Definition 1 (limit of a sequence, p.26)**.** Let $\{x_n\}$ be a sequence and $x_0 \in \mathbb{R}$. Then $\{x_n\}$ converges to x_0, written as $\lim_{n\to\infty} x_n = x_0$ or $x_n \to x_0$, if and only if the following is true:

For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have $|x_n - x_0| < \varepsilon$.

Definition 2 (limit of a function, p.82)**.** Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, $x_0 \in \mathbb{R}$, and $L \in \mathbb{R}$. Then $\lim_{x\to x_0} f(x) = L$ if and only if both of the following are true:

(i) x_0 is a limit point of D.

(ii) For all sequences $\{x_n\}$ in $D\backslash\{x_0\}$, if $x_n \to x_0$, then $f(x_n) \to L$.

Theorem 3. *Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, $x_0 \in \mathbb{R}$, and $L \in \mathbb{R}$. Then $\lim_{x\to x_0} f(x) = L$ if and only if both of the following are true:*

(i) x_0 is a limit point of D.

(ii) For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in D\backslash\{x_0\}$, if $|x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$.

Proof. Exercise. (Mimic the proof of Theorem 3.20 on p.72.) □

Definition 4 (continuity, p.53)**.** Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $x_0 \in D$. Then f is continuous at x_0 if and only if the following is true:

For all sequences $\{x_n\}$ in D, if $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

Theorem 5. *Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $x_0 \in D$. Then the following are equivalent:*

(i) f is continuous at x_0.

(ii) For all sequences $\{x_n\}$ in $D\backslash\{x_0\}$, if $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

(iii) For all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x \in D$, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

(iv) Either x_0 is not a limit point of D, or $\lim_{x\to x_0} f(x) = f(x_0)$.

Proof. Clearly, (i) implies (ii). The proof that (ii) implies (i) is an exercise. (Assume (ii) and then prove (i) by contradiction. Exercise 2.4.10 may be helpful.) The equivalence of (i) and (iii) is Theorem 3.20 on p.72. The equivalence of (i) and (iv) is Exercise 3.7.8 on p.85. □

Definition 6 (differentiability, p.88)**.** Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $x_0 \in D$. Then f is differentiable at x_0 if and only if both of the following are true:

(i) There exists a neighborhood (i.e. an open interval) I such that $x_0 \in I$ and $I \subset D$.

(ii) There exists $L \in \mathbb{R}$ such that $\lim_{x\to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = L$.

In this case, we use the notation $f'(x_0)$ to denote this number L.