Exer. 3.2

(i) $\tilde{P}(\Omega) = \sum_{\omega \in \Omega} Z(\omega) P(\omega) \stackrel{\text{def}}{=} E[Z] = 1.$

(ii) $\tilde{E}[Y] = \sum_{\omega \in \Omega} Y(\omega) \tilde{P}(\omega)$

$$= \sum_{\omega \in \Omega} Y(\omega) Z(\omega) P(\omega) = E[YZ].$$

(iii) Assume $P(A) = 0$. Then

$$0 = P(A) = \sum_{\omega \in A} P(\omega) \Rightarrow P(\omega) = 0 \quad \forall \omega \in A$$

$$\therefore \tilde{P}(A) = \sum_{\omega \in A} \tilde{P}(\omega) = \sum_{\omega \in A} Z(\omega) \overset{0}{\cancel{P(\omega)}} = 0.$$

(iv) Assume $P(Z>0)=1$ and $\tilde{P}(A)=0$. Then

$$P(A) = P(A \cap \{Z>0\}) + \underbrace{\cancel{P(A \cap \{Z=0\})}}_{0 \text{ b/c } P(Z=0)=0}$$

$$= \sum_{\omega \in A \cap \{Z>0\}} P(\omega)$$

$$= \sum_{\omega \in A \cap \{Z>0\}} \frac{\overset{0 \text{ b/c } \omega \in A}{\cancel{\tilde{P}(\omega)}}}{Z(\omega)}$$

$$= 0.$$

(v) $P(A)=1$ iff $P(A^c)=0$

iff $\tilde{P}(A^c)=0$

iff $\tilde{P}(A)=1$.

(vi) Let $\Omega = \{H, T\}$, $P(H) = P(T) = \frac{1}{2}$,

and $Z(\omega) = \begin{cases} 2 & \text{if } \omega = H, \\ 0 & \text{if } \omega = T. \end{cases}$

Then $P(Z \geq 0) = 1$ and $E[Z] = 1$.

But $\tilde{P}(T) = Z(T)P(T) = 0$, and

$P(T) = \frac{1}{2}$,

so they are not equivalent.

Exer. 3.4

(i) $$\zeta_3 = \frac{Z_3}{(1+r)^3} = \left(\frac{4}{5}\right)^3 Z_3 = \frac{64}{125} Z_3$$

$$\zeta_3(H,H,H) = \frac{64}{125} \cdot \frac{27}{64} = \frac{27}{125}$$

$$\zeta_3(H,H,T) = \frac{64}{125} \cdot \frac{27}{32} = \frac{54}{125}$$

$$\zeta_3(H,T,T) = \frac{64}{125} \cdot \frac{27}{16} = \frac{108}{125}$$

$$\zeta_3(T,T,T) = \frac{64}{125} \cdot \frac{27}{8} = \frac{216}{125}$$

(ii) $V_3 = \left(\frac{1}{4} \sum_{n=0}^{3} S_n - 4 \right)^+$

$$V_0 = E[\zeta V_3]$$

$$= \frac{8}{27} \cdot \frac{27}{125} \left(\frac{4+8+16+32}{4} - 4 \right)^+$$

$$+ \frac{4}{27} \cdot \frac{54}{125} \left(\frac{4+8+16+8}{4} - 4 \right)^+$$

$$+ \frac{4}{27} \cdot \frac{54}{125} \left(\frac{4+8+4+8}{4} - 4 \right)^+$$

$$+ \frac{4}{27} \cdot \frac{54}{125} \left(\frac{4+2+4+8}{4} - 4 \right)^+$$

$$+\frac{2}{27}\cdot\frac{108}{125}\left(\frac{4+8+4+2}{4}-4\right)^+$$

$$+\frac{2}{27}\cdot\frac{108}{125}\left(\frac{4+2+4+2}{4}-4\right)^+$$

$$+\frac{2}{27}\cdot\frac{108}{125}\left(\frac{4+2+1+2}{4}-4\right)^+$$

$$+\frac{1}{27}\cdot\frac{216}{125}\left(\frac{4+2+1+\frac{1}{2}}{4}-4\right)^+$$

