notin W'$. Thus, by contradiction, $\mathbf{x} \in F_o$.

Since $\mathbf{x}$ was arbitrary, $W_{N+1} - W' \subset F_o$. Hence, by induction, $W_n - W' \subset F_o$ for all non-negative integers n. Therefore, $W - W' = \left(\bigcup_{i=0}^{\infty} W_i\right) - W' = \bigcup_{i=0}^{\infty} (W_i - W') \subset F_o$.

Now let $\mathbf{x} \in F_o$ be arbitrary. By lemma 3, part (i), $\mathbf{x} \in W$ and $\mathbf{x} \notin W'$, i.e. $\mathbf{x} \in W - W'$. Since $\mathbf{x}$ was arbitrary, $F_o \subset W - W'$. Therefore, $W - W' = F_o$.

Part (ii): First, note that $W'_0 - W = \{\mathbf{0}\} \subset F_e$. Now let N be an arbitrary non-negative integer and assume that $W'_n - W \subset F_e$ for all $n \leq N$.

Let $\mathbf{x} \in W'_{N+1} - W$ be arbitrary. Suppose $\mathbf{x} \notin F_e$. Since $\mathbf{x} \notin F_e$, $\mathbf{x} \neq \mathbf{0}$, i.e. $s(\mathbf{x}) \neq \varnothing$. Let $\mathbf{y} \in s(\mathbf{x})$ be arbitrary.

Assume $\mathbf{y} \in F_e$. Then, since $\mathbf{x} \in W'_{N+1}$ and $\mathbf{y} \in s(\mathbf{x})$, $\exists\, \mathbf{z} \in s(\mathbf{y}) \cap \left(\bigcup_{i=0}^{N} W'_i \right)$. But since $\mathbf{y} \in F_e$, by lemma 2, $s(\mathbf{y}) \subset F_o$. Thus, $\mathbf{z} \in F_o \subset F$. Also, $\mathbf{z} \in \left(\bigcup_{i=0}^{N} W'_i \right) \subset W'$, i.e. $\mathbf{z} \in W' \cap F = F_e$, by lemma 3. Hence, $\mathbf{z} \in F_o \cap F_e = \varnothing$. So, by contradiction, $\mathbf{y} \notin F_e$.

Assume $\mathbf{y} \in F_o$. By supposition, $\mathbf{x} \notin F_e$. Since $\mathbf{x} \in W'$, by lemma 3, $\mathbf{x} \notin F_o$. Thus, $\mathbf{x} \notin F$. Therefore, by lemma 4, since $\mathbf{y} \in s(\mathbf{x}) \cap F_o$, $s(\mathbf{x}) \cap F_e \neq \varnothing$. Find $\mathbf{y}' \in s(\mathbf{x}) \cap F_e$. Then, since $\mathbf{x} \in W'_{N+1}$ and $\mathbf{y}' \in s(\mathbf{x})$, $\exists\, \mathbf{z} \in s(\mathbf{y}') \cap \left(\bigcup_{i=0}^{N} W'_i \right)$. But since $\mathbf{y}' \in F_e$, by lemma 2, $s(\mathbf{y}) \subset F_o$, Thus, $\mathbf{z} \in F_o \subset F$. Also, $\mathbf{z} \in \left(\bigcup_{i=0}^{N} W'_i \right) \subset W'$, i.e. $\mathbf{z} \in W' \cap F = F_e$, by lemma 3. Hence, $\mathbf{z} \in F_o \cap F_e = \varnothing$. So, by contradiction, $\mathbf{y} \notin F_o$.

Therefore, since $F_e \cup F_o = F$, $\mathbf{y} \notin F$. Now, since $\mathbf{x} \in W'_{N+1}$, find $\mathbf{z} \in s(\mathbf{y}) \cap \left(\bigcup_{i=0}^{N} W'_i \right)$.

