MAP 4934 Ch. 4 solns

Exer. 4.1

(i) $V_3^P = v_3^P(S_3)$,

$$v_3^P(s) = g_P(s) = (4-s)^+$$

$$v_n^P(s) = g_P(s) \vee \left(\frac{1}{1+r} \left(\tilde{p}\, v_{n+1}^P(2s) + \tilde{q}\, v_{n+1}^P\left(\tfrac{s}{2}\right)\right)\right)$$

$$= g_P(s) \vee \left(\frac{2}{5} \left(v_{n+1}^P(2s) + v_{n+1}^P\left(\tfrac{s}{2}\right)\right)\right)$$

$$v_2^P(16) = g_P(16) \vee \left(\frac{2}{5} \left(v_3^P(32) + v_3^P(8)\right)\right)$$

$$= 0 \vee \left(\frac{2}{5}(0+0)\right) = 0$$

$$v_2^P(4) = 0 \vee \left(\frac{2}{5}(0+2)\right) = \frac{4}{5}$$

$$v_2^P(1) = 3 \vee \left(\frac{2}{5}\left(2+\frac{7}{2}\right)\right)$$

$$= 3 \vee \left(\frac{2}{5}\cdot\frac{11}{2}\right) = 3 \vee \frac{11}{5} = 3$$

$$v_1^P(8) = g_P(8) \vee \left(\frac{2}{5}\left(v_2^P(16) + v_2^P(4)\right)\right)$$

$$= 0 \vee \left(\frac{2}{5}\left(0 + \frac{4}{5}\right)\right) = \frac{8}{25}$$

$$v_1^P(2) = 2 \vee \left(\frac{2}{5}\left(\frac{4}{5} + 3\right)\right)$$

$$= 2 \vee \left(\frac{2}{5} \cdot \frac{19}{5}\right) = 2 \vee \frac{38}{25} = 2$$

$$V_0^P = v_0^P(4) = g_P(4) \vee \left(\frac{2}{5}\left(v_1^P(8) + v_1^P(2)\right)\right)$$

$$= 0 \vee \left(\frac{2}{5}\left(\frac{8}{25} + 2\right)\right)$$

$$= \frac{2}{5} \cdot \frac{58}{25} = \boxed{\frac{116}{125}}$$

(ii) Same as European:

$$V_0^c = \tilde{E}\left[\frac{V_3^c}{(1+r)^3}\right] = \frac{64}{125}\tilde{E}\left[(S_3-4)^+\right]$$

$$= \frac{64}{125}\left(\frac{1}{8}\cdot 28 + \frac{3}{8}\cdot 4 + \frac{3}{8}\cdot 0 + \frac{1}{8}\cdot 0\right)$$

$$= \frac{8}{125}\cdot(28+12) = \frac{8}{\cancel{125}_{25}}\cdot \cancel{40}^{8} = \boxed{\frac{64}{25}}$$

(iii) $g_S(s) = (s-4)^+ + (4-s)^+ = |s-4|$

$$v_2^S(16) = g_S(16) \vee \left(\frac{2}{5}\left(v_3^S(32) + v_3^S(8)\right)\right)$$

$$= 12 \vee \left(\frac{2}{5}(28+4)\right)$$

$$= 12 \vee \frac{64}{5} = \frac{64}{5}$$

$$v_2^S(4) = 0 \vee \left(\frac{2}{5}(4+2)\right) = \frac{12}{5}$$

$$v_2^S(1) = 3 \vee \left(\frac{2}{5}\left(2+\frac{7}{2}\right)\right) = 3$$

$$v_1^S(8) = g_S(8) \vee \left(\frac{2}{5}\left(v_2^S(16) + v_2^S(4)\right)\right)$$

$$= 4 \vee \left(\frac{2}{5}\left(\frac{64}{5} + \frac{12}{5}\right)\right)$$

$$= 4 \vee \frac{152}{25} = \frac{152}{25}$$

$$v_1^S(2) = 2 \vee \left(\frac{2}{5}\left(\frac{12}{5} + 3\right)\right)$$

$$= 2 \vee \left(\frac{2}{5} \cdot \frac{27}{5}\right) = 2 \vee \frac{54}{25} = \frac{54}{25}$$

$$V_0^S = v_0^S(4) = g_S(4) \vee \left(\frac{2}{5}\left(v_1^S(8) + v_1^S(2)\right)\right)$$

