Practice problems

1. Let $\{p_n\}$ be a Cauchy sequence in a metric space X. Suppose that $p \in X$ is a subsequential limit of $\{p_n\}$. Prove that $\lim_{n\to\infty} p_n = p$.

2. Let $E \subset \mathbb{R}^n$ be open, and suppose $f : E \to \mathbb{R}$ is differentiable. Show that if f has a local maximum at a point $x \in E$, then $f'(x) = 0$.

3. Let m be Lebesgue measure on $\mathbb{R}$. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable sets. We define $\liminf A_n := \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$ and $\limsup A_n := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$. Prove that $m(\liminf A_n) \leq \liminf m(A_n)$.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space. For each n, let $f_n : X \to \mathbb{C}$ be measurable, with $f_n \to f$ a.e. Suppose there exists $g \in L^p(X)$, $p \in [1, \infty)$, such that $|f_n| \leq g$ a.e. Prove that $f_n \to f$ in $L^p(X)$.

5. If $f : \mathbb{R} \to \mathbb{R}$ and $y \in \mathbb{R}$, then we define $\tau_y f(x) = f(x - y)$. If $f \in L^1(\mathbb{R})$, then $\|\tau_y f\|_1 = \|f\|_1$ and $\|\tau_y f - f\|_1 \to 0$ as $y \to 0$. Also,
$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{dt}{(t^2+1)^2} = 1.$$
   You may use all of the above without justification.

   Suppose $f \in L^1(\mathbb{R})$. For each $n \in \mathbb{N}$, let
$$f_n(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{n f(x-t)}{(n^2t^2+1)^2}\, dt.$$
   Prove that $f_n \in L^1(\mathbb{R})$ and that $f_n \to f$ in $L^1(\mathbb{R})$.

6. For each n, let $f_n \in L^1([0,1])$. Suppose that $f_n \to 1$ a.e. and that
$$\lim_{n\to\infty} \int_0^1 |f_n(x)|\, dx = 2.$$
   Prove that
$$\lim_{n\to\infty} \int_0^1 |f_n(x) - 1|\, dx = 1.$$

7. Let X and Y be topological spaces, and let $\pi_1 : X \times Y \to X$ be the projection map, i.e. $\pi_1(x, y) = x$. Prove that π_1 is an open map. That is, prove that if $U \subset X \times Y$ is open in the product topology, then $\pi_1(U)$ is open in X.

8. Let $k : [0,1]^2 \to \mathbb{R}$ be continuous and define $K \in \mathcal{B}(C([0,1]))$ by
$$Kf(x) = \int_0^1 k(x,y) f(y)\, dy.$$
   Prove that K is compact.

9. Let X be a Banach space and $K \in \mathcal{B}(X)$. Prove that if $\|K\| < 1$, then $I - K$ is invertible, and that

$$(I-K)^{-1} = \sum_{n=0}^{\infty} K^n,$$

where the above series converges uniformly in $\mathcal{B}(X)$.

10. Let f be a positive, Lebesgue measurable function on $(0,1)$. Suppose that f and $\log f$ are both integrable on $(0,1)$. Prove that

$$\int_0^1 f(x)\log f(x)\,dx \geq \left(\int_0^1 f(x)\,dx\right)\left(\int_0^1 \log f(x)\,dx\right).$$

11. Let X and Y be (nontrivial) normed spaces. Prove that if $\mathcal{B}(X,Y)$, the space of bounded operators from X to Y, is complete, then Y is a Banach space.

12. Let $p \in [1,\infty)$. Let $\{f_n\}$ be a sequence in $L^p(\mathbb{R})$ and $f \in L^p(\mathbb{R})$. Suppose that $f_n \to f$ pointwise. Prove that $f_n \to f$ in $L^p(\mathbb{R})$ if and only if $\|f_n\|_p \to \|f\|_p$.

13. Let $\mathcal{H}$ be a Hilbert space, and let $A : \mathcal{H} \to \mathcal{H}$ be linear. Suppose that $\langle x, Ay\rangle = \langle Ax, y\rangle$ for all $x, y \in \mathcal{H}$. Prove that A is bounded.

14. Let T be the distributional derivative of p.v.$\frac{1}{x}$. Prove that for all $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\langle T, \varphi\rangle = \lim_{\varepsilon \downarrow 0} \int_{|x|>\varepsilon} \left(-\frac{1}{x^2}\right)(\varphi(x) - \varphi(0))\,dx.$$

15. Prove that if $f \in L^2(\mathbb{R}^n)$, $a \in \mathbb{R}$, and $g(x) = f(ax)$, then $g \in L^2(\mathbb{R}^n)$ and

$$\widehat{g}(k) = \frac{1}{|a|^n}\widehat{f}\left(\frac{k}{a}\right).$$

16. Let S be a linear subspace of $L^q([0,1])$ that is closed as a subspace of $L^p([0,1])$, where $1 < p < q < \infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in S. Prove that $\{f_n\}_{n=1}^{\infty}$ is convergent in $(L^q([0,1]), \|.\|)$ iff $\{f_n\}_{n=1}^{\infty}$ is convergent in $(L^p([0,1]), \|.\|)$.