Assume $\mathbf{z} \in F_e$. Then, since $\mathbf{y} \notin F$, by lemma 4, $\exists\, \mathbf{z}' \in s(\mathbf{y}) \cap F_o$. By lemma 3, part (i), $F_o \subset W$, i.e. $\mathbf{z}' \in W$. Thus, $s(\mathbf{y}) \cap W \neq \varnothing$.

Assume $\mathbf{z} \notin F_e$. Then, since $\mathbf{z} \in W'_n$ for some $n \le N$, then by hypothesis $\mathbf{z} \in W$. Hence, $s(\mathbf{y}) \cap W \neq \varnothing$.

Since either $\mathbf{z} \in F_e$ or $\mathbf{z} \notin F_e$, it follows that $s(\mathbf{y}) \cap W \neq \varnothing$. Since $\mathbf{y}$ was arbitrary and since $s(\mathbf{x}) \neq \varnothing$, then by lemma 1, part (i), $\mathbf{x} \in W$. But we have defined $\mathbf{x} \in W'_{N+1} - W$, i.e. $\mathbf{x} \notin W$. Thus, by contradiction, $\mathbf{x} \in F_e$.

Since $\mathbf{x}$ was arbitrary, $W'_{N+1} - W \subset F_e$. Hence, by induction, $W'_n - W \subset F_e$ for all non-negative integers n. Therefore, $W' - W = \left(\bigcup_{i=0}^{\infty} W'_i \right) - W = \bigcup_{i=0}^{\infty} (W'_i - W) \subset F_e$.

Now let $\mathbf{x} \in F_e$ be arbitrary. By lemma 3, part (ii), $\mathbf{x} \in W'$ and $\mathbf{x} \notin W$, i.e. $\mathbf{x} \in W' - W$. Since $\mathbf{x}$ was arbitrary, $F_e \subset W' - W$. Therefore, $W' - W = F_e$. ■

Part (i) of the above theorem states that if a position is a winner, but not a contrary winner, then it is fundamental and, in particular, odd. Part (ii) states that if a position is a contrary winner, but not a winner, then it is fundamental and, in particular, even. The following conclusion can be immediately drawn: if a position is not fundamental, then it is either both a winner and a contrary winner, or it is neither. This fact is stated in the following corollary.

Corollary 1

Let G be the game space of order k, W the set of winners, W' the set of contrary winners, and F the set of fundamental positions. If $\mathbf{x} \notin$ F, then $\mathbf{x} \in$ W if and only if $\mathbf{x} \in$ W'.

Proof

Let $\mathbf{x} \notin$ F and $\mathbf{x} \in$ W. Suppose $\mathbf{x} \notin$ W'. Then $\mathbf{x} \in$ W – W' = $F_o \subset$ F. Thus, by contradiction, $\mathbf{x} \in$ W'.

Let $\mathbf{x} \notin$ F and $\mathbf{x} \in$ W'. Suppose $\mathbf{x} \notin$ W. Then $\mathbf{x} \in$ W' – W = $F_e \subset$ F. Thus, by contradiction, $\mathbf{x} \in$ W. ■

Part II

Exclusive Or and The General Solution

In the previous section, the exact relationship between winners and contrary winners was specified. For this reason, we need only concern ourselves with one of these categories, which, as stated before, for mathematical simplicity, will be contrary winners. Hence, the adjective "contrary" will be dropped from the informal part of this discussion. Any reference (not in a definition, lemma, theorem, or corollary) from this point on to a winner will be meant to be either a winner or a contrary winner.

Now, the set of winners was defined with the intention that if you can create a winner in the course of play, victory is ensured. This is because once you create a winner, you can continue to create winners indefinitely. Since the game must end in a finite number of moves, it will end with you creating a winner. Some questions, however, remain unanswered. If the initial position of the game is not a winner, can you be guaranteed that you will be able to create a winner from it? If you create a winner, could it be possible for your opponent to create from it a winner?

