The Feynman-Kac representation

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1 Introduction

Suppose that \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d \) are continuous functions that satisfy the linear growth condition

\[
|b(x)| + |\sigma(x)| \leq K(1 + |x|)
\]

for some constant \( K \) and all \( x \in \mathbb{R}^d \). Consider the stochastic differential equation

\[
X(t) = X(0) + \int_0^t \sigma(X(s)) dB(s) + \int_0^t b(X(s)) ds.
\] (1.1)

We shall assume that for each \( x \in \mathbb{R}^d \), there exists a pair of \( \mathbb{R}^d \)-valued processes, \( X \) and \( B \), defined on some probability space \( (\Omega, \mathcal{F}, P) \), such that \( P(X(0) = x) = 1 \), \( B \) is a standard \( d \)-dimensional Brownian motion under \( P \), and (1.1) is satisfied. We also assume that the law of \((X, B)\) is uniquely determined.

Let \( L \) denote the generator of \( X \). That is, \( L \) is the differential operator

\[
L f(x) = \frac{1}{2} \sum_{i,j} a_{ij}(x) \partial_{ij}^2 f(x) + b(x)^T \nabla f(x),
\]

where \( f \in C^2(\mathbb{R}^d) \) and \( a(x) = \sigma(x)\sigma(x)^T \).

The Feynman-Kac representation asserts that, under appropriate conditions, the solution to the initial value problem

\[
\partial_t u = Lu, \quad u(0, x) = f(x)
\] (1.2)

is given by

\[
u(t, x) = E^x[f(X(t))].
\] (1.3)

We will first present a heuristic derivation of this result, and then state the full theorem, whose proof can be found in the references.

Suppose that (1.2) has a solution \( u(t, x) \). Fix \( T > 0 \) and define \( v(t, x) = u(T - t, x) \). Then \( \partial_t v = -Lv \). Define \( Y(t) = v(t, X(t)) \). By Itô's rule,

\[
Y(t) = Y(0) + M(t) + \int_0^t (\partial_t v(s, X(s)) + L v(s, X(s))) ds = Y(0) + M(t),
\]
where $M(t)$ is a local martingale. Hence,
\[ u(T - t, X(t)) = u(T, X(0)) + M(t). \]
If $M$ is in fact a martingale, then taking expectations gives
\[ E^x[u(T - t, X(t))] = u(T, x). \]
Assuming that we can justify letting $t \to T$ under the expectation, this gives
\[ E^x[u(0, X(T))] = u(T, x). \]
Since $u(0, x) = f(x)$ and since this is true for all $T > 0$, we have derived (1.3).

2 The Feynman-Kac representation theorem

The full theorem is more general than what is described in the introduction. We will actually consider the initial value problem
\[ \partial_t u = Lu - ku + g, \quad u(0, x) = f(x), \quad (2.1) \]
where $k, g : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ are continuous. We assume that $k \geq 0$ and that $g$ satisfies the following growth condition: for each $T > 0$, there exist constants $L$ and $r$ such that
\[ \sup_{0 \leq t \leq T} |g(t, x)| \leq L(1 + |x|^r) \]
for all $x \in \mathbb{R}^d$.

**Theorem 2.1** Assume that $u(t, x)$ is continuous on $[0, \infty) \times \mathbb{R}^d$, and that $\partial_t u$ and $\partial_{ij}^2 u$ are continuous on $(0, \infty) \times \mathbb{R}^d$ for all $i$ and $j$. Assume that $u$ satisfies (2.1), and that $u$ satisfies the same growth condition as $g$. Let
\[ Z(t) = \exp\left\{ - \int_0^t k(s, X(s)) \, ds \right\}. \]
Then
\[ u(t, x) = E^x\left[ f(X(t))Z(t) + \int_0^t g(s, X(s))Z(s) \, ds \right]. \quad (2.2) \]

In particular, such a solution to (2.1) is unique.