17. Let X be the metric space $(\mathbb{R}, d)$ where

$$d(x,y) = \frac{|x-y|}{1+|x-y|}.$$

Show there is a decreasing sequence of nonempty closed bounded sets with empty intersection.

18. Let K be a continuous function on the unit square $Q = [0,1] \times [0,1]$ with the property that $|K(x,y)| < 1$ for all $(x,y) \in Q$. Show that there is a continuous function g defined on $[0,1]$ so that

$$g(x) + \int_0^1 K(x,y)g(y)\,dy = \frac{e^x}{1+x^2}$$

for $x \in [0,1]$.

19. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on $[0,1]$, and assume $\int_0^1 |f_n(x)|^2\,dx \le \frac{1}{n^2}$.

   (a) Fix $\varepsilon > 0$ and let
   $$A_N = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x : |f_n(x)| \ge \varepsilon\}.$$
   Prove that $m(A_N) = 0$, where m is Lebesgue measure.

   (b) Use part (a) to show that $f_n \to 0$ a.e. on $[0,1]$.

20. Let $\{f_n\}$ be a sequence in $C^1([0,1])$ such that $\|f_n'\|_\infty \le 1$ for all $n \in \mathbb{N}$. Suppose that there exists a complex number a and a measurable function $g : [0,1] \to \mathbb{C}$ such that $f_n(0) \to a$ and $f_n' \to g$ a.e. on $[0,1]$. Show that there exists a continuous function $f : [0,1] \to \mathbb{C}$ such that $f_n \to f$ uniformly.

21. Let $X = C([0,1])$ with the uniform norm. Define $K : X \to X$ by
$$Kf(x) = \int_0^1 t\cos(tx) f(t)\,dt.$$
Show that K is a bounded linear operator with $\|K\| = 1/2$.

22. Let V be the space of complex-valued sequences $a = (a_1, a_2, \ldots)$ which satisfy
$$\sum_{n=1}^{\infty} n|a_n| < \infty.$$
With the norm $\|a\| = \sum_{n=1}^{\infty} n|a_n|$, the vector space V becomes a Banach space (this you may assume). Consider the bounded linear operator $B : V \to V$ defined by
$$B(a_1, a_2, \ldots) = (a_2, a_3, \ldots).$$
Show that if $|\lambda| > 1$, then $B - \lambda I$ is invertible.

23. Suppose $A \subset \mathbb{R}$ is Lebesgue measurable and satisfies $m(A \cap (a,b)) \le (b-a)/2$ for all $a < b$, where m is Lebesgue measure on $\mathbb{R}$. Prove that $m(A) = 0$.

24. Suppose that $\mathcal{H}$ is a separable Hilbert space.

   (a) Prove that if $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator such that $\|I - T\| < 1$, then T is invertible.

   (b) Assume that $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$. Prove that if $\{f_n\}$ is an orthonormal set in $\mathcal{H}$ such that
   $$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1,$$
   then $\{f_n\}$ is also an orthonormal basis for $\mathcal{H}$. (Hint: Let $Tx = \sum_{n=1}^{\infty} \langle x, f_n\rangle\, e_n$. Prove that T is a well-defined, bounded linear operator on $\mathcal{H}$, and apply part (a).)

25. For $x \in \mathbb{R}$, let

$$f_n(x) = \begin{cases} nx^n & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that f_n is a tempered distribution on $\mathbb{R}$, and find a tempered distribution $T \in \mathcal{S}^*(\mathbb{R})$ such that $f_n \to T$.

26. Consider the operator $T : C([0,1]) \to C([0,1])$ defined by

$$(Tf)(t) = \int_0^1 \frac{f(s)}{1+s+t}\, ds.$$

   (a) Show that T is a bounded linear operator.

   (b) Show that $S : C([0,1]) \to C([0,1])$ defined by $Sf = f - Tf$ is invertible and its inverse is bounded.

27. Let $\mathcal{F}$ be the collection of twice continuously differentiable functions on $\mathbb{R}$ satisfying $f \ge 0$ on $\mathbb{R}$ and $f'' \le 1$ on $\mathbb{R}$. Find the smallest constant $C \in (0, \infty)$ such that for each $f \in \mathcal{F}$ and for each $x \in \mathbb{R}$, we have

$$(f'(x))^2 \le Cf(x). \tag{1}$$

Prove that your chosen constant works in (1), and show by example that the constant constant cannot be improved.