$$=\frac{8}{125}\left(11+5+2+\frac{1}{2}+\frac{1}{2}+0+0+0\right)$$

$$=\frac{8}{125}(19)=\frac{152}{125}\ \checkmark$$

(iii) $$Y_2=\frac{Z_2}{(1+r)^2}=\frac{16}{25}Z_2$$

$$Y_2(H,T)=\frac{16}{25}\cdot\frac{9}{8}=\frac{18}{25}$$

(iv) $$V_2 = \frac{1}{\zeta_2} E_2[\zeta_3 V_3]$$

$$V_2(H,T) = \frac{25}{18} E[\zeta_3 V_3 \mid \xi_1 = H,\ \xi_2 = T]$$

$$= \frac{25}{18} E[\zeta_3(H,T,\xi_3)\, V_3(H,T,\xi_3)]$$

$$= \frac{25}{18}\left(\frac{2}{3}\cdot\frac{54}{125}\cdot 2 + \frac{1}{3}\cdot\frac{108}{125}\cdot\frac{1}{2}\right)$$

$$= \frac{25}{18}\left(\frac{4\cdot 18}{125} + \frac{18}{125}\right)$$

$$= \frac{4}{5} + \frac{1}{5} = 1$$

$$V_2(T,H) = \frac{25}{18} E[\zeta_3(T,H,\xi_3)\, V_3(T,H,\xi_3)]$$

$$= \frac{25}{18}\left(\frac{2}{3}\cdot\frac{54}{125}\cdot\frac{1}{2} + \cancel{\frac{1}{3}\cdot\frac{54}{125}\cdot 0}\right)$$

$$= \frac{1}{5}$$

Exer 3.6

$$U(x) = \log x$$

$$U'(x) = \frac{1}{x} = y \Rightarrow x = \frac{1}{y}$$

$$I(y) = \frac{1}{y}$$

$$X_0 = E[\zeta I(\lambda\zeta)] = E\left[\zeta \cdot \frac{1}{\lambda\zeta}\right] = \frac{1}{\lambda}$$

$$\Rightarrow \lambda = \frac{1}{X_0}$$

$$X_N = I(\lambda\zeta) = \frac{1}{\lambda\zeta} = \frac{X_0}{\zeta}.$$

$\left\{\frac{X_n}{(1+r)^n}\right\}$ a $\tilde{P}$-mart.

$$\Rightarrow \frac{X_n}{(1+r)^N} = \tilde{E}_n\left[\frac{X_N}{(1+r)^N}\right]$$

$$\Rightarrow X_n = \frac{1}{(1+r)^{N-n}} \tilde{E}_n[X_N]$$

$$\therefore \quad X_n = \frac{1}{(1+r)^{N-n}} \tilde{E}_n\left[\frac{X_0}{\zeta}\right]$$

$$= \frac{1}{(1+r)^{N-n}} \tilde{E}_n\left[\frac{(1+r)^N X_0}{Z}\right]$$

$$= (1+r)^n \tilde{E}_n\left[\frac{X_0}{Z}\right]$$

$$= (1+r)^n \cdot \frac{1}{Z_n} E_n\left[Z \cdot \frac{X_0}{Z}\right]$$

$$= \frac{(1+r)^n}{Z_n} \cdot X_0 = \frac{X_0}{\zeta}$$

Exer. 3.7

$$U(x) = \frac{1}{p} x^p$$

$$U'(x) = x^{p-1} = y \implies x = y^a, \quad a = \frac{1}{p-1}$$

$$I(y) = y^a$$

$$X_0 = E[\zeta I(\lambda\zeta)] = \lambda^a E[\zeta^{a+1}]$$

$$\Rightarrow \quad \lambda = \frac{X_0^{1/a}}{\left(E[\zeta^{a+1}]\right)^{1/a}}$$

$$X_N = I(\lambda\zeta) = \lambda^a \zeta^a$$

$$= \frac{X_0 \zeta^a}{E[\zeta^{a+1}]}$$

$$= \frac{X_0 Z^a}{(1+r)^{Na}} \cdot \frac{1}{E\left[\frac{Z^{a+1}}{(1+r)^{N(a+1)}}\right]}$$

$$= \frac{X_0 Z^a}{(1+r)^{Na}} \cdot \frac{(1+r)^{N(a+1)}}{E[Z^{a+1}]}$$

$$= \frac{X_0 (1+r)^N Z^a}{E[Z^{a+1}]}$$

$$= \frac{X_0 (1+r)^N Z^{1/(p-1)}}{E[Z^{p/(p-1)}]}.$$