Supplemental theorems for the Wick product approach to SPDEs

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1 Introduction

A "standard" approach to SPDEs can be found in Walsh [11]. Upon first reading Walsh's notes, I became stuck at the point where I needed to understand the notion of an integral of a real-valued function against a vector-valued measure. Walsh refers to this as the "Bochner integral." Diligently, I studied all references to the Bochner integral that I could find. Strangely, though, every one of them defined the Bochner integral as an integral of a vector-valued function against a real-valued measure, just the opposite of what Walsh meant. At the time, I assumed there must be some alternate definition of the Bochner integral hidden in some obscure reference that I had yet to discover. Only later did I come to realize that the integral Walsh is using is not a Bochner integral. Rather than studying texts and papers on the Bochner integral, I should have been reading the literature on vector-valued measures. One such reference is Diestel and Uhl [4].

Seeking an alternative approach to SPDEs, I first considered Da Prato and Zabczyk [3], who give a more functional-analytic approach to the subject. At the time, I found that material quite difficult. Soon, however, I discovered Holden, Øksendal, Ubøe, and Zhang [7]. They present an approach to SPDEs which uses the Wick product. The text was not too difficult to digest and is mostly self-contained. A few key preliminary results, however, are missing from the text and, yet, are essential to the development of the material. The following notes are a summary of some of these needed results complete with references, and proofs where references are unavailable. These notes are meant to be read in conjunction with [7].

2 Measure Theory

To paraphrase Laurent Schwartz in the preface to [10], the standard approach to measure theory consists of two parts: the general theory on arbitrary measure spaces and the theory of Radon measures on locally compact Hausdorff spaces. The latter is a much more powerful theory, but unfortunately the spaces that arise in probability are not locally compact. For this reason, the theory of Radon measures on arbitrary topological spaces is needed in the study of SPDEs. Some of the required theorems are included here.

Let $\mathcal{P}$ and $\mathcal{E}$ be the algebras defined in the remark preceding Theorem 2.1.3 in [7]. Moreover, let $\mathcal{E}_r$ denote the algebra generated by functions $f : \mathcal{S}'(\mathbb{R}^d) \to \mathbb{R}$ of the form

$$f(\omega) = \exp[\langle \omega, \phi \rangle] \text{ where } \phi \in \mathcal{S}(\mathbb{R}^d).$$

If $\eta = \{\eta_j\}_{j\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ is any orthonormal basis of $L^2(\mathbb{R}^d)$, let $\mathcal{P}_\eta$ denote the algebra generated by functions $f : \mathcal{S}'(\mathbb{R}^d) \to \mathbb{R}$ of the form

$$f(\omega) = \langle \omega, \eta_j \rangle \text{ where } j \in \mathbb{N}.$$

Theorem 2.1 *The algebras $\mathcal{P}$, $\mathcal{E}$, $\mathcal{E}_r$, and $\mathcal{P}_\eta$ are dense in $L^p(\mu_1)$, for all $p \in [1, \infty)$. Moreover, $\mathcal{E}$ is dense in $L^\infty(\mu_1)$.*

Proof. This is Theorem 1.9 in [6]; they include a sketch of the proof. For further details, the reader is referred to [10]. The idea of the proof is to use the Stone-Weierstrass Theorem. The difficulty lies in the fact that $\mathcal{S}'(\mathbb{R}^d)$ is not a locally compact Hausdorff space. Overcoming this hurdle relies on the theory of Radon measures on arbitrary topological spaces. □

Comparing the above with the version presented as Theorem 2.1.3 in [7], one finds that the above version is more complete. In particular, [7] fails to mention that $\mathcal{E}$ is dense in L^∞. This fact, for example, seems to be necessary to prove that a function in $L^1(\mu_1)$ is uniquely determined by its Fourier transform.

