Elementary properties of functions with one-sided limits

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1 Definition and basic properties

Let (X, d) be a metric space. A function $f : \mathbb{R} \to X$ is said to have one-sided limits if, for each $t \in \mathbb{R}$, the limits $f(t+) = \lim_{s\to t^+} f(s)$ and $f(t-) = \lim_{s\to t^-} f(s)$ both exist. These functions are more well-behaved than one might initially expect, as the following theorems demonstrate.

Theorem 1.1. *A function with one-sided limits is bounded on compact sets.*

Proof. Let f have one-sided limits and let $K \subset \mathbb{R}$ be compact. Fix any $p \in X$. We want to show that there exists $r > 0$ such that $f(K) \subset B_r(p)$.

Fix $t \in K$. Since $f(t+)$ exists, there exists $\delta_{t+} > 0$ such that $d(f(s), f(t+)) < 1$ for all $s \in (t, t + \delta_{t+})$. Thus, if $r_{t+} = 1 + d(f(t+), p)$, then

$$d(f(s), p) \le d(f(s), f(t+)) + d(f(t+), p) < r_{t+}.$$

In other words, $f(s) \in B_{r_{t+}}(p)$, for all $s \in (t, t + \delta_{t+})$.

Similarly, since $f(t-)$ exists, there exists $\delta_{t-} > 0$ such that $f(s) \in B_{r_{t-}}(p)$ for all $s \in (t - \delta_{t-}, t)$, where $r_{t-} = 1 + d(f(t-), p)$. Thus, for all $s \in U_t = (t - \delta_{t-}, t + \delta_{t+})$, we have that $f(s) \in B_{r_t}(p)$, where $r_t = \max\{r_{t-}, r_{t+}, d(f(t), p) + 1\}$.

Since $\{U_t : t \in K\}$ is an open cover of K, there exists $\{t_1, \dots, t_n\} \subset K$ such that $K \subset U_1 \cup \dots \cup U_n$. It follows that, for all $s \in K$, we have $f(s) \in B_r(p)$, where $r = \max\{r_{t_1}, \dots, r_{t_n}\}$. That is, $f(K) \subset B_r(p)$. □

This next theorem shows that a function with one-sided limits cannot have large discontinuities which accumulate.

Theorem 1.2. *Let f have one-sided limits. Then for all $t \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d(f(s+), f(s)) + d(f(s), f(s-)) < \varepsilon,$$

whenever $s \neq t$ and $|t - s| < \delta$.

Proof. Suppose not. Then there exists $t \in \mathbb{R}$, $\varepsilon > 0$, and a sequence $\{s_n\}$ of real numbers such that, for all n, we have $s_n \neq t$, $|t - s_n| < 1/n$, and

$$d(f(s_n+), f(s_n)) + d(f(s_n), f(s_n-)) \ge \varepsilon,$$

Consider the following four sets:

$$\begin{aligned}
S_1 &= \{n : s_n > t \text{ and } d(f(s_n+), f(s_n)) \geq \varepsilon/2\}, \\
S_2 &= \{n : s_n > t \text{ and } d(f(s_n), f(s_n-)) \geq \varepsilon/2\}, \\
S_3 &= \{n : s_n < t \text{ and } d(f(s_n+), f(s_n)) \geq \varepsilon/2\}, \\
S_4 &= \{n : s_n < t \text{ and } d(f(s_n), f(s_n-)) \geq \varepsilon/2\}.
\end{aligned}$$

Since these sets cover $\mathbb{N}$, at least one of them is infinite. By passing to a subsequence, we may assume that the entire sequence $\{s_n\}$ is contained in one of these sets.

First assume that each $s_n \in S_1$. For each n, choose $u_n \in (s_n, s_n + 1/n)$ such that $d(f(u_n), f(s_n+)) < \varepsilon/4$. Then

$$\frac{\varepsilon}{2} \leq d(f(s_n+), f(s_n)) \leq d(f(s_n+), f(u_n)) + d(f(u_n), f(s_n)) < \frac{\varepsilon}{4} + d(f(u_n), f(s_n)).$$

But $s_n \to t^+$ and $u_n \to t^+$, so $d(f(u_n), f(s_n)) \to d(f(t+), f(t+)) = 0$, a contradiction.

