On the Variance of Pure Jump Processes

Jason Swanson

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Theorem 1 *Let $N(t)$ be a pure jump process. Suppose that $\lambda(t)$ is $\{\mathcal{F}_t^N\}$-adapted and that*

$$M(t) = N(t) - \int_0^t \lambda(s)\,ds$$

is an $\{\mathcal{F}_t^N\}$-martingale. Then

$$\operatorname{var}(N(t)) = EN(t) + 2\int_0^t \operatorname{cov}(N(s), \lambda(s))\,ds$$

for all $t \geq 0$.

Proof. First note that if U is $\{\mathcal{F}_t^N\}$-adapted, then

$$\begin{aligned}
E\left[N(t)\int_0^t U(s)\,ds\right] &= \int_0^t E[U(s)E[N(t)|\mathcal{F}_s]]\,ds \\
&= \int_0^t E\left[U(s)\left(M(s) + E\left[\int_0^t \lambda(u)\,du\middle|\mathcal{F}_s\right]\right)\right]\,ds \\
&= \int_0^t E\left[U(s)\left(N(s) + E\left[\int_s^t \lambda(u)\,du\middle|\mathcal{F}_s\right]\right)\right]\,ds \\
&= \int_0^t E[U(s)N(s)]\,ds + E\int_0^t\int_s^t U(s)\lambda(u)\,du\,ds.
\end{aligned}$$

Taking $U(s) = \lambda(s)$ gives

$$E\left[N(t)\int_0^t \lambda(s)\,ds\right] = \int_0^t E[N(s)\lambda(s)]\,ds + \frac{1}{2}E\left[\left(\int_0^t \lambda(s)\,ds\right)^2\right],$$

and taking $U(s) = E\lambda(s)$ gives

$$E\left[N(t)\int_0^t E\lambda(s)\,ds\right] = \int_0^t EN(s)E\lambda(s)\,ds + \frac{1}{2}\left(\int_0^t E\lambda(s)\,ds\right)^2.$$

Now note that $EN(t) = \int_0^t E\lambda(s)\,ds$, so that

$$\begin{aligned}
\operatorname{var}(N(t)) &= E\left[\left(N(t) - \int_0^t E\lambda(s)\,ds\right)^2\right] \\
&= EN(t)^2 - 2E\left[N(t)\int_0^t E\lambda(s)\,ds\right] + \left(\int_0^t E\lambda(s)\,ds\right)^2 \\
&= EN(t)^2 - 2\int_0^t EN(s)E\lambda(s)\,ds.
\end{aligned}$$

On the other hand, $N(t) = [M]_t$, so that $EN(t) = EM(t)^2$, which gives

$$
\begin{aligned}
EN(t) &= E\left[\left(N(t) - \int_0^t \lambda(s)\,ds\right)^2\right] \\
&= EN(t)^2 - 2E\left[N(t)\int_0^t \lambda(s)\,ds\right] + E\left(\int_0^t \lambda(s)\,ds\right)^2 \\
&= EN(t)^2 - 2\int_0^t E[N(s)\lambda(s)]\,ds.
\end{aligned}
$$

Subtracting these two formulas completes the proof. □

Corollary 2 *Let $N(t)$ be a pure birth process with birth rate $\lambda(t) = f(N(t))$, where f is a decreasing function. Then* $\operatorname{var}(N(t)) \leq EN(t)$, *with equality if and only if f is a constant.*

Proof. Note that

$$
\begin{aligned}
E[N(s)f(N(s))] &= EN(s)Ef(N(s)) + E[(N(s) - EN(s))f(N(s))] \\
&= EN(s)Ef(N(s)) + E[(N(s) - EN(s))(f(N(s)) - f(\lfloor EN(s) \rfloor))].
\end{aligned}
$$

Hence, $\operatorname{cov}(N(s), \lambda(s))$ is the expectation of a nonpositive random variable, and is zero if and only if $f(N(s))$ is constant a.s. □