Definition 2.2 *Let $(X, \mathcal{M}, \nu)$ be a σ-finite measure space. A function $Z(t) : X \to (\mathcal{S})^*$ is called $(\mathcal{S})^*$-integrable if $\langle Z(t), f \rangle \in L^1(X)$ for all $f \in (\mathcal{S})$.*

Remark 2.3 *Compare this with Definition 2.5.5 in [7].*

Proposition 2.4 *Let $(X, \mathcal{M}, \nu)$ be a σ-finite measure space and let $Z(t) : X \to (\mathcal{S})^*$ be $(\mathcal{S})^*$-integrable. Then the map $I : (\mathcal{S}) \to \mathbb{R}$ given by*

$$\langle I, f \rangle = \int_X \langle Z(t), f \rangle \, d\nu(t) \text{ for all } f \in (\mathcal{S}) \tag{2.1}$$

belongs to $(\mathcal{S})^*$.

Proof. This is Proposition 8.1 in [6]. □

Definition 2.5 *Let $(X, \mathcal{M}, \nu)$ be a σ-finite measure space and let $Z(t) : X \to (\mathcal{S})^*$ be $(\mathcal{S})^*$-integrable. The Pettis-integral of $Z(t)$ is the (unique) element $I \in (\mathcal{S})^*$ that satisfies (2.1) and is denoted by $\int_X Z(t)\, d\nu(t)$.*

Lemma 2.6 *Let $(X, \mathcal{M}, \nu)$ be a σ-finite measure space and let $Z(t) : X \to (\mathcal{S})^*$ be $(\mathcal{S})^*$-integrable. Write*

$$Z(t) = \sum c_\alpha(t) H_\alpha.$$

Then $c_\alpha(t) \in L^1(X)$ for all α, and $\int_X Z(t)\, d\nu(t) = \sum (\int_X c_\alpha(t)\, d\nu(t)) H_\alpha$.

Proof. Trivial. □

Proposition 2.7 *Let $(X, \mathcal{M}, \nu)$ be a σ-finite measure space and let $Z : X \to (\mathcal{S})^*$ be $(\mathcal{S})^*$-integrable with $I = \int Z\, d\nu$. Then*

$$(\mathcal{S}I)(\lambda\phi) = \int_X (\mathcal{S}Z(t))(\lambda\phi)\, d\nu(t)$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ and all $\lambda \in \mathbb{C}$.

Proof. Unwinding definitions, we have

$$(\mathcal{S}I)(\lambda\phi) = \langle I, \exp^\diamond[\langle\cdot, \lambda\phi\rangle]\rangle = \int_X \langle Z(t), \exp^\diamond[\langle\cdot, \lambda\phi\rangle]\rangle\, d\nu(t) = \int_X (\mathcal{S}Z(t))(\lambda\phi)\, d\nu(t)$$

for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $\lambda \in \mathbb{C}$. □

Proposition 2.8 *Let $(X, \mathcal{M}, \nu)$ be a σ-finite measure space and let $Z : X \to (\mathcal{S})^*$ be $(\mathcal{S})^*$-integrable with $I = \int Z\, d\nu$. Suppose there exists $q \in (1, \infty)$ and $\delta > 0$ such that*

$$M(t) = \sup_{z \in \mathbb{K}_q(\delta)} |\widetilde{Z}(t; z)| \in L^1(X).$$

Then $\widetilde{I}(z) = \int_X \widetilde{Z}(t; z)\, d\nu(t)$ for all $z \in \mathbb{K}_{3q}(\delta)$.

Remark 2.9 *Compare this with Lemma 2.8.5 in [7].*

Proof. Write $Z(t) = \sum c_\alpha(t) H_\alpha$. By Proposition 2.6.8 in [7], $\sum |c_\alpha(t) z^\alpha| \leq M(t) A(q)$ for all $z \in \mathbb{K}_{3q}(\delta)$, where $A(q) = \sum (2\mathbb{N})^{-q\alpha} < \infty$, since $q > 1$. Thus, by Lemma 2.6 and Fubini's Theorem,

$$\widetilde{I}(z) = \sum \left(\int_X c_\alpha(t)\, d\nu(t) \right) z^\alpha = \int_X \sum c_\alpha(t) z^\alpha\, d\nu(t) = \int_X \widetilde{Z}(t; z)\, d\nu(t)$$

for all $z \in \mathbb{K}_{3q}(\delta)$. □

The reason for this more general treatment of the Pettis integral can be seen in Chapter 4 of [7]. In particular, the construction of the solution (4.3.5) to the SPDE (4.3.2) requires us to take the expectation of a random variable taking values in $(\mathcal{S})^*$. For this, we need a way to interpret the integral of this random variable over the underlying probability space.