Next assume that each $s_n \in S_2$. For each n, choose $u_n \in (t, s_n)$ such that $d(f(u_n), f(s_n-)) < \varepsilon/4$. Then

$$\frac{\varepsilon}{2} \leq d(f(s_n), f(s_n-)) \leq d(f(s_n), f(u_n)) + d(f(u_n), f(s_n-)) < d(f(u_n), f(s_n)) + \frac{\varepsilon}{4}.$$

But $s_n \to t^+$ and $u_n \to t^+$, so $d(f(u_n), f(s_n)) \to d(f(t+), f(t+)) = 0$, a contradiction.

Next assume that each $s_n \in S_3$. For each n, choose $u_n \in (s_n, t)$ such that $d(f(u_n), f(s_n+)) < \varepsilon/4$. Then

$$\frac{\varepsilon}{2} \leq d(f(s_n+), f(s_n)) \leq d(f(s_n+), f(u_n)) + d(f(u_n), f(s_n)) < \frac{\varepsilon}{4} + d(f(u_n), f(s_n)).$$

But $s_n \to t^-$ and $u_n \to t^-$, so $d(f(u_n), f(s_n)) \to d(f(t-), f(t-)) = 0$, a contradiction.

Finally assume that each $s_n \in S_4$. For each n, choose $u_n \in (s_n - 1/n, s_n)$ such that $d(f(u_n), f(s_n-)) < \varepsilon/4$. Then

$$\frac{\varepsilon}{2} \leq d(f(s_n), f(s_n-)) \leq d(f(s_n), f(u_n)) + d(f(u_n), f(s_n-)) < d(f(u_n), f(s_n)) + \frac{\varepsilon}{4}.$$

But $s_n \to t^-$ and $u_n \to t^-$, so $d(f(u_n), f(s_n)) \to d(f(t-), f(t-)) = 0$, a contradiction. $\square$

Theorem 1.3. *A function with one-sided limits has at most countably many discontinuities.*

Proof. Let f have one-sided limit. Then f is continuous at t if and only if $f(t-) = f(t+) = f(t)$, which happens if and only if

$$d(f(t+), f(t)) + d(f(t), f(t-)) = 0.$$

Let

$$A_n = \{t \in \mathbb{R} : d(f(t+), f(t)) + d(f(t), f(t-)) \geq 1/n\}.$$

Then $A = \bigcup_{n=1}^{\infty} A_n$ is the set of discontinuities of f.

Fix $M, n \in \mathbb{N}$. Fix $t \in [-M, M]$. By Theorem 1.2 with $\varepsilon = 1/n$, there exists $\delta_t > 0$ such that $((t - \delta_t, t) \cup (t, t + \delta_t)) \cap A_n = \emptyset$. Thus, if $U_t = (t - \delta_t, t + \delta_t)$, then $U_t \cap A_n \subset \{t\}$. Since $[-M, M]$ is compact, and $\{U_t : t \in [-M, M]\}$ is an open cover of $[-M, M]$, it follows that there exists $\{t_1, \ldots, t_k\} \subset [-M, M]$ such that $[-M, M] \subset U_{t_1} \cup \cdots \cup U_{t_k}$. Hence, $[-M, M] \cap A_n \subset \{t_1, \ldots, t_k\}$. In particular, $[-M, M] \cap A_n$ is finite.

