# MAP 4934 Ch. 2 solns

## Exer. 2.3

If $M_n$ is a func. of $\xi_1, \dots, \xi_n$, then
$\varphi(M_n)$ is a func. of $\xi_1, \dots, \xi_n$,
so $\{M_n\}_{n=0}^{N}$ is adapted. ✓

By Thm 2.3.2 (v) (conditional Jensen's inequ.),

$$E_n[\varphi(M_{n+1})] \geq \varphi(E_n[M_{n+1}])$$

Since $\{M_n\}_{n=0}^{N}$ is a mart.,

$$E_n[M_{n+1}] = M_n .$$

Therefore,

$$E_n[\varphi(M_{n+1})] \geq \varphi(M_n),$$

so $\{\varphi(M_n)\}_{n=0}^{N}$ is a submart.

Exer. 2.4

(i) $E_n[M_{n+1}] = E_n[M_n + X_{n+1}]$

$\overset{(i)}{=} E_n[M_n] + E_n[X_{n+1}]$

$\overset{(ii)}{=} M_n + E_n[X_{n+1}]$

$\overset{(iv)}{=} M_n + E[X_{n+1}]$

$= M_n + 0 = M_n.$

(ii) $E_n[S_{n+1}] = E_n\left[e^{\sigma(M_n + X_{n+1})}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1}\right]$

$= \left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1} E_n\left[e^{\sigma M_n} e^{\sigma X_{n+1}}\right]$

$\overset{(ii)+(iv)}{=} \left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1} e^{\sigma M_n} E\left[e^{\sigma X_{n+1}}\right]$

$= \left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right) S_n\left(\frac{1}{2}e^{\sigma} + \frac{1}{2}e^{-\sigma}\right) = S_n.$

## Exer. 2.5

(i) $$M_n^2 = \sum_{j=0}^{n-1} \left(M_{j+1}^2 - M_j^2\right)$$

$$= \sum_{j=0}^{n-1} (M_{j+1} - M_j)(M_{j+1} + M_j)$$

$$= \sum_{j=0}^{n-1} (M_{j+1} - M_j)\left((M_{j+1} - M_j) + 2M_j\right)$$

$$= \sum_{j=0}^{n-1} (M_{j+1} - M_j)^2 + 2\sum_{j=0}^{n-1} M_j (M_{j+1} - M_j)$$

$$= \left(\sum_{j=0}^{n-1} X_{j+1}^2\right) + 2 I_n$$

$$= \left(\sum_{j=0}^{n-1} 1\right) + 2 I_n = n + 2 I_n$$

$$\Rightarrow \quad I_n = \frac{1}{2} M_n^2 - \frac{n}{2}$$

(ii) $E_n[f(I_{n+1})]$

$$= E_n\left[f\left(\frac{M_{n+1}^2 - n}{2}\right)\right]$$

$$= E_n\left[f\left(\frac{(M_n + X_{n+1})^2 - n}{2}\right)\right] = h(M_n),$$

where

$$h(m) = E\left[f\left(\frac{(m + X_{n+1})^2 - n}{2}\right)\right]$$

$$= \frac{1}{2} f\left(\frac{(m+1)^2 - n}{2}\right) + \frac{1}{2} f\left(\frac{(m-1)^2 - n}{2}\right).$$

Note that

$$h(-m) = \frac{1}{2} f\left(\frac{(-m+1)^2 - n}{2}\right) + \frac{1}{2} f\left(\frac{(-m-1)^2 - n}{2}\right)$$

$$= \frac{1}{2} f\left(\frac{(m-1)^2 - n}{2}\right) + \frac{1}{2} f\left(\frac{(m+1)^2 - n}{2}\right)$$

$$= h(m),$$

so $h(m) = h(|m|)$ for all $m$.

Define $g(i) = h(\sqrt{2i+n})$.

Since $|M_n| = \sqrt{2I_n + n}$, we get

$$g(I_n) = h(|M_n|) = h(M_n) = E_n[f(I_{n+1})].$$

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Exer. 2.6

By the mart. prop., $E_n[M_{n+1}] = M_n$.

Also, $E_n[M_n] = M_n$.

So $E_n[M_{n+1} - M_n] = 0$. Now...

$$\begin{aligned}
E_n[I_{n+1}] &= E_n[I_n + \Delta_n (M_{n+1} - M_n)] \\
&= E_n[I_n] + E_n[\Delta_n (M_{n+1} - M_n)] \\
&= I_n + \Delta_n E_n[M_{n+1} - M_n] \\
&= I_n + \Delta_n \cdot 0 = I_n
\end{aligned}$$

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## Exer. 2.7

$S_0 = 4,\ u = 2,\ d = \frac{1}{2},\ r = 0,\ N = 2$

$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{3}{2}\right)} = \frac{1}{3}, \quad \tilde{q} = \frac{2}{3}$$

Let $\Delta_0 = 0, \quad \Delta_1 = 1_{\{\xi_1 = H\}}$.

Since $r = 0$, $\{S_n\}$ is a mart. under $\tilde{P}$.

By Exer. 2.6, if $I_0 = 0$ and

$$I_n = \sum_{j=0}^{n-1} \Delta_j (S_{j+1} - S_j),$$

then $\{I_n\}$ is a mart. under $\tilde{P}$.

We will show $\{I_n\}$ is <u>not</u> Markov under $\tilde{P}$.