3 Analytic Functions

The Hermite transform of an element of $(\mathcal{S})_{-1}$ is a power series of infinitely many complex variables. Namely, it is a series of the form $f(z) = \sum a_\alpha z^\alpha$, where $z \in \mathbb{C}^\mathbb{N}$, $\alpha \in (\mathbb{Z}_+^\mathbb{N})_c$, and $z^\alpha = \prod z_j^{\alpha_j}$, where $z_j^0 = 1$. The function f is said to be analytic on

$$\mathbb{K}_q(R) = \left\{ (z_1, z_2, \ldots) \in \mathbb{C}^\mathbb{N} : \sum_{\alpha \neq 0} |z^\alpha|^2 (2\mathbb{N})^{q\alpha} < R^2 \right\}$$

if the series converges absolutely for all $z \in \mathbb{K}_q(R)$. Here, $2\mathbb{N}$ denotes the element $w \in \mathbb{C}^{\mathbb{N}}$ with $w_j = 2j$ for all j, that is, $2\mathbb{N} = (2, 4, 6, \ldots)$.

Many of the results in the theory of several complex variables have analogues in this setting, some of which can be found in Section 2.6 of [7]. What follows is a proposition which is a generalization of Montel's Theorem to analytic functions on $\mathbb{C}^{\mathbb{N}}$. This proposition is then applied to the situation in Theorem 4.1.1 of [7] to produce a result which is useful in verifying the hypotheses of that theorem and which is used implicitly throughout all of Chapter 4.

Lemma 3.1 *Let $G \subset \mathbb{C}^k$ be a neighborhood of the origin and let $f_n(z) = \sum_\alpha a_\alpha^{(n)} z^\alpha$ be a sequence of analytic functions on G with $|f_n(z)| \le M$ for all $z \in G$ and all $n \in \mathbb{N}$. Suppose $f_n(z) \to f(z)$ as $n \to \infty$ for each $z \in G$. Then f is analytic on G and $f(z) = \sum_\alpha a_\alpha z^\alpha$, where $a_\alpha = \lim_{n\to\infty} a_\alpha^{(n)}$.*

Proof. By Montel's Theorem (Proposition 6, Chapter 1 of [9]), there is a subsequence $\{f_{n_j}\}$ which converges uniformly on compact subsets of G. By Weierstrass's Theorem (Proposition 5, Chapter 1 of [9]), f is analytic on G and $D^\alpha f_{n_j}$ converges to $D^\alpha f$, uniformly on compact subsets of G. In particular,

$$a_\alpha^{(n_j)} = \frac{1}{\alpha!} D^\alpha f_{n_j}(0) \to \frac{1}{\alpha!} D^\alpha f(0) = a_\alpha.$$

Applying this argument to a subsequence shows that every subsequence of $\{a_\alpha^{(n)}\}$ has a subsequence that converges to a_α; hence, $a_\alpha = \lim_{n\to\infty} a_\alpha^{(n)}$. □

Lemma 3.2 *Let $f_n(z) = \sum a_\alpha^{(n)} z^\alpha$ be analytic on $\mathbb{K}_q(R)$, where $q > 1$. Suppose $|f_n(z)| \le M$ for all $z \in \mathbb{K}_q(R)$ and all $n \in \mathbb{N}$. If, for each α, $a_\alpha^{(n)} \to a_\alpha$ as $n \to \infty$, then $f(z) = \sum a_\alpha z^\alpha$ is analytic on $\mathbb{K}_{3q}(R)$ and $f_n(z) \to f(z)$ for all $z \in \mathbb{K}_{3q}(R)$.*