Therefore,

$$A = \bigcup_{n=1}^{\infty} \bigcup_{M=1}^{\infty} [-M, M] \cap A_n$$

is a countable set. □

2 Càdlàg functions

If f has one-sided limits, we define $f_+ : \mathbb{R} \to \mathbb{R}$ and $f_- : \mathbb{R} \to \mathbb{R}$ by $f_+(t) = f(t+)$ and $f_-(t) = f(t-)$. Note that a function f with one-sided limits is right-continuous if and only if $f(t+) = f(t)$ for all $t \in \mathbb{R}$, which is equivalent to saying that $f_+ = f$. If f has one-sided limits and is right-continuous, then we say that f is càdlàg. This is an acronym for the French phrase, "continu à droite, limite à gauche". If f has one-sided limits and is left-continuous, that is, if $f(t-) = f(t)$ for all $t \in \mathbb{R}$ (which is equivalent to $f_- = f$), then we say that f is càglàd.

If f has one-sided limits, we also define the function $\Delta f : \mathbb{R} \to \mathbb{R}$ by $\Delta f = f_+ - f_-$. Note that, by Theorem 1.3, the set $\{t : \Delta f(t) \neq 0\}$ is countable.

Given any $f : \mathbb{R} \to \mathbb{R}$, let us define $Rf : \mathbb{R} \to \mathbb{R}$ by $Rf(t) = f(-t)$.

Lemma 2.1. *If f has one-sided limits, then so does Rf. Moreover, $(Rf)_+ = Rf_-$ and $(Rf)_- = Rf_+$.*

Proof. Let f have one-sided limits. Then

$$\lim_{s \to t^+} Rf(s) = \lim_{s \to t^+} f(-s) = \lim_{z \to (-t)^-} f(z) = f_-(-t),$$

and

$$\lim_{s \to t^-} Rf(s) = \lim_{s \to t^-} f(-s) = \lim_{z \to (-t)^+} f(z) = f_+(-t),$$

which shows that Rf has one-sided limits, and that $(Rf)_+ = Rf_-$ and $(Rf)_- = Rf_+$. □

Lemma 2.2. *If $f : \mathbb{R} \to \mathbb{R}$ is nondecreasing, then f has one-sided limits, and f_+ and f_- are both nondecreasing.*

Proof. Fix $t \in \mathbb{R}$ and fix some strictly increasing sequence $\{t_n\}$ with $t_n \to t$. Then $\{f(t_n)\}$ is a nondecreasing sequence of real numbers, bounded above by $f(t)$. Hence, there exists $L \in \mathbb{R}$ such that $f(t_n) \to L$.

Now let $\{s_n\}$ be any other strictly increasing sequence with $s_n \to t$. As above, $f(s_n) \to L'$ for some $L' \in \mathbb{R}$. Now fix $m \in \mathbb{N}$. Since $s_m < t$ and $t_n \to t$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $s_m < t_n$. This implies $f(s_m) \leq f(t_n)$. Letting $n \to \infty$, we have $f(s_m) \leq L$. But this holds for all m, so letting $m \to \infty$, we have $L' \leq L$. A similar argument shows that $L \le L'$. Thus, $L' = L$, so that $f(s_n) \to L$. Since this holds for any such sequence $\{s_n\}$, we have that $L = \lim_{s \to t^-} f(s)$, and so $f(t-)$ exists.

A similar argument shows that $f(t+)$ exists for all $t \in \mathbb{R}$.

Now let $s < t$. Choose a strictly decreasing sequence $\{s_n\} \subset (s, t)$ such that $s_n \to s$, and choose a strictly decreasing sequence $\{t_n\}$ such that $t_n \to t$. Then $s_n < t < t_n$ for all n. Hence, $f(s_n) \le f(t_n)$ for all n. Letting $n \to \infty$ gives $f(s+) \le f(t+)$, showing that f_+ is nondecreasing. A similar argument shows that f_- is nondecreasing. □

Theorem 2.3. *If f has one-sided limits, then*

$$\begin{aligned} f_+(t+) &= f_-(t+) = f(t+), \text{ and} \\ f_+(t-) &= f_-(t-) = f(t-), \end{aligned}$$

for all $t \in \mathbb{R}$. In other words, $(f_+)_+ = (f_-)_+ = f_+$ and $(f_+)_- = (f_-)_- = f_-$. In particular, f_+ is càdlàg and f_- is càglàd.