If the answer to the first question is yes, more questions follow. What are the conditions under which a non-winner cannot be transformed through reduction into a winner? Under what conditions can the resulting positions be transformed into winners? How can we augment our set of winners to include positions that cannot be transformed through reduction into winners, yet all their reductions can be so transformed?

Also, if the answer to the second question is yes, even more questions follow. If a winner can be transformed into a winner, then who will ultimately win the game? Is the ultimate winner determined only by the initial setup of the game or do the moves made by each player affect who will reach the zero position first? How should we modify our definition of the set of winners to take these possibilities into account?

Fortunately, the answer to both of these questions is no, as is demonstrated in the following theorem.

Theorem 2

Let G be the game space of order k, W' the set of contrary winners, and $\mathbf{x} \in$ G. Then $\mathbf{x} \in$ W' if and only if $s(\mathbf{x}) \cap \mathrm{W}' = \varnothing$.

Proof

Let $\mathbf{x} \in \mathrm{W}'_0$, i.e. $\mathbf{x} = \mathbf{0}$. Then $s(\mathbf{x}) = \varnothing$, i.e. $s(\mathbf{x}) \cap \mathrm{W}' = \varnothing$. Now let N be an arbitrary non-negative integer and assume that for all $n \le N$, $\mathbf{x} \in \mathrm{W}'_n$ implies $s(\mathbf{x}) \cap \mathrm{W}' = \varnothing$.

Let $\mathbf{x} \in \mathrm{W}'_{N+1}$ be arbitrary. Suppose $s(\mathbf{x}) \cap \mathrm{W}' \neq \varnothing$. Find $\mathbf{y} \in s(\mathbf{x}) \cap \mathrm{W}'$. Since $\mathbf{x} \in \mathrm{W}'_{N+1}$ and $\mathbf{y} \in s(\mathbf{x})$, $\exists\, \mathbf{z} \in s(\mathbf{y}) \cap \left(\bigcup_{i=0}^{N} \mathrm{W}'_i\right)$, i.e. $\mathbf{z} \in \mathrm{W}_n$ for some $n \le N$. Thus, by hypothesis, $s(\mathbf{z}) \cap \mathrm{W}' = \varnothing$. But $\mathbf{y} \in \mathrm{W}'$, i.e. $\mathbf{y} \in \mathrm{W}'_m$ for some non-negative integer m. Since $\mathbf{z} \in s(\mathbf{y})$, $s(\mathbf{y}) \neq \varnothing$, i.e. $m \neq 0$. Thus, by definition 6, $s(\mathbf{z}) \cap \left(\bigcup_{i=0}^{m-1} \mathrm{W}'_i\right) \neq \varnothing$, i.e. $s(\mathbf{z}) \cap \mathrm{W}' \neq \varnothing$. Hence, by contradiction, $s(\mathbf{x}) \cap \mathrm{W}' = \varnothing$. Therefore, by induction, for all non-negative integers n, $\mathbf{x} \in \mathrm{W}'_n$ implies $s(\mathbf{x}) \cap \mathrm{W}' = \varnothing$, i.e. if $\mathbf{x} \in \mathrm{W}'$, then $s(\mathbf{x}) \cap \mathrm{W}' = \varnothing$.

Now, define G (n) = {$\mathbf{x} \in$ G : $\sum_{i=1}^{k} x_i = n$}. First note that if $\mathbf{x} \in$ G (0), then $\mathbf{x} = \mathbf{0} \in$ W'. Now let N be an arbitrary non-negative integer and assume that for all $n \le N$, if $\mathbf{x} \in$ G (n) and s ($\mathbf{x}$) $\cap$ W' = $\varnothing$, then $\mathbf{x} \in$ W'.

