Fourier transform fact sheet

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1 The Fourier transform

This small document collects for easy reference some elementary facts about the Fourier transform. Regarding references, there are many excellent textbooks to choose from, and we leave it to the readers to explore for themselves, rather than providing a bibliography.

Let $\mathcal{S}(\mathbb{R}^n)$ denote the space of Schwartz functions with its usual topology, and let $\mathcal{S}^*(\mathbb{R}^n)$ denote its topological dual. That is, $\mathcal{S}^*(\mathbb{R})$ is the space of complex-valued, continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$, also known as tempered distributions.

If f is a measurable function such that

$$|f(x)| \leqslant |g(x)|(1 + |x|^k),$$

for some positive integer k and some integrable function g, then f determines a tempered distribution whose action is given by $\langle f, \varphi \rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$.

For $\varphi \in \mathcal{S}$, we define the Fourier transform in this document with the following normalization:

$$\mathcal{F}\varphi(\xi) = \widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{i\xi\cdot x}\varphi(x)\,dx,$$

where $\xi \cdot x$ denotes the usual dot product in $\mathbb{R}^n$. For $T \in \mathcal{S}^*$, the Fourier transform, $\widehat{T}$, is the tempered distribution satisfying $\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle$.

Example 1.1. Let X be an $\mathbb{R}^n$-valued random variable with law μ, so that μ is a probability measure on $\mathbb{R}^n$ with $\mu(A) = P(X \in A)$. Then μ determines a tempered distribution whose action is $\langle \mu, \psi \rangle = \int_{\mathbb{R}^n} \psi\,d\mu$. We then have

$$\begin{aligned}\langle \widehat{\mu}, \psi \rangle = \langle \mu, \widehat{\psi} \rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{it\cdot\xi}\psi(t)\,dt\,\mu(d\xi)\\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{it\cdot\xi}\,\mu(d\xi)\right)\psi(t)\,dt = \int_{\mathbb{R}^n} E[e^{it\cdot X}]\psi(t)\,dt\end{aligned}$$

In other words, $\widehat{\mu}(t) = E[e^{it\cdot X}]$, and this is what probabilists call the characteristic function of X. Note that if μ has a density f with respect to Lebesgue measure, then μ and f determine the same tempered distribution, and hence $\widehat{f}(t) = E[e^{it\cdot X}]$. □

Example 1.2. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and define

$$\varphi(x) = \exp\left(-\frac{1}{2}x \cdot Ax\right).$$

Then

$$\widehat{\varphi}(\xi) = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp\left(-\frac{1}{2}\xi \cdot A^{-1}\xi\right).$$

In particular, if X is a real-valued, standard normal random variable, then it has density $f(x) = (2\pi)^{-1/2}e^{-x^2/2}$, and so its characteristic function is $E[e^{itX}] = e^{-t^2/2}$. $\square$

2 Operations on Schwartz functions

Let $\mathbb{Z}_+$ denote the set of nonnegative integers. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ is a multi-index, then the order of α is $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $\varphi \in \mathcal{S}$, then

$$\partial^\alpha \varphi = \frac{\partial^{|\alpha|}\varphi}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}.$$

The reflection of φ is $R\varphi(x) = \varphi(-x)$. The translation of φ by $h \in \mathbb{R}^n$ is $\tau_h\varphi(x) = \varphi(x - h)$.

If $\varphi, \psi \in \mathcal{S}$, then their convolution is

$$\varphi * \psi(x) = \int_{\mathbb{R}^n} \varphi(y)\psi(x-y)\,dy = \int_{\mathbb{R}^n} \varphi(x-y)\psi(y)\,dy.$$

The convolution has the property that if $\varphi, \psi, f \in \mathcal{S}$, then

$$\int_{\mathbb{R}^n} (\varphi * f)(x)\psi(x)\,dx = \int_{\mathbb{R}^n} f(x)((R\varphi) * \psi)(x)\,dx.$$

3 Operations on tempered distributions

If $\alpha \in \mathbb{Z}_+^n$ and $T \in \mathcal{S}^*$, then $\partial^\alpha T$ is the tempered distribution satisfying $\langle \partial^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, \partial^\alpha\varphi\rangle$. The reflection of T is defined by $\langle RT, \varphi\rangle = \langle T, R\varphi\rangle$, and the translation of T by $h \in \mathbb{R}^n$ is defined by $\langle \tau_h T, \varphi\rangle = \langle T, \tau_{-h}\varphi\rangle$. It is easily verified that $\partial^\alpha R = (-1)^{|\alpha|}R\partial^\alpha$ and $\tau_h R = R\tau_{-h}$.

If $\varphi \in \mathcal{S}$ and $T \in \mathcal{S}^*$, then their convolution, $\varphi * T$, is the tempered distribution defined by

$$\langle \varphi * T, \psi\rangle = \langle T, (R\varphi) * \psi\rangle.$$

If $f \in C^\infty(\mathbb{R}^n)$ has polynomial growth and $T \in \mathcal{S}^*$, then fT is the tempered distribution defined by $\langle fT, \varphi\rangle = \langle T, f\varphi\rangle$.

4 Properties of the Fourier transform

The following theorem collects many important properties of the Fourier transform.

Theorem 4.1. *Let $\varphi \in \mathcal{S}$, $T \in \mathcal{S}^*$, $\alpha \in \mathbb{Z}_+^n$, and $h \in \mathbb{R}^n$. Then:*

(a) $\partial^\alpha \widehat{T} = \widehat{(ix)^\alpha T}$.

(b) $\widehat{\partial^\alpha T} = (-i\xi)^\alpha \widehat{T}$.

(c) $\widehat{\tau_h T} = e^{i\xi \cdot h} \widehat{T}$.

(d) $\tau_h \widehat{T} = \widehat{e^{-ix\cdot h} T}$.

(e) $\widehat{\varphi * T} = \widehat{\varphi} \widehat{T}$.

(f) The inverse Fourier transform is given by

$$\mathcal{F}^{-1}\varphi(x) = \check{\varphi}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} \varphi(\xi)\, d\xi,$$

and more generally by $\mathcal{F}^{-1} = (2\pi)^{-n} R\mathcal{F}$.

(g) $\mathcal{F}R = R\mathcal{F}$, $\mathcal{F}^{-1}R = R\mathcal{F}^{-1}$, *and* $\mathcal{F} = (2\pi)^n R\mathcal{F}^{-1}$.

Let $C_0(\mathbb{R}^n)$ denote the space of continuous functions from $\mathbb{R}^n$ to $\mathbb{C}$ that tend to 0 as $|x| \to \infty$. The following is the Riemann-Lebesgue lemma.

Theorem 4.2. *If $f \in L^1(\mathbb{R}^n)$, then $\widehat{f} \in C_0(\mathbb{R}^n)$ and $\|\widehat{f}\|_\infty \leqslant \|f\|_1$.*

Let

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx$$

denote the inner product in $L^2(\mathbb{R}^n)$. The following is the Plancherel theorem.

Theorem 4.3. *If $f, g \in L^2(\mathbb{R}^n)$, then $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}^n)$ and $\langle \widehat{f}, \widehat{g} \rangle_{L^2} = (2\pi)^n \langle f, g \rangle$.*