MAP 4934 Ch. 1 HW solns

Exer. 1.1

$$X_0 = 0 \Rightarrow X_1 = \Delta_0 S - (1+r)\Delta_0 S_0$$
$$= \Delta_0 (S_1 - (1+r) S_0)$$

$$S_1 - (1+r) S_0 = \begin{cases} (u-(1+r)) S_0 > 0 & \text{if } \omega = H, \\ (d-(1+r)) S_0 < 0 & \text{if } \omega = T. \end{cases}$$

Case 1. $\Delta_0 = 0$.

Then $X_1 = 0$ and $P(X_1 < 0) = P(X_1 > 0) = 0$

Case 2. $\Delta_0 > 0$.

Then $P(X_1 > 0) = P(H) = p > 0$,
$P(X_1 < 0) = P(T) = 1-p > 0$.

Case 3. $\Delta_0 < 0$.

Then $P(X_1 > 0) = P(T) = 1-p > 0,$

$P(X_1 < 0) = P(H) = p > 0.$

Since we never have

$P(X_1 < 0) = 0$ and $P(X_1 > 0) > 0,$

there is no arbitrage.

Exer. 1.3

$$V_0 = \frac{1}{1+r}\left(\tilde{p}\, V_1(H) + \tilde{q}\, V_1(T)\right)$$

$$= \frac{1}{1+r}\left(\frac{1+r-d}{u-d}\, S_1(H) + \frac{u-1-r}{u-d}\, S_1(T)\right)$$

$$= \frac{1}{1+r}\left(\frac{1+r-d}{u-d} \cdot u S_0 + \frac{u-1-r}{u-d} \cdot d S_0\right)$$

$$= \frac{u + ru - ud + ud - d - rd}{(1+r)(u-d)}\, S_0$$

$$= \frac{u-d+r(u-d)}{(1+r)(u-d)} S_0$$

$$= \frac{(u-d)(1+r)}{(1+r)(u-d)} S_0 = S_0$$

Exer. 1.6

The bank should short $\frac{1}{2}$ share, receive \$2, then invest that \$2 in the money market.

- If stock price goes up, they collect \$3 payoff from option, withdraw \$2.50 from money market, use \$4 to pay back $\frac{1}{2}$ share of stock, and have \$1.50 left over.

- If stock price goes down, option is worthless, they withdraw \$2.50 from money market, use \$1 to pay back $\frac{1}{2}$ share of stock, and have \$1.50 left over.

Exer. 1.8

(i) $V_n(s,y) = \frac{1}{1+r}\left(\tilde{p}\, V_{n+1}(2s, y+2s) + \tilde{q}\, V_{n+1}\left(\frac{s}{2}, y+\frac{s}{2}\right)\right)$

$$V_n(s,y) = \frac{2}{5}\left(V_{n+1}(2s, y+2s) + V_{n+1}\left(\frac{s}{2}, y+\frac{s}{2}\right)\right)$$

(ii) $V_3(s,y) = \left(\frac{y}{4} - 4\right)^+$

$$V_2(s,y) = \frac{2}{5}\left(V_3(2s, y+2s) + V_3\left(\frac{s}{2}, y+\frac{s}{2}\right)\right)$$

$$V_2(s,y) = \frac{2}{5}\left(\left(\frac{y}{4} + \frac{s}{2} - 4\right)^+ + \left(\frac{y}{4} + \frac{s}{8} - 4\right)^+\right)$$

$$V_0(4,4) = \frac{2}{5}\left(V_1(8,12) + V_1(2,6)\right)$$

$$V_1(8,12) = \frac{2}{5}\left(V_2(16,28) + V_2(4,16)\right)$$

$$V_1(2,6) = \frac{2}{5}\left(V_2(4,10) + V_2(1,7)\right)$$

$$V_2(16,28) = \frac{2}{5}\left((7+8-4)^+ + (7+2-4)^+\right)$$

$$= \frac{2}{5}(11+5) = \frac{32}{5}$$

$$V_2(4,16) = \frac{2}{5}\left((4+2-4)^+ + (4+\tfrac{1}{2}-4)^+\right)$$

$$= \frac{2}{5}\left(2+\tfrac{1}{2}\right) = 1$$

$$V_2(4,10) = \frac{2}{5}\left((\tfrac{5}{2}+2-4)^+ + (\tfrac{5}{2}+\tfrac{1}{2}-4)^+\right)$$

$$= \frac{2}{5}\left(\tfrac{1}{2}+0\right) = \frac{1}{5}$$

$$V_2(1,7) = \frac{2}{5}\left((\tfrac{7}{4}+\tfrac{1}{2}-4)^+ + (\tfrac{7}{4}+\tfrac{1}{8}-4)^+\right) = 0$$

$$V_1(8,12) = \frac{2}{5}\left(\frac{32}{5} + 1\right) = \frac{2 \cdot 37}{25} = \frac{74}{25}$$

$$V_1(2,6) = \frac{2}{5}\left(\frac{1}{5} + 0\right) = \frac{2}{25}$$

$$V_0(4,4) = \frac{2}{5}\left(\frac{74}{25} + \frac{2}{25}\right) = \frac{2 \cdot 76}{125} = \frac{152}{125}$$

(iii)

$$\delta_n(s,y) = \frac{V_{n+1}(2s, y+2s) - V_{n+1}\left(\frac{s}{2}, y + \frac{s}{2}\right)}{2s - \frac{s}{2}}$$

$$\delta_n(s,y) = \frac{2}{3s}\left(V_{n+1}(2s, y+2s) - V_{n+1}\left(\frac{s}{2}, y + \frac{s}{2}\right)\right)$$