Proof. Fix $t \in \mathbb{R}$ and let $\{t_n\}$ be a strictly decreasing sequence of real numbers such that $t_n \to t$. Let $\varepsilon > 0$ be arbitrary. Using Theorem 1.2 and the fact that $f(t_n) \to f(t+)$, we may choose $N \in \mathbb{N}$ such that for all $n \ge N$, we have $d(f(t_n+), f(t_n)) < \varepsilon$, $d(f(t_n-), f(t_n)) < \varepsilon$, and $d(f(t_n), f(t+)) < \varepsilon$. By the triangle inequality, this implies that $d(f(t_n+), f(t+)) < 2\varepsilon$ and $d(f(t_n-), f(t+)) < 2\varepsilon$. Since ε was arbitrary, this shows that $f_+(t_n) = f(t_n+) \to f(t+)$ and $f_-(t_n) = f(t_n-) \to f(t+)$. Since the sequence $\{t_n\}$ was arbitrary, this shows that $f_+(t+) = f(t+)$ and $f_-(t+) = f(t+)$. Since this holds for all $t \in \mathbb{R}$, we have $(f_+)_+ = (f_-)_+ = f_+$.

Now let $g = Rf$. We have already shown that $(g_+)_+ = (g_-)_+ = g_+$. By Lemma 2.1, we have $g_+ = Rf_-$. Therefore, $(g_+)_+ = R(f_-)_-$ and similarly, $(g_-)_+ = R(f_+)_-$. Hence, $R(f_+)_- = R(f_-)_- = Rf_-$, which implies $(f_+)_- = (f_-)_- = f_-$.

Lastly, since $(f_+)_+ = f_+$, it follows that f_+ is càdlàg, and since $(f_-)_- = f_-$, it follows that f_- is càglàd. □

Remark 2.4. By Theorem 2.3, if f is càdlàg (or any function with one-sided limits), then $g = f_-$ is càglàd. Conversely, if g is any càglàd function, then $g = g_- = (g_+)_-$. In other words, $g = f_-$, where $f = g_+$ is a càdlàg function. What this shows is that a function $g : \mathbb{R} \to \mathbb{R}$ is càglàd if and only if $g = f_-$ for some càdlàg function f.

3 Functions of bounded variation

Given $G : [a, b] \to \mathbb{R}$ we define

$$T_G(a, b) = \sup \left\{ \sum_{j=1}^{n} |G(t_j) - G(t_{j-1})| : n \in \mathbb{N}, a = t_0 < \cdots < t_n = b \right\}.$$

The quantity $T_G(a, b)$ is called the total variation of G on $[a, b]$. If $T_G(a, b) < \infty$, then G is said to have bounded variation on $[a, b]$, and the set of all such functions G is denoted by $BV[a, b]$.

Given $G : \mathbb{R} \to \mathbb{R}$, we define $T_G = \sup\{T_G(a, b) : -\infty < a < b < \infty\}$. If $T_G < \infty$, then we say that G has bounded variation, and the set of all such functions G is denoted by BV.

Suppose $G : [a, b] \to \mathbb{R}$ is in $BV[a, b]$. Let us extend G to be defined on the whole real line by defining $G(t) = G(a)$ for $t < a$, and $G(t) = G(b)$ for $t > b$. Then $G \in BV$ and $T_G = T_G(a, b)$. In this way, all facts about the set BV give rise to corresponding facts about the set $BV[a, b]$. We will therefore focus our attention on the set BV.

A function $G : \mathbb{R} \to \mathbb{R}$ is in BV if and only if G can be written as $G = G_1 - G_2$, where each G_j is a bounded, nondecreasing function (see Theorem 3.27 in [1]). Since nondecreasing functions have one-sided limits, every BV function has one-sided limits. Moreover, by Lemma 2.2, this shows that G_+ and G_- are both BV functions.

