Final exam, due Monday, April 21, 2014

Instructions

I will choose a subset of these problems to grade. I will choose at least one problem from each of Parts A, B, and C. Graded problems will be worth at most 10 points. Ungraded problems will be given up to two points for reasonable effort.

Part A

1. Let $\mathcal{F}$ be a σ-algebra on a set Ω. Prove that either $\mathcal{F}$ is finite or $\mathcal{F}$ is uncountable.

2. Let $X_1, X_2, \dots$ be random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. Suppose there exist constants $C < \infty$ and $p > 1$ such that $|E[X_i X_j]| \leqslant C(|i-j| \vee 1)^{-p}$ for all i and j. Show that $(X_1 + \cdots + X_n)/n \to 0$ in probability.

3. Let X_0 be uniformly distributed on $(0,1)$ and define $X_n > 0$ inductively by declaring that X_{n+1} is chosen at random from the interval $(X_n, 2X_n)$, i.e., $(X_{n+1} - X_n)/X_n$ is uniformly distributed on $(0,1)$ and independent of $X_1, \dots, X_n$. Show that $n^{-1} \log X_n \to c$ a.s. and compute c.

Part B

Let $X_1, X_2, \dots$ be i.i.d. $\mathbb{Z}$-valued random variables. Let $S_n = X_1 + \cdots + X_n$ and $S_0 = 0$. Let $\alpha_0 = 0$ and for $k \in \mathbb{Z}_+$, let $\alpha_{k+1} = \inf\{n > \alpha_k : S_n > S_{\alpha_k}\}$. (Note that if $\alpha_k = \infty$, then $\{n > \alpha_k : S_n > S_{\alpha_k}\} = \varnothing$ regardless of how S_{α_k} is defined.) Let $\alpha = \alpha_1$ and recall that $P(\alpha_k < \infty) = P(\alpha < \infty)^k$. Similarly, let $\beta_0 = 0$ and for $k \in \mathbb{Z}_+$, let $\beta_{k+1} = \inf\{n > \beta_k : S_n < S_{\beta_k}\}$. (Note that if $\beta_k = \infty$, then $\{n > \beta_k : S_n < S_{\beta_k}\} = \varnothing$ regardless of how S_{β_k} is defined.) Let $\beta = \beta_1$ and recall that $P(\beta_k < \infty) = P(\beta < \infty)^k$.

1. (i) Prove that if $P(\alpha < \infty) < 1$, then $P(\sup_n S_n < \infty) = 1$.

   (ii) Prove that if $P(\alpha < \infty) = 1$, then $P(\sup_n S_n = \infty) = 1$.

2. Let $p_\alpha = P(\alpha < \infty)$ and $p_\beta = P(\beta < \infty)$. Prove that the four possibilities in Theorem 4.1.2 correspond, in some order, to the four cases: (a) $p_\alpha < 1$ and $p_\beta < 1$, (b) $p_\alpha < 1$ and $p_\beta = 1$, (c) $p_\alpha = 1$ and $p_\beta < 1$, and (d) $p_\alpha = 1$ and $p_\beta = 1$.

3. Suppose $P(X_1 = 1) = p > 1/2$ and $P(X_1 = -1) = 1 - p$. You may use, without proof, the fact that in this case, $\alpha_k = \inf\{n \geqslant 0 : S_n \geqslant k\}$, $\beta_k = \inf\{n \geqslant 0 : S_n \leqslant -k\}$, $S_{\alpha_k} = k$ on the event $\{\alpha_k < \infty\}$, and $S_{\beta_k} = -k$ on the event $\{\beta_k < \infty\}$.

   (i) Use Problem 2 to prove that $p_\alpha = 1$ and $p_\beta < 1$.

   (ii) Let $Y = \inf_n S_n$. Prove that $P(Y \leqslant -k) = P(\beta < \infty)^k$.

   (iii) Prove that $E\alpha = 1/(2p - 1)$. (Hint: Apply Wald's equation to $\alpha \wedge n$ and take a limit.)

Part C

1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, X a square integrable random variable, $\mathcal{G} \subset \mathcal{F}$ a σ-algebra, and $\varepsilon > 0$. Prove the conditional Chebyshev's inequality:

$$P(|X| \geqslant \varepsilon \mid \mathcal{G}) \leqslant \frac{E[X^2 \mid \mathcal{G}]}{\varepsilon^2} \text{ a.s.}$$

2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G} \subset \mathcal{F}$ a σ-algebra. Let $A \in \mathcal{F}$ and $G \in \mathcal{G}$.

   (i) Prove that $P(G \mid A) = \dfrac{E[P(A \mid \mathcal{G})1_G]}{E[P(A \mid \mathcal{G})]}$.

   (ii) Prove that if $\mathcal{G} = \sigma(\{G_j\}_{j=1}^n)$, where $\Omega = \biguplus_{j=1}^n G_j$ and $P(G_j) > 0$ for all j, then the above formula gives

$$P(G_i \mid A) = \frac{P(A \mid G_i)P(G_i)}{\sum_{j=1}^n P(A \mid G_j)P(G_j)}.$$

   (Remark: This is Bayes' formula, so the equation in Part (i) is a generalization of Bayes' formula to an infinite σ-algebra.)

3. Use Bayes' formula to solve the following problem: Mary, who is pregnant, is tested to see if her unborn child has a chromosomal abnormality. The test checks only for trisomy 18 (Edwards syndrome) and trisomy 21 (Down syndrome). Among women Mary's age, trisomy 18 occurs in 1 in 2,700 pregnancies, and trisomy 21 occurs in 1 in 900 pregnancies. Assume that no fetus can have both disorders simultaneously.

   The test will only report the results "positive" or "negative". The test reports positive in 92% of the cases in which the fetus has trisomy 18; it reports positive in 85% of the cases of trisomy 21; and it gives a false positive in 5% of healthy fetuses.

   If the test comes back positive, then what is the probability that Mary's child has trisomy 18? has trisomy 21? is healthy? What if the test comes back negative?