Homework #7 for MAA 6229, due Friday, March 2, 2012

1. Let $(X, \mathcal{T})$ be a topological space and $G \subset X$. Show that $G \in \mathcal{T}$ if and only if for all $x \in G$, there exists a neighborhood V of x such that $x \in V \subset G$.

2. Applied Analysis, Exercise 4.1.

3. Applied Analysis, Exercise 4.4.

4. Show that if $\{\mathcal{T}_\alpha\}_{\alpha \in \Lambda}$ is a family of topologies on a set X, then $\mathcal{T} = \bigcap_{\alpha \in \Lambda} \mathcal{T}_\alpha$ is a topology on X.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space and $p \in [1, \infty)$. Suppose that

$$f \in L^p(X, \mathcal{M}, \mu) \cap L^\infty(X, \mathcal{M}, \mu).$$

   Show that $f \in L^q$ for all $q > p$ and that $\lim_{q \to \infty} \|f\|_q = \|f\|_\infty$.

6. Let $X = L^\infty([0,1], m)$, where m is Lebesgue measure. Let $\overline{B}$ be the closed unit ball in X. That is,

$$\overline{B} = \{f \in X : \|f\|_\infty \leq 1\}.$$

   Prove that $\overline{B}$ is not compact.