If $G \in BV$, then there exists a unique signed Borel measure μ_{G_+} on $\mathbb{R}$ such that $\mu_{G_+}((s, t]) = G_+(t) - G_+(s)$ for all $s < t$ (see Theorem 3.29 in [1]). Note that

$$\begin{aligned}
\mu_{G+}(\{t\}) &= \lim_{s \to t^-} \mu_{G_+}((s, t]) \\
&= \lim_{s \to t^-} (G_+(t) - G_+(s)) \\
&= G_+(t) - G_+(t-) \\
&= G_+(t) - G_-(t),
\end{aligned}$$

by Theorem 2.3. If we recall that $\Delta G = G_+ - G_-$, then $\mu_{G_+}(\{t\}) = \Delta G(t)$.

We define the Lebesgue-Stieltjes integral of a Borel measurable function f with respect to a BV function G by

$$\int_A f \, dG = \int_A f \, d\mu_{G_+}.$$

The following theorem illustrates a relationship between the Lebesgue-Stieltjes integral and classical Riemann sums.

Theorem 3.1. *Let $G \in BV$ and let f be a function with one-sided limits. Fix $a < b$. For each $m \in \mathbb{N}$, let $\mathcal{P}_m = \{t_j^{(m)}\}_{j=0}^{n(m)}$ be a strictly increasing, finite sequence of real numbers with $a = t_0^{(m)} < \ldots < t_{n(m)}^{(m)} = b$. Assume that $\|\mathcal{P}_m\| = \max\{|t_j^{(m)} - t_{j-1}^{(m)}| : 1 \le j \le n(m)\} \to 0$ as $m \to \infty$. Let*

$$\begin{aligned}
I_-^{(m)} &= \sum_{j=1}^{n(m)} f(t_{j-1}^{(m)})(G(t_j^{(m)}) - G(t_{j-1}^{(m)})), \text{ and} \\
I_+^{(m)} &= \sum_{j=1}^{n(m)} f(t_j^{(m)})(G(t_j^{(m)}) - G(t_{j-1}^{(m)})).
\end{aligned}$$

Then:

(i) If G is càdlàg, then $I_-^{(m)} \to \int_{(a,b]} f_- \, dG$ as $m \to \infty$.

(ii) If f and G are both càdlàg, then $I_+^{(m)} \to \int_{(a,b]} f_+ \, dG = \int_{(a,b]} f \, dG$ as $m \to \infty$.

(iii) If G is càglàd, then $I_+^{(m)} \to \int_{[a,b)} f_+ \, dG$ as $m \to \infty$.

(iv) If f and G are both càglàd, then $I_-^{(m)} \to \int_{[a,b)} f_- \, dG = \int_{[a,b)} f \, dG$ as $m \to \infty$.

Proof. In this proof, for notational simplicity, we will suppress the dependence of n, t_j, and $I_\pm$ on m.

Let us first assume that G is càdlàg. Then $G = G_+$, and so

$$I_- = \sum_{j=1}^n f(t_{j-1}) \mu_{G+}((t_{j-1}, t_j]) = \int_{(a,b]} f_m^{(1)} \, d\mu_{G_+},$$

where

$$f_m^{(1)}(t) = \sum_{j=1}^n f(t_{j-1}) 1_{(t_{j-1}, t_j]}(t).$$

For each fixed t, we have $f_m^{(1)}(t) = f(t_{j-1})$, where $t_{j-1} < t$ and $|t - t_{j-1}| \le \|\mathcal{P}_m\|$. Thus, $f_m^{(1)}(t) \to f_-(t)$. By Theorem 1.1, there exists $M < \infty$ such that $|f_m^{(1)}| \le M$ for all m. Thus, by dominated convergence, $I_- \to \int_{(a,b]} f_- \, d\mu_{G_+} = \int_{(a,b]} f_- \, dG$, and this proves (i).