This is a special case of Theorem 5.7.6 in [1]. (The full result in [1] concerns the case when $b$ and $\sigma$ also depend on $t$.) Note that when $k = 0$, we have $Z(t) = 1$ and (2.2) reduces to
\[ u(t, x) = E^x\left[ f(X(t)) + \int_0^t g(s, X(s)) \, ds \right]. \]
In particular, if $k = g = 0$, then (2.2) reduces to (1.3).
3 The killed process

The process $X(t)$ can be thought of as representing the location of a particle which is moving about randomly in $\mathbb{R}^d$. In this section, we modify the process $X$ so that the particle is “killed” at a random time $\rho$. Specifically, we define

$$\rho = \inf\left\{ t \geq 0 : \int_0^t k(s, X(s)) \, ds \geq \tau \right\},$$

(3.1)

where $\tau$ is independent of $X$ and is exponentially distributed with mean 1. The killed process is defined as

$$\tilde{X}(t) = \begin{cases} X(t) & \text{if } t < \rho, \\ \Delta & \text{if } t \geq \rho, \end{cases}$$

(3.2)

where $\Delta$ is a so-called “cemetery” state which is outside of $\mathbb{R}^d$.

The function $k(t, x)$ is interpreted as the killing rate. Informally, this means that if, at time $t$, the particle is alive and is situated at the point $x$, then the probability that it dies in the next $h$ units of time is approximately $k(t, x)h$ when $h$ is small. Symbolically,

$$P(\rho \leq t + h | \rho > t, X(t) = x) \approx k(t, x)h.$$

(3.3)

To see this more formally, first recall that $X$ is a Markov process with respect to a filtration $\mathcal{F}_t$. Since $\tau$ and $X$ are independent,

$$P(\rho > t + h | \mathcal{F}_\infty) = P\left( \int_0^{t+h} k(s, X(s)) \, ds < \tau \bigg| \mathcal{F}_\infty \right) = Z(t+h),$$

where $Z$ is defined as in Theorem 2.1. Hence,

$$P(\rho > t + h | \mathcal{F}_t) = E[Z(t+h) | \mathcal{F}_t]
= Z(t)E\left[ \exp\left\{ - \int_t^{t+h} k(s, X(s)) \, ds \right\} \bigg| \mathcal{F}_t \right]
= Z(t)E^{X(t)}\left[ \exp\left\{ - \int_0^h k(t+s, X(s)) \, ds \right\} \right],$$

where we have used the Markov property in the last equality. Finally, then,

$$P(\rho > t + h | X(t)) = E\left[ Z(t)E^{X(t)}\left[ \exp\left\{ - \int_0^h k(t+s, X(s)) \, ds \right\} \right] \bigg| X(t) \right]
= E^{X(t)}\left[ \exp\left\{ - \int_0^h k(t+s, X(s)) \, ds \right\} \right] E[Z(t) | X(t)].$$

Therefore,

$$P(\rho \leq t + h | \rho > t, X(t) = x) = 1 - P(\rho > t + h | \rho > t, X(t) = x)
= 1 - \frac{P(\rho > t + h | X(t) = x)}{P(\rho > t | X(t) = x)}
= 1 - E^{X(t)}\left[ \exp\left\{ - \int_0^h k(t+s, X(s)) \, ds \right\} \right].$$
Under suitable conditions on $k$ and $X$, we may differentiate under the expectation, which yields (3.3) as the first order linear approximation.

The connection between the killed process and the Feynman-Kac representation is given by the following theorem.

**Theorem 3.1** Suppose that (2.1) has a solution $u$ which satisfies the assumptions of Theorem 2.1. Let $X$ denote the solution to (1.1) and let $\tilde{X}$ be the killed process given by (3.1) and (3.2). Then

$$u(t, x) = E_x \left[ f(\tilde{X}(t)) + \int_0^t g(s, \tilde{X}(s)) ds \right],$$

where $f$ and $g$ are extended so that $f(\Delta) = 0$ and $g(t, \Delta) = 0$.

**Proof.** Let $\varphi(t, x)$ be a measurable function such that $E_x[\varphi(t, X(t))Z(t)]$ exists. Extend $\varphi$ so that $\varphi(t, \Delta) = 0$. Then

$$E_x[\varphi(t, X(t))Z(t)] = E_x \left[ \varphi(t, X(t))P \left( \int_0^t k(s, X(s)) ds < \tau \big| \mathcal{F}_\infty \right) \right]$$

$$= E_x[\varphi(t, X(t))P(t < \rho \big| \mathcal{F}_\infty)]$$

$$= E_x[\varphi(t, X(t))1_{\{t < \rho\}}]$$

$$= E_x[\varphi(t, \tilde{X}(t))].$$

The theorem now follows directly from Theorem 2.1. 

**References**