28. Let $n \in \mathbb{N}$. Define $P : \mathbb{R} \to \mathbb{R}$ by

$$P(x) = \frac{d^n}{dx^n}((x^2-1)^n).$$

Prove that if $x \in \mathbb{R}$ satisfies $P(x) = 0$, then $x \in (-1, 1)$.

29. Give an example of a normed vector space $(X, \|\cdot\|)$ and a linear functional φ on X such that φ does not belong to the dual space X^*.

30. A subset S of a Banach space X is called *weakly bounded* if, for each $\lambda \in X^*$, we have $\sup_{x \in S} |\lambda(x)| < \infty$. The set S is *strongly bounded* if $\sup_{x \in S} \|x\| < \infty$. Prove that a subset of a Banach space is strongly bounded if and only if it is weakly bounded.

31. Let $\{s_n\}$ be a sequence of complex numbers such that $\lim\limits_{n\to\infty} s_n$ exists. Prove that

$$s = \lim_{n\to\infty} \frac{s_1 + s_2 + \cdots + s_n}{n}$$

exists, and that $s = \lim\limits_{n\to\infty} s_n$.

32. Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{H} \to \mathcal{H}$ a linear operator. Suppose that for all sequences $\{x_n\}_{n=1}^\infty$ in $\mathcal{H}$, if $x_n \to x$ in norm, then $Ax_n \to Ax$ weakly. Prove that A is bounded.

33. Let X be a connected metric space, Y a metric space, and $f : X \to Y$ continuous. Suppose that for all $p \in X$, there exists $\varepsilon > 0$ such that f is constant on $B_\varepsilon(p)$. Prove that f is constant.

34. Let d be the Euclidean metric on $\mathbb{R}^n$ and let $\{e_1, \dots, e_n\}$ denote the standard basis in $\mathbb{R}^n$. Suppose $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ satisfies $d(Ae_j, e_j) < n^{-1}$ for all $j \in \{1, \dots, n\}$. Prove that A is invertible.

35. Let $E \subset \mathbb{R}$ be Lebesgue measurable with $0 < m(E) < \infty$. Prove that for every $\varepsilon > 0$, there exists a nonempty open interval I such that

$$\frac{m(E \cap I)}{m(I)} > 1 - \varepsilon.$$

36. Let $f : [0, 1] \to \mathbb{R}$ be continuously differentiable. Prove that

$$\int_0^1 f(x)\, dx \geq f(1) - \sqrt[3]{\frac{4}{25} \int_0^1 |f'(x)|^3\, dx}.$$

37. Let $f : [0, 1] \to \mathbb{R}$ be bounded and Lebesgue measurable. Suppose that for every $0 \leq a < b \leq 1$, we have

$$\int_a^b f(x)\, dx = 0.$$

Prove that $f = 0$ a.e.

38. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and $f : X \to \mathbb{R}$ and integrable function. Compute and justify the limit

$$\lim_{n \to \infty} \int_X |f(x)|^{1/n}\, \mu(dx).$$

39. Let $f : [0, 1] \to [0, \infty)$ be Lebesgue measurable with $f \in L^p([0, 1], \mathcal{L}, m)$ for all $p \in [1, \infty)$. Suppose that

$$\int_0^1 (f(x))^n\, dx = \int_0^1 f(x)\, dx,$$

for all $n \in \mathbb{N}$. Prove that $f = \chi_E$ a.e. for some measurable set $E \subset [0, 1]$.

40. Let $\mathscr{H}$ be a Hilbert space and let $x_n, x, y_n, y \in \mathscr{H}$. Suppose $x_n \to x$ weakly and $y_n \to y$ in norm. Prove that $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

41. Let $f \in L^2(\mathbb{R}, \mathcal{L}, m)$ and suppose that

$$\int_{\mathbb{R}} f(y) e^{-(x-y)^2/2}\, dy = 0,$$

for all $x \in \mathbb{R}$. Prove that $f = 0$ a.e.

42. Let $\mathscr{H}$ be a Hilbert space, and let $T \in \mathcal{B}(\mathscr{H})$ with $\|T\| \leq 1$. Let $x \in \mathscr{H}$ and suppose that $Tx = x$. Prove that $T^*x = x$.

43. Let $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ be Banach spaces, and suppose there exists $K \in \mathbb{R}$ such that $\|x\|_1 \leq K\|x\|_2$ for all $x \in X$. Prove that the two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, are equivalent.

44. (a) Find all distributions $T \in S^*(\mathbb{R})$ such that $xT = 0$.

    (b) Find all distributions $T \in S^*(\mathbb{R})$ such that $x^2T = 0$.