Let $\mathbf{x} \in$ G (N + 1) and s ($\mathbf{x}$) $\cap$ W' = $\varnothing$. Suppose $\mathbf{x} \notin$ W'. Then, by lemma 1, part (ii), $\exists$ $\mathbf{y} \in s$ ($\mathbf{x}$) such that s ($\mathbf{y}$) $\cap$ W' = $\varnothing$. Since $\mathbf{y}$ is a reduction of $\mathbf{x}$, $\mathbf{y} \in$ G (n) for some $n \le N$. Hence, by hypothesis, $\mathbf{y} \in$ W'. But $\mathbf{y} \in s$ ($\mathbf{x}$) as well and s ($\mathbf{x}$) $\cap$ W' = $\varnothing$. So, by contradiction, $\mathbf{x} \in$ W'.

Therefore, by induction, for all non-negative integers n, if $\mathbf{x} \in$ G (n) and s ($\mathbf{x}$) $\cap$ W' = $\varnothing$, then $\mathbf{x} \in$ W'. Since G = $\bigcup_{n=0}^{\infty} G(n)$, it follows that s ($\mathbf{x}$) $\cap$ W' = $\varnothing$ implies $\mathbf{x} \in$ W'. ■

This theorem answers the remaining questions in the third paragraph of this paper, namely, is there a solution and, if so, is it unique. Let's consider the original problem with Adam and Brian. Brian is looking for some non-negative integer n such that $\mathbf{x}$ = (9, 18, 34, n) $\in$ W'. (He actually wants it to be in W, but since the solution won't be a fundamental position, it will also be in W'.) Let's assume for the moment that no solution exists. Then, by the preceding theorem, all such positions, $\mathbf{x}$ = (9, 18, 34, n), will have a reduction in W'. None of these reductions will involve reducing the fourth row, since this will produce another position of the same form, none of which, by assumption, are in W'. Now, there are 9 + 18 + 34 = 61 reductions using the first three rows. Thus, at least two of (9, 18, 34, 0) through (9, 18, 34, 61) will be able to be made into winners by removing the same number of pennies from the same row. This will produce two winners that differ only in their fourth row, i.e. two winners, one of which is a reduction of the other. The preceding theorem shows that this is not possible. Hence, assuming there is no solution has produced a logical contradiction, i.e. there is a solution.

The uniqueness of the solution follows much easier. Suppose there are numbers a and b such that (9, 18, 34, a) and (9, 18, 34, b) are both winners. If $a \ne b$, then one is greater. Suppose $a > b$. Then (9, 18, 34, b) is a reduction of (9, 18, 34, a) which is not possible since they are both winners. Thus $a = b$ and the solution is unique. These facts are stated and proved formally in the following theorem.

Theorem 3

Let G be the game space of order k and W' the set of contrary winners. If $\{a_i\}_{i=1}^{k-1}$ are non-negative integers, then there exists a unique non-negative integer a_k such that $\mathbf{a} = \{a_1, a_2, \ldots, a_k\} \in$ W'.

Proof

Let G be the game space of order k and $\{a_i\}_{i=1}^{k-1}$ non-negative integers. Assume that $\mathbf{a}_n = \{a_1, a_2, \ldots, a_{k-1}, n\} \notin$ W' for all $n \in \mathbb{Z}$, $n \ge 0$.

For each $\mathbf{a}_n$, by theorem 2, find $\mathbf{b}_n \in s$ ($\mathbf{a}_n$) $\cap$ W'. For each $\mathbf{b}_n$, find p_n such that $a_{p_n} > b_{p_n}$ and $a_i = b_i$ for $i \ne p_n$. Let $q_n = a_{p_n} - b_{p_n} > 0$. First note that for all n, $p_n \ne k$. This is because if

$p_n = k$, then $\mathbf{b}_n = \{a_1, a_2, \ldots, a_{k-1}, n - q_n\} \in$ W', which contradicts the initial assumption. Hence, $q_n = a_{p_n} - b_{p_n} \le M = \max\{a_1, a_2, \ldots, a_{k-1}\}$, i.e. there are at most $M(k-1)$ unique ordered pairs (p_n, q_n).