Similarly,

$$I_+ = \sum_{j=1}^n f(t_j) \mu_{G+}((t_{j-1}, t_j]) = \int_{(a,b]} f_m^{(2)} \, d\mu_{G_+},$$

where

$$f_m^{(2)}(t) = \sum_{j=1}^n f(t_j) 1_{(t_{j-1}, t_j]}(t).$$

For each fixed t, we have $f_m^{(2)}(t) = f(t_j)$, where $t \le t_j$ and $|t - t_j| \le \|\mathcal{P}_m\|$. Because of the possibility that $t = t_j$, we cannot conclude that $f_m^{(2)}(t) \to f_+(t)$ as $m \to \infty$. However, if we make the further assumption that f is càdlàg, so that $f_+ = f$, then we do obtain $f_m^{(2)}(t) \to f(t)$ as $m \to \infty$, and again by dominated convergence, we have $I_+ \to \int_{(a,b]} f \, d\mu_{G_+} = \int_{(a,b]} f \, dG$, and this proves (ii).

Next assume that G is càglàd. Then $G = G_- = G_+ - \Delta G$, and so for any $s < t$, we have

$$\begin{aligned} G(t) - G(s) &= G_+(t) - G_+(s) - \Delta G(t) + \Delta G(s) \\ &= \mu_{G_+}((s,t]) - \mu_{G_+}(\{t\}) + \mu_{G_+}(\{s\}) \\ &= \mu_{G_+}([s,t)). \end{aligned}$$

Hence,

$$I_+ = \sum_{j=1}^n f(t_j) \mu_{G+}([t_{j-1}, t_j)) = \int_{[a,b)} f_m^{(3)} \, d\mu_{G_+},$$

where

$$f_m^{(3)}(t) = \sum_{j=1}^n f(t_j) 1_{[t_{j-1}, t_j)}(t).$$

For each fixed t, we have $f_m^{(3)}(t) = f(t_j)$, where $t < t_j$ and $|t - t_j| \leq \|\mathcal{P}_m\|$. Thus, $f_m^{(3)}(t) \to f_+(t)$. Again, by dominated convergence, $I_+ \to \int_{[a,b)} f_+ \, d\mu_{G_+} = \int_{[a,b)} f_+ \, dG$, and this proves (iii).

Similarly,

$$I_- = \sum_{j=1}^{n} f(t_{j-1})\mu_{G+}([t_{j-1}, t_j)) = \int_{[a,b)} f_m^{(4)} \, d\mu_{G_+},$$

where

$$f_m^{(4)}(t) = \sum_{j=1}^{n} f(t_{j-1}) 1_{[t_{j-1}, t_j)}(t).$$

For each fixed t, we have $f_m^{(4)}(t) = f(t_{j-1})$, where $t_{j-1} \leq t$ and $|t - t_j| \leq \|\mathcal{P}_m\|$. Because of the possibility that $t = t_{j-1}$, we cannot conclude that $f_m^{(4)}(t) \to f_-(t)$ as $m \to \infty$. However, if we make the further assumption that f is càglàd, so that $f_- = f$, then we do obtain $f_m^{(4)}(t) \to f(t)$ as $m \to \infty$, and again by dominated convergence, we have $I_- \to \int_{[a,b)} f \, d\mu_{G_+} = \int_{[a,b)} f \, dG$, and this proves (iv). $\square$

Remark 3.2. In (ii) and (iv) of Theorem 3.1, the assumptions on f cannot be omitted. For example, let $f = 1_{(0,\infty)}$, $G = 1_{[0,\infty)}$, $a = -1$, and $b = 1$. In this case, G is càdlàg and f is càglàd, but $I_+^{(m)}$ need not converge to anything.

To see this, let $\{\mathcal{P}_m\}$ be a sequence of partitions with $\|\mathcal{P}_m\| \to 0$, satisfying the following conditions:

(i) If m is even, then there exists $k = k(m)$ such that $t_k^{(m)} = 0$.

(ii) If m is odd, then then there exists $k = k(m)$ such that $t_{k-1}^{(m)} < 0 < t_k^{(m)}$.