First,

$$I_0 = 0$$

$$I_1 = \Delta_0 (S_1 - S_0) = 0$$

$$I_2 = \Delta_0 (S_1 - S_0) + \Delta_1 (S_2 - S_1)$$

$$= \Delta_1 (S_2 - S_1)$$

$$= \begin{cases} S_2 - S_1 & \text{if } \xi_1 = H, \\ 0 & \text{if } \xi_1 = T. \end{cases}$$

$$I_2(H,H) = S_2(H,H) - S_1(H,H)$$

$$= 16 - 8 = 8$$

$$I_2(H,T) = S_2(H,T) - S_1(H,T)$$

$$= 4 - 8 = -4$$

$$I_2(T,H) = I_2(T,T) = 0$$

Suppose $\{I_n\}$ is Markov under $\tilde{P}$.

Then

$$\tilde{E}_1[I_2^2] = g(I_1) = \underbrace{g(0)}_{\text{constant, not random}} \quad \text{for some } g.$$

$$\therefore \tilde{E}_1[I_2^2](H) = \tilde{E}_1[I_2^2](T).$$

$$\tilde{E}_1[I_2^2](H) = \tilde{E}[I_2^2 \mid \xi_1 = H]$$

$$= \tilde{E}[I_2(H, \xi_2)^2]$$

$$= \tilde{p}\, I_2(H,H)^2 + \tilde{q}\, I_2(H,T)^2$$

$$= \frac{1}{3}\cdot 8^2 + \frac{2}{3}(-4)^2$$

$$= \frac{64}{3} + \frac{32}{3} = \frac{96}{3} = 32$$

$$\tilde{E}_1[I_2^2](T) = \tilde{p}\, I_2(T,H)^2 + \tilde{q}\, I_2(T,T)^2$$
$$= 0,$$

contradiction.

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## Exer. 2.11

(i) For all $x \in \mathbb{R}$

$$(x-K) + (K-x)^+ = \begin{cases} x-K+K-x & \text{if } K > x, \\ x-K & \text{if } K \leq x \end{cases}$$

$$= \begin{cases} x-K & \text{if } x \geq K, \\ 0 & \text{if } x < K \end{cases}$$

$$= (x-K)^+.$$

Therefore,

$$F_N + P_N = (S_N - K) + (K - S_N)^+$$

$$= (S_N - K)^+ = C_N$$

(ii)

$$F_n + P_n = \widetilde{E}_n\left[\frac{F_N}{(1+r)^{N-n}}\right] + \widetilde{E}_n\left[\frac{P_N}{(1+r)^{N-n}}\right]$$

$$= \widetilde{E}_n\left[\frac{F_N + P_N}{(1+r)^{N-n}}\right]$$

$$= \widetilde{E}_n\left[\frac{C_N}{(1+r)^{N-n}}\right] = C_n$$

(iii)

$$F_0 = \widetilde{E}\left[\frac{F_N}{(1+r)^N}\right] = \widetilde{E}\left[\frac{S_N - K}{(1+r)^N}\right]$$

$$= \widetilde{E}\left[\frac{S_N}{(1+r)^N}\right] - \frac{K}{(1+r)^N} = S_0 - \frac{K}{(1+r)^N}$$

because $\left\{\frac{S_n}{(1+r)^n}\right\}$ is a mart. under $\widetilde{P}$

## Exer. 2.13

(i) Since $S_{n+1} = Y_{n+1} S_n$ in class, let's use $T_n$ instead for the sum, i.e.

$T_n = \sum_{k=0}^{n} S_k$ and we want to show

that $\{(S_n, T_n)\}$ is Markov (under $\tilde{P}$).

$$\tilde{E}_n[f(S_{n+1}, T_{n+1})]$$

$$= \tilde{E}_n[f(Y_{n+1} S_n, T_n + S_{n+1})]$$

$$= \tilde{E}_n[f(Y_{n+1} S_n, T_n + Y_{n+1} S_n)]$$

$$= g(S_n, T_n),$$

where

$$g(s,t) = \widetilde{E}\left[f(Y_{n+1}s,\ t + Y_{n+1}s)\right]$$

$$= \tilde{p} f(us, t+us) + \tilde{q} f(ds, t+ds)$$

(ii) $$V_N = f\left(\frac{1}{N+1}\sum_{n=0}^{N} S_n\right) = f\left(\frac{T_N}{N+1}\right)$$

$$\boxed{v_n(s,t) = f\left(\frac{t}{N+1}\right)}$$

$$V_n = \frac{1}{1+r}\widetilde{E}_n\left[V_{n+1}\right]$$

$$= \frac{1}{1+r}\widetilde{E}_n\left[v_{n+1}(S_{n+1}, T_{n+1})\right]$$

$$= \frac{1}{1+r}\widetilde{E}_n\left[v_{n+1}(Y_{n+1}S_n,\ T_n + Y_{n+1}S_n)\right]$$

$$v_n(s,t) = \frac{1}{1+r}\widetilde{E}\left[v_{n+1}(Y_{n+1}s,\ t + Y_{n+1}s)\right]$$

$$\boxed{v_n(s,t) = \frac{1}{1+r}\left(\tilde{p}\, v_{n+1}(us, t+us) + \tilde{q}\, v_{n+1}(ds, t+ds)\right)}$$