$$= 0 \vee \left(\frac{2}{5}\left(\frac{152}{25} + \frac{54}{25}\right)\right)$$

$$= \frac{2}{5} \cdot \frac{206}{25} = \boxed{\frac{412}{125}}$$

(iv) $V_0^S = \dfrac{412}{125}$

$$V_0^P + V_0^C = \frac{116}{125} + \frac{64}{25}$$

$$= \frac{116 + 320}{125} = \frac{436}{125}$$

Why:

$$V_0^S = \max_{\tau \in \mathcal{S}_0} \widetilde{E}\left[1_{\{\tau<\infty\}} \frac{G_\tau^S}{(1+r)^\tau} \right]$$

$$= \max_{\tau \in \mathcal{S}_0} \widetilde{E}\left[1_{\{\tau<\infty\}} \frac{G_\tau^P + G_\tau^C}{(1+r)^\tau} \right]$$

$$= \max_{\tau \in \mathcal{S}} \left(\widetilde{E}\left[1_{\{\tau<\infty\}} \frac{G_\tau^P}{(1+r)^\tau} \right] + \widetilde{E}\left[1_{\{\tau<\infty\}} \frac{G_\tau^C}{(1+r)^\tau} \right] \right)$$

$$< \max_{\tau \in \mathcal{S}_0} \widetilde{E}\left[1_{\{\tau<\infty\}} \frac{G_\tau^P}{(1+r)^\tau} \right] + \max_{\tau \in \mathcal{S}_0} \widetilde{E}\left[1_{\{\tau<\infty\}} \frac{G_\tau^C}{(1+r)^\tau} \right]$$

because cannot use the same τ to maximize both of them.

$$= V_0^P + V_0^C$$

Exer 4.2

When selling the option, you hedge with Δ (calculated already in Expl 4.2.1).

So when buying the option, you can hedge with $-\Delta$.

The agent should hedge with $-\Delta$ and exercise at $\tau^* = \inf\{n : \xi_n = T\}$.

Specifically:

- $-\Delta_0 = 0.4\overline{3}$, $S_0 = 4$

  At time 0, buy $0.4\overline{3}$ shares

  Portfolio: stock: $0.4\overline{3}$

  m.m.: $-1.36 - 0.4\overline{3}(4)$

  $= -3.09\overline{3}$

- If $\xi_1 = T$: $(S_1 = 2)$

  Exercise option: $+3$

  Sell stock: $+0.4\overline{3}(2)$

  Pay off m.m.: $-3.09\overline{3}(1.25)$

  Total: 0 ✓

- If $\xi_1 = H$: $(S_1 = 8)$

  $-\Delta_1(H) = \frac{1}{12} = 0.08\overline{3}$

  Sell 0.35 shares

  Portfolio: stock: $0.4\overline{3} - 0.35 = 0.08\overline{3}$

  m.m.: $-3.09\overline{3}(1.25) + 0.35(8)$

  $= -1.0\overline{6}$

- If $\xi_1 = H$, $\xi_2 = T$: $(S_2 = 4)$

  Exercise option: $+1$

  Sell shares: $+0.08\overline{3}(4)$

  Pay off m.m.: $-1.0\overline{6}(1.25)$

  Total: 0 ✓

- If $\xi_1 = H$, $\xi_2 = H$: $(S_2 = 16)$

  Discard option: 0

  Sell shares: $+0.08\overline{3}(16)$

  Pay off m.m.: $-1.0\overline{6}(1.25)$

  Total: 0 ✓

Exer. 4.3

$$G_n = \left(4 - \frac{1}{n+1}\sum_{j=0}^{n} S_j\right)^+$$

$G_1(T) = 1$

$G_2(T,H) = \frac{2}{3}$, $G_2(T,T) = \frac{5}{3}$

$G_3(T,H,T) = 1$, $G_3(T,T,H) = \frac{7}{4}$,

$$G_3(T,T,T) = \frac{17}{8}$$

All others are 0.

$$V_3 = G_3^+ = G_3$$

$$V_2 = G_2 \vee \left(\frac{1}{1+r}\tilde{E}_2[V_3]\right)$$

$$= G_2 \vee \left(\frac{1}{1+r}\left(\tilde{p}\, V_3(\dots,H) + \tilde{q}\, V_3(\dots,T)\right)\right)$$

$$= G_2 \vee \left(\frac{2}{5}\left(V_3(\dots,H) + V_3(\dots,T)\right)\right)$$

$$V_2(H,H) = G_2(H,H) \vee \left(\frac{2}{5}\left(V_3(H,H,H) + V_3(H,T,T)\right)\right)$$