Proof. Let $z \in \mathbb{K}_{3q}(R)$. Define $w \in \mathbb{C}^{\mathbb{N}}$ by $w_j = (2j)^q z_j$. Then

$$\sum_\alpha |w^\alpha|^2 (2\mathbb{N})^{q\alpha} = \sum_\alpha \prod_{j=1}^\infty w_j^{2\alpha_j} \prod_{j=1}^\infty (2j)^{q\alpha_j} = \sum_\alpha \prod_{j=1}^\infty (2j)^{3q\alpha_j} z_j^{2\alpha_j} = \sum_\alpha |z^\alpha|^2 (2\mathbb{N})^{3q\alpha} < R^2$$

and $w \in \mathbb{K}_q(R)$. Also note that $z^\alpha = w^\alpha (2\mathbb{N})^{-q\alpha}$. By Proposition 2.6.8 in [7], $|a_\alpha^{(n)} w^\alpha| \le \widetilde{M}$, where $\widetilde{M} = M \sum_\alpha (2\mathbb{N})^{-q\alpha} < \infty$ since $q > 1$. Thus, $|a_\alpha^{(n)} z^\alpha| \le \widetilde{M} (2\mathbb{N})^{-q\alpha}$. Letting n go to infinity shows that f is given by a convergent power series on $\mathbb{K}_{3q}(R)$ and is therefore analytic. Moreover, by dominated convergence

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \sum_\alpha a_\alpha^{(n)} z^\alpha = \sum_\alpha a_\alpha z^\alpha = f(z)$$

and $f_n \to f$ pointwise on $\mathbb{K}_{3q}(R)$. □

Proposition 3.3 *Let $f_n(z) = \sum a_\alpha^{(n)} z^\alpha$ be analytic on $\mathbb{K}_q(R)$, where $q > 1$. Suppose $|f_n(z)| \le M$ for all $z \in \mathbb{K}_q(R)$ and all $n \in \mathbb{N}$. If $f_n \to f$ pointwise on $\mathbb{K}_q(R)$, then f is analytic on $\mathbb{K}_{3q}(R)$ and $f(z) = \sum a_\alpha z^\alpha$, where $a_\alpha = \lim_{n\to\infty} a_\alpha^{(n)}$.*

Proof. For each $k \in \mathbb{N}$, we may restrict the functions in question to a neighborhood of the origin in $\mathbb{C}^k$ and apply Lemma 3.1 to conclude that $a_\alpha = \lim_{n\to\infty} a_\alpha^{(n)}$ are well-defined. Thus, by Lemma 3.2, $\sum a_\alpha z^\alpha$ defines an analytic function on $\mathbb{K}_{3q}(R)$ which is the pointwise limit of the sequence $\{f_n\}$ and therefore agrees with f on $\mathbb{K}_{3q}(R)$. □

Proposition 3.4 *Let $G \subset \mathbb{R}^d$ be a bounded open set. Suppose $u(x,z) : G \times \mathbb{K}_q(R) \to \mathbb{C}$ and $\alpha \in \mathbb{N}^d$ satisfy*

(i) $u(x,\cdot)$ is analytic on $\mathbb{K}_q(R)$ for each $x \in G$, and

(ii) $\partial_x^\alpha u(x,z)$ is uniformly bounded for $(x,z) \in G \times \mathbb{K}_q(R)$.

Then there exists $\widetilde{q}$ such that $\partial_x^\alpha u(x,\cdot)$ is analytic on $\mathbb{K}_{\widetilde{q}}(R)$.

Proof. Without loss of generality, we may assume $q > 1$. Note that if $\beta < \alpha$, then by the mean value theorem, $\partial_x^\beta u(x,z)$ is uniformly bounded for $(x,z) \in G \times \mathbb{K}_q(R)$. Thus, by induction, it suffices to assume that $|\alpha| = 1$, that is, $\partial_x^\alpha = \partial_{x_j}$ for some $1 \le j \le d$. Fix $x \in G$. For each $z \in \mathbb{K}_q(R)$, $\partial_{x_j} u(x,z)$ is the limit of difference quotients which are analytic in z by (i) and uniformly bounded by (ii) and the mean value theorem. The conclusion therefore follows by Proposition 3.3. □

4 Estimates for PDEs

The general method for solving SPDEs as presented in [7] is to first take the Hermite transform of the SPDE, then solve the resulting PDE, then finally take the inverse Hermite transform of the solution. Before proceeding with the final stage of this procedure, one must verify that the solution to the PDE satisfies a certain boundedness condition given by Theorem 4.1.1 in [7]. For this, we need certain tools from the theory of (deterministic) PDEs.