In this case, $G(t_j^{(m)}) - G(t_{j-1}^{(m)}) = 1$ if $j = k(m)$, and 0 otherwise. Thus,

$$I_+^{(m)} = f(t_k^{(m)}) = \begin{cases} 0 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd,} \end{cases}$$

and so $I_+^{(m)}$ does not converge. Similarly, if $f = 1_{[0,\infty)}$, $G = 1_{(0,\infty)}$, and $\{\mathcal{P}_m\}$ are the above partitions, then $I_-^{(m)}$ does not converge.

Remark 3.3. If f and G are both càdlàg, then

$$\int_{(a,b]} f \, dG - \int_{(a,b]} f_- \, dG = \int_{(a,b]} (f - f_-) \, dG = \int_{(a,b]} \Delta f \, d\mu_{G_+}.$$

Since Δf vanishes outside a countable set, we have

$$\int_{(a,b]} \Delta f \, d\mu_{G_+} = \sum_{t \in (a,b]} \Delta f(t) \mu_{G_+}(\{t\}) = \sum_{t \in (a,b]} \Delta f(t) \Delta G(t),$$

where this sum is, in fact, a countable sum. In particular, this shows that $I_-^{(m)}$ and $I_+^{(m)}$ need not converge to the same limit. More specifically,

$$I_+^{(m)} - I_-^{(m)} = \sum_{j=1}^{n} (f(t_j) - f(t_{j-1}))(G(t_j) - G(t_{j-1})) \to \sum_{t \in (a,b]} \Delta f(t) \Delta G(t),$$

as $m \to \infty$. This quantity is called the covariation of f and G.

If f has one-sided limits, but is not of bounded variation, then the integral $\int_{(a,b]} G\, df$ is undefined. More specifically, the map $(s,t] \mapsto f_+(t) - f_+(s)$ cannot be extended to a signed measure. But, even though the integral is undefined, we can still obtain convergence of the Riemann sums in Theorem 3.1, provided that the integrand is of bounded variation.

Theorem 3.4. *Let $G \in BV$ and let f be a function with one-sided limits. Assume f and G are both càdlàg. Fix $a < b$. For each $m \in \mathbb{N}$, let $\mathcal{P}_m = \{t_j^{(m)}\}_{j=0}^{n(m)}$ be a strictly increasing, finite sequence of real numbers with $a = t_0^{(m)} < \ldots < t_{n(m)}^{(m)} = b$. Assume that $\|\mathcal{P}_m\| = \max\{|t_j^{(m)} - t_{j-1}^{(m)}| : 1 \le j \le n(m)\} \to 0$ as $m \to \infty$. Let*

$$J_-^{(m)} = \sum_{j=1}^{n(m)} G(t_{j-1}^{(m)})(f(t_j^{(m)}) - f(t_{j-1}^{(m)})), \text{ and}$$
$$J_+^{(m)} = \sum_{j=1}^{n(m)} G(t_j^{(m)})(f(t_j^{(m)}) - f(t_{j-1}^{(m)})).$$

Then

$$J_-^{(m)} \to f(b)G(b) - f(a)G(b) - \int_{(a,b]} f_-\, dG - \sum_{t \in (a,b]} \Delta f(t) \Delta G(t), \text{ and} \tag{3.1}$$

$$J_+^{(m)} \to f(b)G(b) - f(a)G(b) - \int_{(a,b]} f\, dG + \sum_{t \in (a,b]} \Delta f(t) \Delta G(t), \tag{3.2}$$

as $m \to \infty$.

Proof. As before, for notational simplicity, we will suppress the dependence of n, t_j, and $J_\pm$ on m.

