The Feynman-Kac representation

Jason Swanson

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1 Introduction

Suppose that $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ are continuous functions that satisfy the linear growth condition

$$|b(x)| + |\sigma(x)| \leq K(1 + |x|)$$

for some constant K and all $x \in \mathbb{R}^d$. Consider the stochastic differential equation

$$X(t) = X(0) + \int_0^t \sigma(X(s))\, dB(s) + \int_0^t b(X(s))\, ds. \tag{1.1}$$

We shall assume that for each $x \in \mathbb{R}^d$, there exists a pair of $\mathbb{R}^d$-valued processes, X and B, defined on some probability space $(\Omega, \mathcal{F}, P)$, such that $P(X(0) = x) = 1$, B is a standard d-dimensional Brownian motion under P, and (1.1) is satisfied. We also assume that the law of (X, B) is uniquely determined.

Let L denote the generator of X. That is, L is the differential operator

$$Lf(x) = \frac{1}{2} \sum_{i,j} a_{ij}(x) \partial_{ij}^2 f(x) + b(x)^T \nabla f(x),$$

where $f \in C^2(\mathbb{R}^d)$ and $a(x) = \sigma(x)\sigma(x)^T$.

The Feynman-Kac representation asserts that, under appropriate conditions, the solution to the initial value problem

$$\partial_t u = Lu, \quad u(0, x) = f(x) \tag{1.2}$$

is given by

$$u(t, x) = E^x[f(X(t))]. \tag{1.3}$$

We will first present a heuristic derivation of this result, and then state the full theorem, whose proof can be found in the references.

Suppose that (1.2) has a solution $u(t, x)$. Fix $T > 0$ and define $v(t, x) = u(T - t, x)$. Then $\partial_t v = -Lv$. Define $Y(t) = v(t, X(t))$. By Itô's rule,

$$Y(t) = Y(0) + M(t) + \int_0^t (\partial_t v(s, X(s)) + Lv(s, X(s)))\, ds = Y(0) + M(t),$$

where $M(t)$ is a local martingale. Hence,

$$u(T-t, X(t)) = u(T, X(0)) + M(t).$$

If M is in fact a martingale, then taking expectations gives

$$E^x[u(T-t, X(t))] = u(T, x).$$

Assuming that we can justify letting $t \to T$ under the expectation, this gives

$$E^x[u(0, X(T))] = u(T, x).$$

Since $u(0, x) = f(x)$ and since this is true for all $T > 0$, we have derived (1.3).

2 The Feynman-Kac representation theorem

The full theorem is more general than what is described in the introduction. We will actually consider the initial value problem

$$\partial_t u = Lu - ku + g, \quad u(0, x) = f(x), \tag{2.1}$$

where $k, g : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ are continuous. We assume that $k \geq 0$ and that g satisfies the following growth condition: for each $T > 0$, there exist constants L and r such that

$$\sup_{0 \leq t \leq T} |g(t, x)| \leq L(1 + |x|^r)$$

for all $x \in \mathbb{R}^d$.

Theorem 2.1 *Assume that $u(t, x)$ is continuous on $[0, \infty) \times \mathbb{R}^d$, and that $\partial_t u$ and $\partial_{ij}^2 u$ are continuous on $(0, \infty) \times \mathbb{R}^d$ for all i and j. Assume that u satisfies* (2.1), *and that u satisfies the same growth condition as g. Let*

$$Z(t) = \exp\left\{ -\int_0^t k(s, X(s))\, ds \right\}.$$

Then

$$u(t, x) = E^x\left[f(X(t))Z(t) + \int_0^t g(s, X(s))Z(s)\, ds \right]. \tag{2.2}$$

In particular, such a solution to (2.1) *is unique.*

This is a special case of Theorem 5.7.6 in [1]. (The full result in [1] concerns the case when b and σ also depend on t.) Note that when $k = 0$, we have $Z(t) = 1$ and (2.2) reduces to

$$u(t, x) = E^x\left[f(X(t)) + \int_0^t g(s, X(s))\, ds \right].$$

In particular, if $k = g = 0$, then (2.2) reduces to (1.3).