4.1 Elliptic Equations

A priori estimates for elliptic equations are used in section 4.2 of [7] when solving the stochastic Dirichlet problem. The authors refer to [1] as the source of their estimates. These estimates, however, are also listed without proof in sections 2.2.14 and 2.2.17 of [5], which is a nice general reference for classical PDE theory. The source for these estimates, as listed in [5], is [2]. Looking at [2], however, we find that proofs are omitted even there and the authors refer to the original papers of J. Schauder. The following discussion is taken from Chapter IV, §7 of [2].

J. Schauder derived certain a priori estimates for solutions u of linear elliptic equations of the form

$$L[u](x) \equiv \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_j(x)\frac{\partial u(x)}{\partial x_j} + c(x)u(x) = f(x) \tag{4.1}$$

in a bounded domain G. The estimates hold for uniformly elliptic equations (4.1) with bounded Hölder continuous coefficients, that is, for equations satisfying the following conditions: there exist positive constants m, M, and $\alpha \in (0,1)$ such that

(i) $\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq m\|\xi\|^2$ for all $x \in G$ and all $\xi \in \mathbb{R}^d$,

(ii) $|a_{ij}(x)|, |b_j(x)|, |c(x)| \leq M$ for all $x \in G$, $(i, j = 1, 2, \ldots, d)$, and

(iii) the coefficients a_{ij}, b_j, c satisfy a Hölder condition with exponent α and coefficient M.

A function μ is said to satisfy a Hölder condition with exponent α and coefficient M if $\sup_{x,y\in G} |\mu(x) - \mu(y)|/|x - y|^\alpha \leq M$.

Recall that the space $C^k(\overline{G})$ is the space of functions on $\overline{G}$ that are k times continuously differentiable and that the space $C^{k+\alpha}(\overline{G})$ consists of those functions in $C^k(\overline{G})$ whose k-th order derivatives satisfy a Hölder condition in $\overline{G}$ with exponent α. The norms in these spaces are

$$\|u\|_k = \sum_{|\alpha|\leq k} \sup_{x\in\overline{G}} |\partial^\alpha u(x)|$$

and

$$\|u\|_{k+\alpha} = \|u\|_k + \sum_{|\alpha|=k} \sup_{x,y\in\overline{G}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}. \tag{4.2}$$

For the estimates we will present, we also require that the domain G and the boundary values φ of u be sufficiently smooth. We call a domain G smooth if its boundary is smooth, that is, if we can cover ∂G by a finite number of balls which have the property that, singling out one coordinate, say x_d, one can express the part of the boundary contained in each of these balls in the form $x_d = g(x_1, x_2, \ldots, x_{d-1})$, where g is assumed to have Hölder continuous second derivatives with exponent α. In addition, in terms of the local parameters $x_1, x_2, \ldots, x_{d-1}$, the boundary values φ are also assumed to be smooth, that is, to have Hölder continuous second derivatives with exponent α. Using the fixed finite number of local parameter systems on the boundary and the norms $\|\varphi\|_{2+\alpha}$ in each ball, we may define a norm $\|\varphi\|'_{2+\alpha}$ for the function φ as the maximum of the $\|\varphi\|_{2+\alpha}$.

In this setting, we are finally prepared to state the Schauder estimates. Let u be a solution in $C^{2+\alpha}(\overline{G})$ of (4.1) in a smooth domain G with smooth boundary values φ. Then

$$\|u\|_{2+\alpha} \leq K(\|u\|_0 + \|f\|_\alpha + \|\varphi\|'_{2+\alpha}),$$

where K is a constant depending only on m, α, M, and the domain G. Moreover,

$$\|u\|_{2+\alpha} \leq K(\|f\|_\alpha + \|\varphi\|'_{2+\alpha})$$

whenever $c \leq 0$ in (4.1).

