Moment estimates for Itô diffusions

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Consider the stochastic differential equation,

$$X(t) = \xi + \int_0^t b(X(s), s)\, ds + \int_0^t \sigma(X(s), s)\, dW(s), \tag{1}$$

where $X(t)$ is an $\mathbb{R}^d$-valued process, $b : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^{d \times r}$, W is an r-dimensional, standard Brownian motion, and ξ is an $\mathbb{R}^d$-valued random variable, independent of W. In what follows, $|\cdot|$ will always denote the Euclidean norm. The following is an improved version of [1, Theorem 5.2.9].

Theorem 1. *Suppose there exists a constant K such that*

$$|b(x,t) - b(y,t)| + |\sigma(x,t) - \sigma(y,t)| \leqslant K|x - y|, \text{ and} \tag{2}$$

$$|b(x,t)|^2 + |\sigma(x,t)|^2 \leqslant K^2(1 + |x|^2), \tag{3}$$

for all x, y, t. Also assume that $E|\xi|^{2m} < \infty$ for some real $m \geqslant 1$. Then there exists a continuous process X, adapted to the augmented filtration of W, which is a strong solution of (1). Moreover, for any $T > 0$, there exists a positive constant C, depending only on K, T, and m, such that

$$E|X(t)|^{2m} \leqslant C(1 + E|\xi|^{2m})e^{Ct}, \tag{4}$$

for all $0 \leqslant t \leqslant T$.

In the case $m = 1$, Theorem 1 is just a restatement of [1, Theorem 5.2.9]. We will generalize to $m > 1$ using the Burkholder-Davis-Gundy inequalities, which can be found in [1, Theorem 3.2.28]. The part of that theorem which is relevant to us is included below.

Theorem 2. *For every real $m > 0$, there exists a finite constant K_m such that for all continuous local martingales M and all $t \geqslant 0$,*

$$E|M(t)|^{2m} \leqslant K_m E \langle M \rangle_t^m, \tag{5}$$

where $\langle M \rangle$ is the quadratic variation of M.

Proof of Theorem 1. By Jensen's inequality, $E|\xi|^{2m} < \infty$ implies $E|\xi|^2 < \infty$, so the hypotheses of [1, Theorem 5.2.9] hold. In particular, we have the existence of X, the strong solution of (1).

Let $X_0(t) = \xi$ and

$$X_{k+1}(t) = \xi + \int_0^t b(X_k(s), s)\, ds + \int_0^t \sigma(X_k(s), s)\, dW(s).$$

It is shown in the proof of [1, Theorem 5.2.9] that these processes are well-defined, and for P-a.e. $\omega \in \Omega$, we have $X_k(\cdot, \omega) \to X(\cdot, \omega)$ uniformly on compact subsets of $[0, \infty)$.

In what follows, $\|\cdot\|_p$ will denote the norm in $L^p(\Omega)$, so that $\|Z\|_p = (E[|Z|^p])^{1/p}$. Note that (5) can be reformulated as $\|M(t)\|_{2m}^2 \leqslant K'_m \| \langle M \rangle_t \|_m$, where $K'_m = K_m^{1/m}$.

If c_j are nonnegative constants and $p > 0$, then

$$\left(\sum_{j=1}^n c_j \right)^p \leqslant (n \max\{c_1, \dots, c_n\})^p = n^p \max\{c_1^p, \dots, c_n^p\} \leqslant n^p \sum_{j=1}^n c_j^p. \tag{6}$$

Using this inequality and Theorem 2, we have

$$\begin{aligned} \|X_{k+1}(t)\|_{2m}^2 &\leqslant 9\|\xi\|_{2m}^2 + 9 \left\| \int_0^t b(X_k(s), s)\, ds \right\|_{2m}^2 + 9 \left\| \int_0^t \sigma(X_k(s), s)\, dW(s) \right\|_{2m}^2 \\ &\leqslant 9\|\xi\|_{2m}^2 + 9 \left\| \int_0^t b(X_k(s), s)\, ds \right\|_{2m}^2 + 9K'_m \left\| \int_0^t |\sigma(X_k(s), s)|^2\, ds \right\|_m. \end{aligned}$$