$$= 0 \vee \left(\frac{2}{5}(0+0)\right) = 0$$

$$V_2(H,T) = 0 \vee \left(\frac{2}{5}(0+0)\right) = 0$$

$$V_2(T,H) = \frac{2}{3} \vee \left(\frac{2}{5}(0+1)\right) = \frac{2}{3}$$

$$V_2(T,T) = \frac{5}{3} \vee \left(\frac{2}{5}\left(\frac{7}{4} + \frac{17}{8}\right)\right)$$

$$= \frac{5}{3} \vee \left(\frac{2}{5} \cdot \frac{31}{8}\right) = \frac{5}{3} \vee \frac{31}{20} = \frac{5}{3}$$

$$V_1(H) = G_1(H) \vee \left(\frac{2}{5}\left(V_2(H,H) + V_2(H,T)\right)\right)$$

$$= 0 \vee \left(\frac{2}{5}(0+0)\right) = 0$$

$$V_1(T) = 1 \vee \left(\frac{2}{5}\left(\frac{2}{3} + \frac{5}{3}\right)\right)$$

$$= 1 \vee \left(\frac{2}{5} \cdot \frac{7}{3}\right) = 1 \vee \frac{14}{15} = 1$$

$$V_0 = G_0 \vee \left(\frac{2}{5} \left(V_1(H) + V_1(T) \right) \right)$$

$$= 0 \vee \left(\frac{2}{5} (0 + 1) \right) = \boxed{\frac{2}{5}}$$

$$\tau^* = \inf \{ V_n = G_n \}$$

$$V_0 = \frac{2}{5}, \quad G_0 = 0$$

$$V_1(H) = 0 = G_1(H)$$

$$V_1(T) = 1 = G_1(T)$$

$\boxed{\tau^* = 1}$ Exercise at time 1 no matter what

Exer. 4.6 (i)

$$\widetilde{E}_n\left[\frac{G_{n+1}}{(1+r)^{n+1}}\right] = \frac{K}{(1+r)^{n+1}} - \widetilde{E}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right]$$

b/c $\left\{\frac{S_n}{(1+r)^n}\right\}$ is a $\widetilde{P}$- mart

$$\stackrel{\downarrow}{=} \frac{K}{(1+r)^{n+1}} - \frac{S_n}{(1+r)^n}$$

$$= \frac{K}{(1+r)^{n+1}} - \frac{K}{(1+r)^n} + \frac{K-S_n}{(1+r)^n}$$

$$= \frac{K-K(1+r)}{(1+r)^n} + \frac{G_n}{(1+r)^n}$$

$$= \frac{-Kr}{(1+r)^n} + \frac{G_n}{(1+r)^n}$$

$$\leq \frac{G_n}{(1+r)^n}.$$

So $\left\{ \frac{G_n}{(1+r)^n} \right\}$ is a $\widetilde{P}$ - supermart.

If $\tau \in \mathcal{S}_0$ and $\tau \le N$, then

$$\widetilde{E}\left[\frac{G_\tau}{(1+r)^\tau} \right] = \widetilde{E}\left[\frac{G_{\tau \wedge N}}{(1+r)^{\tau \wedge N}} \right]$$

optional stopping thm

$$\le \widetilde{E}\left[\frac{G_0}{(1+r)^0} \right]$$

$$= G_0 = K - S_0$$

So $\tau^* = 0$ maximizes (4.8.3) and $V_0 = K - S_0$.

Exer. 4.7

$$\tilde{E}_n\left[\frac{G_{n+1}}{(1+r)^{n+1}}\right] = \tilde{E}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] - \frac{K}{(1+r)^{n+1}}$$

$$= \frac{S_n - K}{(1+r)^n} + \frac{Kr}{(1+r)^{n+1}}$$

$$\geq \frac{G_n}{(1+r)^n}$$

So $\left\{\frac{G_n}{(1+r)^n}\right\}$ is a $\tilde{P}$-submart.

If $\tau \in \mathcal{S}_0$ and $\tau \leq N$, then

$$\tilde{E}\left[\frac{G_\tau}{(1+r)^\tau}\right] = \tilde{E}\left[\frac{G_{\tau\wedge N}}{(1+r)^{\tau\wedge N}}\right]$$

$$\leq \tilde{E}\left[\frac{G_N}{(1+r)^N}\right]$$

$$= \widetilde{E}\left[\frac{S_N}{(1+r)^N}\right] - \frac{K}{(1+r)^N}$$

$$= S_0 - \frac{K}{(1+r)^N}.$$

So $\boxed{\tau^* = N}$ maximizes (4.8.3) and

$$\boxed{V_0 = S_0 - \frac{K}{(1+r)^N}}$$