4.2 Parabolic Equations

A priori estimates for parabolic equations are used in the proof of Theorem 4.3.1 in [7]. The authors cite Theorem 2.78 in [5] and the references therein. It is difficult, however, to understand even the statement of Theorem 2.78. In particular, it is not clear how to define the norms being used on submanifolds (boundaries of regions involved) since these norms involve the use of a distinguished time variable. The original source for these estimates, as cited in [5], is [8], which is much more understandable and from which the following is taken.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $D = (0,T) \times \Omega$. Let $S = (0,T] \times \partial\Omega$ and $\Gamma = S \cup \Omega$. For a function $u(t,x)$, $t \in (0,T)$, $x \in \Omega$, defined on D and $0 < \alpha < 1$, let $|u|_\alpha = \|u\|_\alpha$, where $\|u\|_\alpha$ is its norm in $C^\alpha(D)$ (see (4.2) with $k = 0$). We then define the norms

$$|u|_{1+\alpha} = |u|_\alpha + \sum_{j=1}^{d} \left| \frac{\partial u}{\partial x_j} \right|_\alpha,$$

$$|u|_{2+\alpha} = |u|_\alpha + \sum_{j=1}^{d} \left| \frac{\partial^2 u}{\partial x_j} \right|_\alpha + \sum_{i,j=1}^{d} \left| \frac{\partial u}{\partial x_i \partial x_j} \right|_\alpha + \left| \frac{\partial u}{\partial t} \right|_\alpha .$$

The function $u(t,x)$ belongs to the class $H^{k+\alpha}(D)$ if the norm $|u|_{k+\alpha}$ is finite, $k = 0, 1, 2$. The surface S belongs to the class $A^{k+\alpha}$ if it can be locally expressed as the graph of a $C^{k+\alpha}$ function (that is, if $\partial\Omega$ is of class $C^{k+\alpha}$).

With these preliminaries, we can now state the main a priori estimate for parabolic equations which is Theorem 5 in Chapter 2 of [8].

Theorem 4.1 *Consider the parabolic equation*

$$L(u) \equiv \sum_{i,j=1}^{d} a_{ij}(t,x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_j(t,x) \frac{\partial u}{\partial x_j} + c(t,x)u - \frac{\partial u}{\partial t} = f(t,x), \tag{4.3}$$

where the functions a_{ij}, b_j, c, and f are real, with finite values, $a_{ij} = a_{ji}$, and

$$\sum_{i,j=1}^{d} a_{ij} \xi_i \xi_j \geq \mu \sum_{j=1}^{d} \xi_j^2$$

for all $\xi \in \mathbb{R}^d$. Suppose the coefficients of (4.3) *in the domain $\overline{D}$ satisfy the conditions*

$$|a_{ij}|_\alpha \leq M_1 \quad |b_j|_\alpha \leq M_1 \quad |c|_\alpha \leq M_1$$

and a solution $u(t,x)$ of (4.3) *belongs to the class $H^{2+\alpha}(D)$. Also suppose that S belongs to the class $A^{2+\alpha}$. Then*

$$|u|_{2+\alpha} \leq M(|f|_\alpha + |\psi|_{2+\alpha}),$$

where the function ψ is defined in $\overline{D}$ and $u|_\Gamma = \psi$; the constant M depends only on M_1, μ, and the domain D.

This estimate is utilized at the end of the proof of Theorem 4.3.1 in [7]. Note, however, that making use of this estimate requires knowledge of the $(2 + \alpha)$-norm of the boundary values of u, not just at time $t = 0$, but at later times as well. It is for this reason that it is unclear how the authors of [7] intended to make use of this estimate in their particular setting.

5 Miscellaneous Formulas

The following formula is helpful in determining, among other things, the chaos expansion for functions of Brownian motion.