By Minkowski's inequality, the fact that $\|Z^2\|_m = \|Z\|_{2m}^2$, and Jensen's inequality, we have

$$\begin{aligned} \|X_{k+1}(t)\|_{2m}^2 &\leqslant 9\|\xi\|_{2m}^2 + 9 \left(\int_0^t \|b(X_k(s), s)\|_{2m}\, ds \right)^2 + 9K'_m \int_0^t \|\sigma(X_k(s), s)\|_{2m}^2\, ds \\ &\leqslant 9\|\xi\|_{2m}^2 + 9t \int_0^t \|b(X_k(s), s)\|_{2m}^2\, ds + 9K'_m \int_0^t \|\sigma(X_k(s), s)\|_{2m}^2\, ds. \end{aligned}$$

Next, by (3) and (6), we obtain

$$|b(x,t)|^{2m} \leqslant K^{2m}(1 + |x|^2)^m \leqslant 2^m K^{2m}(1 + |x|^{2m}).$$

Thus,

$$\|b(X_k(s), s)\|_{2m}^2 = (E|b(X_k(s), s)|^{2m})^{1/m} \leqslant 2K^2(1 + E|X_k(s)|^{2m})^{1/m}.$$

Using (6) again, we obtain $\|b(X_k(s), s)\|_{2m}^2 \leqslant 4K^2(1 + \|X_k(s)\|_{2m}^2)$, along with the same inequality for $\sigma(X_k(s), s)$. Therefore,

$$\begin{aligned} \|X_{k+1}(t)\|_{2m}^2 &\leqslant 9\|\xi\|_{2m}^2 + 36K^2(t + K'_m) \int_0^t (1 + \|X_k(s)\|_{2m}^2)\, ds \\ &= 9\|\xi\|_{2m}^2 + 36K^2(t + K'_m)t + 36K^2(t + K'_m) \int_0^t \|X_k(s)\|_{2m}^2\, ds. \end{aligned}$$

Now let $T > 0$ and let $c = \max\{9, 36K^2(T + K'_m)T, 36K^2(T + K'_m)\}$. Then

$$\|X_{k+1}(t)\|_{2m}^2 \leqslant c(1 + \|\xi\|_{2m}^2) + c \int_0^t \|X_k(s)\|_{2m}^2\, ds,$$

for all $0 \leqslant t \leqslant T$. It is straightforward to show by induction that this implies

$$\|X_{k+1}(t)\|_{2m}^2 \leqslant c(1 + \|\xi\|_{2m}^2) \sum_{j=0}^{k+1} \frac{(ct)^j}{j!} \leqslant c(1 + \|\xi\|_{2m}^2)e^{ct},$$

for all $0 \leqslant t \leqslant T$ and all $k \geqslant 0$. Letting $k \to \infty$ and using Fatou's lemma shows that

$$\|X(t)\|_{2m}^2 \leqslant c(1 + \|\xi\|_{2m}^2)e^{ct},$$

for all $0 \leqslant t \leqslant T$. Using (6), this gives

$$E|X(t)|^{2m} \leqslant c^m(1 + \|\xi\|_{2m}^2)^m e^{mct} \leqslant 2^m c^m (1 + \|\xi\|_{2m}^{2m})e^{mct}.$$

Letting $C = \max\{(2c)^m, mc\}$ yields (4). □

Corollary 3. *Under the conditions of Theorem 1, for any $T > 0$, there exists a constant C', depending only on K and T such that*

$$E|X(t)|^r \leqslant C'(1 + (E|\xi|^2)^{r/2})e^{C't},$$

for all $0 \leqslant t \leqslant T$ and all $0 < r < 2$.

Proof. Assume the hypotheses of Theorem 1 hold for some $m \geqslant 1$. By Jensen's inequality, we may assume $m = 1$ without loss of generality. Let $0 < r < 2$. By Jensen's inequality, (4), and (6), we have

$$\begin{aligned} E|X(t)|^r &\leqslant (E|X(t)|^2)^{r/2} \\ &\leqslant C^{r/2}(1 + \|\xi\|_2^2)^{r/2} e^{rCt/2} \\ &\leqslant (2C)^{r/2}(1 + \|\xi\|_2^r)e^{Ct} \\ &\leqslant (2C + 1)(1 + \|\xi\|_2^r)e^{(2C+1)t}. \end{aligned}$$

Taking $C' = 2C + 1$ finished the proof. □

References

[1] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.