Now find $0 \le u < v \le M(k-1)$ such that $(p_u, q_u) = (p_v, q_v)$. It follows then that $\mathbf{b}_u = \{a_1, a_2, \ldots, a_{p_u} - q_u, \ldots, u\} = \{a_1, a_2, \ldots, a_{p_v} - q_v, \ldots, v - (v-u)\}$, i.e. $\mathbf{b}_u \in s(\mathbf{b}_v)$. But $\mathbf{b}_u, \mathbf{b}_v \in$ W' which, by theorem 2, is a contradiction. Therefore, the original assumption is false and $\exists\, n \in \mathbb{Z}, n \ge 0$ such that $\mathbf{a}_n = \{a_1, a_2, \ldots, a_{k-1}, n\} \in$ W'.

Now let $\mathbf{a}_n, \mathbf{a}_m \in$ W'. Assume $n \ne m$. Without loss of generality, let $n < m$. Then $\mathbf{a}_n = \{a_1, a_2, \ldots, a_{k-1}, n\} = \{a_1, a_2, \ldots, a_{k-1}, m - (m-n)\}$, i.e. $\mathbf{a}_n \in s(\mathbf{a}_m)$, which, by theorem 2, is a contradiction. Thus, $n = m$ and the solution is unique. ■

At this point, the questions of existence and uniqueness of the solution have been answered. But the original problem remains unsolved. The number $n \in \mathbb{Z}, n \ge 0$ such that $(9, 18, 34, n) \in$ W' remains to be found. The rest of this paper will be devoted to determining the mathematical structure of the elements of W'.

Theorem 2 states that a winner can't be made into a winner and a non-winner can always be made into a winner. This property of the set of winners can be abstracted and applied to any arbitrary subset of G as is done in the following definition.

Definition 8

Let G be the game space of order k and S ⊂ G any subset of G. If for all $\mathbf{x} \in$ G, $\mathbf{x} \in$ S if and only if $s(\mathbf{x}) \cap \mathrm{S} = \varnothing$, then S is **reduction exclusive**.

With this definition, theorem 2 simply states that W' is reduction exclusive. It turns out that for any game space, there are very few reduction exclusive subsets. In fact, there is only one.

Theorem 4

Let G be the game space of order k. If S ⊂ G is reduction exclusive, then S = W'.

Proof

First, note that $s(\mathbf{0}) = \varnothing$, i.e. $s(\mathbf{0}) \cap \mathrm{S} = \varnothing$. Since S is reduction exclusive, $\mathbf{0} \in$ S.

Thus, $\mathrm{W'}_0 = \{\mathbf{0}\} \subset$ S. Now let N be an arbitrary non-negative integer and assume that $\mathrm{W'}_n \subset$ S for all $n \le N$.

Let $\mathbf{x} \in \mathrm{W'}_{N+1}$ be arbitrary. Suppose $\mathbf{x} \notin$ S. Since S is reduction exclusive, $s(\mathbf{x}) \cap \mathrm{S} \ne \varnothing$. Find $\mathbf{y} \in s(\mathbf{x}) \cap$ S. Since $\mathbf{x} \in \mathrm{W'}_{N+1}$ and $\mathbf{y} \in s(\mathbf{x})$, $s(\mathbf{y}) \cap \left(\bigcup_{i=0}^{N} \mathrm{W'}_i\right) \ne \varnothing$. By hypothesis, $\left(\bigcup_{i=0}^{N} \mathrm{W'}_i\right) \subset$ S, i.e. $s(\mathbf{y}) \cap \mathrm{S} \ne \varnothing$. But then, since S is reduction exclusive, $\mathbf{y} \notin$ S, a contradiction. Thus, $\mathbf{x} \in$ S and, since $\mathbf{x}$ was arbitrary, $\mathrm{W'}_{N+1} \subset$ S.