Proposition 5.1 *Let* $\varphi = \sum a_j \xi_j \in L^2(\mathbb{R})$*. Then*

$$\langle \cdot, \varphi \rangle^{\diamond n} = \int_{\mathbb{R}^n} \varphi^{\otimes n} \, dB^{\otimes n} = \sum_{|\alpha|=n} \frac{n!}{\alpha!} a^\alpha H_\alpha.$$

Proof. By induction, the first equality is an easy consequence of Proposition 2.4.2 in [7]. Now, let $S = \{\delta \in \mathbb{N}^n : \delta_1 \leq \cdots \leq \delta_n\}$ and $T = \{\alpha \in (\mathbb{Z}_+)_c^{\mathbb{N}} : |\alpha| = n\}$. Note that for each $\delta \in S$, there exists a unique $\alpha(\delta) \in T$ such that

$$z^{\alpha(\delta)} = z_\delta \equiv \prod_{j=1}^n z_{\delta_j} \text{ for all } z \in \mathbb{C}^{\mathbb{N}}.$$

Note also that the map $\delta \mapsto \alpha(\delta)$ is a bijection from S to T and that

$$\xi^{\otimes \alpha(\delta)} = \xi_\delta \equiv \xi_{\delta_1} \otimes \cdots \otimes \xi_{\delta_n}.$$

Since $\varphi^{\otimes n} \in L^2(\mathbb{R}^n)$, it has an expansion

$$\varphi^{\otimes n} = \sum_{\delta \in \mathbb{N}^n} c(\delta) \xi_\delta,$$

where

$$c(\delta) = \int \varphi^{\otimes n} \xi_\delta = \prod_{j=1}^n \int \varphi \xi_{\delta_j} = \prod_{j=1}^n a_{\delta_j} = a_\delta.$$

Since $\varphi^{\otimes n}$ is symmetric in the variables $x_1, \ldots, x_n$, we may write $\varphi^{\otimes n} = \sum_{\delta \in \mathbb{N}^n} a_\delta \widehat{\xi_\delta}$, where $\widehat{\xi_\delta}$ is the symmetrization of ξ_δ. Since neither a_δ nor $\widehat{\xi_\delta}$ is affected by permuting the components of δ and since each $\delta \in S$ has $n!/\alpha(\delta)!$ distinct permutations, we have

$$\begin{aligned}
\varphi^{\otimes n} &= \sum_{\delta \in S} \frac{n!}{\alpha(\delta)!} a_\delta \widehat{\xi_\delta} \\
&= \sum_{\delta \in S} \frac{n!}{\alpha(\delta)!} a^{\alpha(\delta)} \widehat{\xi^{\otimes \alpha(\delta)}} \\
&= \sum_{|\alpha|=n} \frac{n!}{\alpha!} a^\alpha \xi^{\widehat{\otimes} \alpha}.
\end{aligned}$$

It now follows from the discussion preceding Theorem 2.2.7 in [7] that

$$\sum_{|\alpha|=n} \frac{n!}{\alpha!} a^\alpha H_\alpha = \int_{\mathbb{R}^n} \varphi^{\otimes n} \, dB^{\otimes n} = \langle \cdot, \varphi \rangle^{\diamond n},$$

which completes the proof. □

For the formulas that follow, we begin with two combinatorial lemmas.

Lemma 5.2 *For $k \in \mathbb{Z}_+$, $(2k-1)!! = (2k)!/(2^k k!)$.*

Proof. Since $(-1)!! = 1$ by convention, the lemma is trivial if $k = 0$ or $k = 1$. If the lemma holds for some $m \in \mathbb{N}$, then

$$\begin{aligned}(2(m+1)-1)!! &= (2m+1)!! \\ &= (2m+1)(2m-1)!! \\ &= (2m+1)\frac{(2m)!}{2^m m!} \\ &= \frac{(2m+2)!}{2^m(2m+2)m!} \\ &= \frac{(2(m+1))!}{2^{m+1}(m+1)!}.\end{aligned}$$

By induction, this completes the proof. □

Lemma 5.3 *Suppose $k, M, n \in \mathbb{N}$. Then*

$$\binom{M}{k}k! + \binom{M}{k-1}\binom{n}{k-1}(k-1)!(M+n+2-2k) - \binom{M-1}{k-1}\binom{n}{k-1}(k-1)!M = \binom{M+1}{k}\binom{n}{k}k!$$

whenever $k \le M \le n$.