We begin by observing that

$$\begin{aligned} J_- &= \sum_{j=1}^{n} G(t_{j-1}) f(t_j) - \sum_{j=0}^{n-1} G(t_j) f(t_j) \\ &= f(b)G(b) - f(a)G(a) - \sum_{j=1}^{n} f(t_j)(G(t_j) - G(t_{j-1})). \end{aligned}$$

By Theorem 3.1, we have $J_- \to f(b)G(b) - f(a)G(b) - \int_{(a,b]} f\,dG$. By Remark 3.3, this prove (3.1).

Next, we write

$$\begin{aligned} J_+ &= \sum_{j=2}^{n+1} G(t_{j-1})f(t_{j-1}) - \sum_{j=1}^{n} G(t_j)f(t_{j-1}) \\ &= f(b)G(b) - f(a)G(a) - \sum_{j=1}^{n} f(t_{j-1})(G(t_j) - G(t_{j-1})). \end{aligned}$$

By Theorem 3.1, we have $J_+ \to f(b)G(b) - f(a)G(b) - \int_{(a,b]} f_-\,dG$. By Remark 3.3, this prove (3.2). □

As a corollary, we obtain the following integration-by-parts formulas.

Corollary 3.5. *If f and G are both càdlàg functions of bounded variation, then*

$$\begin{aligned} \int_{(a,b]} G_-\,df &= f(b)G(b) - f(a)G(a) - \int_{(a,b]} f_-\,dG - \sum_{t\in(a,b]} \Delta f(t)\Delta G(t), \text{ and} \\ \int_{(a,b]} G\,df &= f(b)G(b) - f(a)G(a) - \int_{(a,b]} f\,dG + \sum_{t\in(a,b]} \Delta f(t)\Delta G(t). \end{aligned}$$

Proof. Combine Theorem 3.4 with Theorem 3.1. □

4 The Stratonovich integral for càdlàg functions

If g and h are càdlàg, with $h \in BV$, then let us define the Stratonovich integral of g with respect to h as

$$\int_0^t g(s) \circ dh(s) := \int_{(0,t]} \frac{g_- + g}{2}\,dh$$

By Theorem 3.1, we have

$$\sum_{j=1}^{n} \frac{g(t_{j-1}) + g(t_j)}{2}(h(t_j) - h(t_j)) \to \int_0^t g(s) \circ dh(s),$$

as the mesh of the partition tends to zero.

Theorem 4.1. *Let f, g, and h be càdlàg functions, with $h \in BV$. Let*

$$k(t) = \int_0^t g(s) \circ dh(s).$$

Then k is càdlàg, $k \in BV$, and

$$\int_0^t f(s) \circ dk(s) = \int_0^t f(s)g(s) \circ dh(s) - \frac{1}{4} \sum_{s\in(0,t]} \Delta f(s)\Delta g(s)\Delta h(s).$$

Proof. Since

$$k(t) = \int_{(0,t]} \frac{g_- + g}{2}\, dh,$$

we have

$$\begin{aligned}
\int_0^t f(s) \circ dk(s) = \int_{(0,t]} \frac{f_- + f}{2}\, dk &= \int_{(0,t]} \left(\frac{f_- + f}{2}\right)\left(\frac{g_- + g}{2}\right) dh \\
&= \int_{(0,t]} \left(\frac{f_- g_- + fg}{2} - \frac{(f - f_-)(g - g_-)}{4}\right) dh \\
&= \int_{(0,t]} \frac{f_- g_- + fg}{2}\, dh - \frac{1}{4}\int_{(0,t]} \Delta f \Delta g\, dh \\
&= \int_0^t f(s)g(s) \circ dh(s) - \frac{1}{4}\sum_{s\in(0,t]} \Delta f(s)\Delta g(s)\Delta h(s).
\end{aligned}$$

□

Remark 4.2. This theorem shows that as long as f, g, and h have no simultaneous discontinuities, then the Stratonovich integral satisfies the usual transformation rule of calculus that if $dk = g \circ dh$, then $f \circ dk = fg \circ dh$. In general, however, the transformation rule involves a correction term which represents the triple covariation of the three functions, f, g, and h.

References

[1] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*. Wiley-Interscience, 1999.