Hence, by induction, $W'_n \subset S$ for all non-negative integers n, i.e. $W' \subset S$.

Now let $\mathbf{x} \in S$ be arbitrary. Suppose $\mathbf{x} \notin W'$. Since W' is reduction exclusive, $s(\mathbf{x}) \cap W' \neq \varnothing$. Since $W' \subset S$, $s(\mathbf{x}) \cap S \neq \varnothing$. But then, since S is reduction exclusive, $\mathbf{x} \notin S$. So, by contradiction, $\mathbf{x} \in W$ and, since $\mathbf{x}$ was arbitrary, $S \subset W'$. ■

If we can now construct a set whose elements can be easily enumerated and then show that set to be reduction exclusive, we will have succeeded in finding the precise mathematical structure of W' and, hopefully, will be able to find $n \in \mathbb{Z}$, $n \geq 0$ such that $(9, 18, 34, n) \in W'$

Definition 9

Let a and b be non-negative integers. Let $a = \sum_{i=0}^{m} 2^i a_i$ and $b = \sum_{i=0}^{n} 2^i b_i$ be the unique binary representations of a and b. Let $a_i = 0$ for $i > m$, $b_i = 0$ for $i > n$, and $N = \max\{m, n\}$. The **exclusive or** operator (^) is defined as

$$a \wedge b = c = \sum_{i=0}^{N} 2^i c_i, \text{ where}$$

$$c_i = \begin{cases} 0 & \text{if } a_i = b_i \\ 1 & \text{if } a_i \neq b_i \end{cases}$$

Theorem 5

If a, b, and c are non-negative integers, then

(i) $a \wedge 0 = 0 \wedge a = a$,
(ii) $a \wedge b = 0$ if and only if $a = b$,
(iii) $a \wedge b = b \wedge a$, and
(iv) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

Proof

Parts (i), (ii), and (iii) are obvious consequences of definition 9. Part (iv) follows from the fact that, in definition 9, $c_i = a_i + b_i \pmod 2$. ■

Note the following notation: $\bigwedge_{i=m}^{n} a_i = a_m \wedge a_{m+1} \wedge \ldots \wedge a_n$.

Definition 10

Let G be the game space of order k and $\mathbf{x} \in G$. If $\bigwedge_{i=1}^{k} x_i = 0$, then $\mathbf{x}$ is **symmetric**. The set $\Omega = \{\mathbf{x} \in G : \mathbf{x} \text{ is symmetric}\}$ is the **symmetric subset** of G.

Theorem 6

Let G be the game space of order k. The symmetric subset of G, Ω, is reduction exclusive.

Proof

It must be shown that $\mathbf{x} \in \Omega$ if and only if $s(\mathbf{x}) \cap \Omega = \varnothing$.

Let $\mathbf{x} \in \Omega$. If $s(\mathbf{x}) = \varnothing$, then $s(\mathbf{x}) \cap \Omega = \varnothing$. Assume $s(\mathbf{x}) \neq \varnothing$. Let $\mathbf{y} \in s(\mathbf{x})$ be arbitrary. Find $p \in \mathbb{N}$, $1 \le p \le k$ such that $x_p > y_p$ and $x_i = y_i$ for $i \neq p$. Then

$$\begin{aligned}\bigwedge_{i=1}^{k} y_i &= \left(\bigwedge_{i=1}^{p-1} y_i\right) \wedge y_p \wedge \left(\bigwedge_{i=p+1}^{k} y_i\right) \\ &= \left(\bigwedge_{i=1}^{p-1} x_i\right) \wedge \left(x_p \wedge x_p\right) \wedge y_p \wedge \left(\bigwedge_{i=p+1}^{k} x_i\right) \\ &= \left(\bigwedge_{i=1}^{k} x_i\right) \wedge \left(x_p \wedge y_p\right) \\ &= x_p \wedge y_p\end{aligned}$$

Since $x_p \neq y_p$, $x_p \wedge y_p \neq 0$, i.e. $\mathbf{y} \notin \Omega$. Since $\mathbf{y}$ was arbitrary, $s(\mathbf{x}) \cap \Omega = \varnothing$.