3 The killed process

The process $X(t)$ can be thought of as representing the location of a particle which is moving about randomly in $\mathbb{R}^d$. In this section, we modify the process X so that the particle is "killed" at a random time ρ. Specifically, we define

$$\rho = \inf\left\{t \geq 0 : \int_0^t k(s, X(s))\, ds \geq \tau\right\}, \tag{3.1}$$

where τ is independent of X and is exponentially distributed with mean 1. The *killed process* is defined as

$$\widetilde{X}(t) = \begin{cases} X(t) & \text{if } t < \rho, \\ \Delta & \text{if } t \geq \rho, \end{cases} \tag{3.2}$$

where Δ is a so-called "cemetery" state which is outside of $\mathbb{R}^d$.

The function $k(t, x)$ is interpreted as the *killing rate*. Informally, this means that if, at time t, the particle is alive and is situated at the point x, then the probability that it dies in the next h units of time is approximately $k(t, x)h$ when h is small. Symbolically,

$$P(\rho \leq t + h | \rho > t, X(t) = x) \approx k(t, x)h. \tag{3.3}$$

To see this more formally, first recall that X is a Markov process with respect to a filtration $\mathcal{F}_t$. Since τ and X are independent,

$$P(\rho > t + h | \mathcal{F}_\infty) = P\left(\int_0^{t+h} k(s, X(s))\, ds < \tau \,\middle|\, \mathcal{F}_\infty\right) = Z(t + h),$$

where Z is defined as in Theorem 2.1. Hence,

$$\begin{aligned} P(\rho > t + h | \mathcal{F}_t) &= E[Z(t + h) | \mathcal{F}_t] \\ &= Z(t) E\left[\exp\left\{-\int_t^{t+h} k(s, X(s))\, ds\right\} \,\middle|\, \mathcal{F}_t\right] \\ &= Z(t) E^{X(t)}\left[\exp\left\{-\int_0^h k(t + s, X(s))\, ds\right\}\right], \end{aligned}$$

where we have used the Markov property in the last equality. Finally, then,

$$\begin{aligned} P(\rho > t + h | X(t)) &= E\left[Z(t) E^{X(t)}\left[\exp\left\{-\int_0^h k(t + s, X(s))\, ds\right\}\right] \,\middle|\, X(t)\right] \\ &= E^{X(t)}\left[\exp\left\{-\int_0^h k(t + s, X(s))\, ds\right\}\right] E[Z(t) | X(t)]. \end{aligned}$$

Therefore,

$$\begin{aligned} P(\rho \leq t + h | \rho > t, X(t) = x) &= 1 - P(\rho > t + h | \rho > t, X(t) = x) \\ &= 1 - \frac{P(\rho > t + h | X(t) = x)}{P(\rho > t | X(t) = x)} \\ &= 1 - E^x\left[\exp\left\{-\int_0^h k(t + s, X(s))\, ds\right\}\right]. \end{aligned}$$

Under suitable conditions on k and X, we may differentiate under the expectation, which yields (3.3) as the first order linear approximation.

The connection between the killed process and the Feynman-Kac representation is given by the following theorem.

Theorem 3.1 *Suppose that* (2.1) *has a solution* u *which satisfies the assumptions of Theorem 2.1. Let* X *denote the solution to* (1.1) *and let* $\widetilde{X}$ *be the killed process given by* (3.1) *and* (3.2)*. Then*

$$u(t,x) = E^x\left[f(\widetilde{X}(t)) + \int_0^t g(s,\widetilde{X}(s))\,ds\right],$$

where f *and* g *are extended so that* $f(\Delta) = 0$ *and* $g(t,\Delta) = 0$*.*

Proof. Let $\varphi(t,x)$ be a measurable function such that $E^x[\varphi(t,X(t))Z(t)]$ exists. Extend φ so that $\varphi(t,\Delta) = 0$. Then

$$\begin{aligned} E^x[\varphi(t,X(t))Z(t)] &= E^x\left[\varphi(t,X(t))P\left(\int_0^t k(s,X(s))\,ds < \tau \middle| \mathcal{F}_\infty\right)\right] \\ &= E^x[\varphi(t,X(t))P(t < \rho | \mathcal{F}_\infty)] \\ &= E^x[\varphi(t,X(t))1_{\{t<\rho\}}] \\ &= E^x[\varphi(t,\widetilde{X}(t))]. \end{aligned}$$

The theorem now follows directly from Theorem 2.1. □

References

[1] Ioannis Karatzas and Steven E. Shreve. *Brownian Motion and Stochastic Calculus.* Springer, 1991.