Proof. First observe that

$$\binom{M}{k-1}\binom{n}{k-1}(M+1-k) = \frac{M!n!}{(M-k)!(n-k+1)!(k-1)!^2} = \binom{M-1}{k-1}\binom{n}{k-1}M,$$

so that it suffices to prove

$$\binom{M}{k}\binom{n}{k}k + \binom{M}{k-1}\binom{n}{k-1}(n+1-k) = \binom{M+1}{k}\binom{n}{k}k.$$

Since $\binom{M+1}{k} = \binom{M}{k} + \binom{M}{k-1}$, it suffices to show that $\binom{n}{k-1}(n+1-k) = \binom{n}{k}k$. This, however, is trivial since both sides are equal to $n!/((n-k)!(k-1)!)$. □

In Appendix C of [7], the authors provide the formula

$$x^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}(2k-1)!!h_{n-2k}(x).$$

An interesting observation is the following analogous formula.

Proposition 5.4 *The function*

$$h_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k}(2k-1)!!x^{n-2k}$$

is the n-th Hermite polynomial.

Proof. This follows immediately from (C.2) in [7] and Lemma 5.2. □

The following proposition can be used to prove Itô's rule for $f(B_t)$ in the case that f is analytic.

Proposition 5.5 *Let $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\| = 1$, and define*

$$p_n(x) = i^{-n} h_n(ix) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!! x^{n-2k}.$$

Then $\langle \cdot, \varphi \rangle^n = p_n^{\diamond}(\langle \cdot, \varphi \rangle)$.

Proof. Use (C.4) and (2.4.17) in [7]. □

If it were true that $h_m(x)h_n(x) = h_{m+n}(x)$, then the Wick product and the ordinary product would coincide. Hence, in an attempt to better understand the relationship between the two products, we present the following formula.

Proposition 5.6 *If h_n is the n-th Hermite polynomial, then*

$$h_m(x)h_n(x) = \sum_{k=0}^{m \wedge n} \binom{m}{k} \binom{n}{k} k! h_{m+n-2k}(x)$$

for all $m, n \in \mathbb{Z}_+$.

Proof. Without loss of generality, assume $m \leq n$. For $m = 0$, the proposition is trivial. For $m = 1$, it is an immediate consequence of (C.7) in [7]. Now suppose $n > 1$ and assume the proposition is true for all $m \leq M$, where $M \in \{1, \dots, n-1\}$. Using (C.7) in [7], we have

$$\begin{aligned} h_{M+1}(x)h_n(x) &= (xh_M(x) - Mh_{M-1}(x))h_n(x) \\ &= x \sum_{k=0}^{M} \binom{M}{k} \binom{n}{k} k! h_{M+n-2k}(x) - M \sum_{k=0}^{M-1} \binom{M-1}{k} \binom{n}{k} k! h_{M-1+n-2k}(x) \\ &= \sum_{k=0}^{M} \binom{M}{k} \binom{n}{k} k! h_{M+1+n-2k}(x) + \sum_{k=0}^{M} \binom{M}{k} \binom{n}{k} k! (M+n-2k) h_{M-1+n-2k}(x) \\ &\quad - \sum_{k=0}^{M-1} \binom{M-1}{k} \binom{n}{k} k! M h_{M-1+n-2k}(x). \end{aligned}$$

We now separate off the first term from the first sum and the last term from the second sum and shift the index of summation in the second and third sums to obtain

$$\begin{aligned} h_{M+1}(x)h_n(x) = h_{M+1+n}(x) &+ \binom{n}{M} M!(n-M) h_{n-(M+1)}(x) \\ &+ \sum_{k=1}^{M} \Bigg[\binom{M}{k} \binom{n}{k} k! + \binom{M}{k-1} \binom{n}{k-1} (k-1)!(M+n+2-2k) \\ &\qquad - \binom{M-1}{k-1} \binom{n}{k-1} (k-1)! M \Bigg] h_{M+1+n-2k}(x). \end{aligned}$$

By Lemma 5.3, we can write this as

$$h_{M+1}(x)h_n(x) = \sum_{k=0}^{M+1} \binom{M+1}{k}\binom{n}{k} k! h_{M+1+n-2k}(x),$$

which completes the proof. □

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