For $x \in \mathbb{R}$, let $x^0 := 1$. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, let $x^n := x \cdot x^{n-1}$. For $x \in \mathbb{R}\backslash\{0\}$ and $n \in \mathbb{N}$, let $x^{-n} := 1/x^n$.

Proposition 1. *For all $x \in \mathbb{R}\backslash\{0\}$ and all $m, n \in \mathbb{Z}$, we have $x^n x^m = x^{n+m}$ and $(x^n)^m = x^{nm}$.*

Proof. Exercise 3.6.9. □

Let $n \in \mathbb{N}$ and define $f : [0, \infty) \to \mathbb{R}$ by $f(x) = x^n$. Then f is one-to-one, so f^{-1} exists. Also $f([0, \infty)) = [0, \infty)$, so $f^{-1} : [0, \infty) \to \mathbb{R}$. For $y \in [0, \infty)$, we define $y^{1/n} := f^{-1}(y)$. It then follows that

$$y^{1/n} = x \text{ if and only if } y = x^n. \tag{1}$$

Moreover, for all $y \in [0, \infty)$, we have

$$(y^{1/n})^n = x^n = y. \tag{2}$$

Proposition 2. *For all $x > 0$, all $m \in \mathbb{Z}$, and all $n, k \in \mathbb{N}$, we have $(x^m)^{1/n} = (x^{km})^{1/kn}$.*

Proof. Let $u = x^m$, $y = u^k$, and $z = u^{1/n}$. By Proposition 1, we have $z^{kn} = (z^n)^k$. By (2), we have $z^n = (u^{1/n})^n = u$. Therefore, $z^{kn} = (z^n)^k = u^k = y$. By (1), this implies to $y^{1/kn} = z$.

By Proposition 1, we have $y = u^k = (x^m)^k = x^{mk}$. Therefore, $y^{1/kn} = z$ implies

$$(x^{mk})^{1/kn} = (x^m)^{1/n},$$

which is what we wanted to prove. □

For $x > 0$ and $r \in \mathbb{Q}$, Proposition 2 allows us to define $x^r = (x^m)^{1/n}$, where $m \in \mathbb{Z}$, $n \in \mathbb{N}$, and $r = m/n$.

Proposition 3. *For all $x > 0$, $m \in \mathbb{Z}$, and $n \in \mathbb{N}$, we have $(x^m)^{1/n} = (x^{1/n})^m$.*

Proof. Exercise 3.6.11. □

Proposition 4. *For all $x > 0$ and $r, s \in \mathbb{Q}$, we have $x^r x^s = x^{r+s}$ and $(x^r)^s = x^{rs}$.*

Proof. Exercise 3.6.12. □