Now let $s(\mathbf{x}) \cap \Omega = \varnothing$. Suppose $\mathbf{x} \notin \Omega$. Then $\bigwedge_{i=1}^{k} x_i = c > 0$, i.e. $c = \sum_{i=0}^{N} 2^i c_i$, where $c_N = 1$. Let $x_i = \sum_{j=0}^{m_i} 2^j x_{ij}$, with $x_{ij} = 0$ for $j > m_i$ be the unique binary representations of the components of $\mathbf{x}$. Since $1 = c_N = \sum_{i=1}^{k} x_{iN} \pmod 2$, $\exists\, p \in \mathbb{N}$, $1 \le p \le k$ such that $x_{pN} = 1$. Define $\mathbf{y} \in$ G such that $y_p = x_p \wedge c$ and $y_i = x_i$ for $i \neq p$. Now let $x_p \wedge c = \sum_{j=0}^{m_p} 2^j x'_{pj}$. It then follows that $x_p - y_p = x_p - x_p \wedge c = \sum_{j=0}^{m_p} 2^j \left(x_{pj} - x'_{pj}\right)$. Note that for $j > N$, $c_j = 0$, so that $x'_{pj} = x_{pj}$. Also, $c_N = x_{pN} = 1$, so that $x'_{pN} = 0$. Therefore, $x_p - y_p = 2^N + \sum_{j=0}^{N-1} 2^j \left(x_{pj} - x'_{pj}\right) \ge 2^N - \sum_{j=0}^{N-1} 2^j = 1$, i.e. $x_p > y_p$ and $\mathbf{y} \in s(\mathbf{x})$. Furthermore,

$$\begin{aligned}\bigwedge_{i=1}^{k} y_i &= \left(\bigwedge_{i=1}^{p-1} y_i\right) \wedge y_p \wedge \left(\bigwedge_{i=p+1}^{k} y_i\right) \\ &= \left(\bigwedge_{i=1}^{p-1} x_i\right) \wedge x_p \wedge c \wedge \left(\bigwedge_{i=p+1}^{k} x_i\right) \\ &= \left(\bigwedge_{i=1}^{k} x_i\right) \wedge c \\ &= c \wedge c = 0\end{aligned}$$

Thus, $\mathbf{y} \in \Omega$. However, $s(\mathbf{x}) \cap \Omega = \varnothing$, so by contradiction, $\mathbf{x} \in \Omega$. ■

Corollary 2

Let G be the game space of order *k*. If W' is the set of contrary winners and Ω is the symmetric subset of G, then W' = Ω.

Proof

This follows immediately from theorems 4 and 5. ■

So now the solution comes easily. First, compute 9 ^ 18 ^ 34. We have

$9 = 2^0 (1) + 2^1 (0) + 2^2 (0) + 2^3 (1)$,
$18 = 2^0 (0) + 2^1 (1) + 2^2 (0) + 2^3 (0) + 2^4 (1)$, and
$34 = 2^0 (0) + 2^1 (1) + 2^2 (0) + 2^3 (0) + 2^4 (0) + 2^5 (1)$.

Hence 9 ^ 18 ^ 34 = $2^0 (1) + 2^1 (0) + 2^2 (0) + 2^3 (1) + 2^4 (1) + 2^5 (1) = 57$. Therefore, 9 ^ 18 ^ 34 ^ 57 = 0, i.e. (9, 18, 34, 57) ∈ Ω = W'.

BIBLIOGRAPHY

The Web Wizard's Math Challenge, *www.best.com/~perry/wwizard.shtml*
The CTK Exchange, *www.cut-the-knot.com/exchange/index.html*