Although arbitrary Lebesgue m'ble sets $A \in \mathcal{M}$ cannot be expressed explicitly in terms of intervals, we do have the following result. Note that for two sets A and B, the <u>symmetric difference</u> is defined as $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

<u>Lemma L.88</u> Let $\mathcal{A}$ be an algebra on a set X and μ_0 a premeasure on $(X, \mathcal{A})$. Let $(X, \mathcal{M}, \bar{\mu})$ be the extension of μ_0 given in Thm L.86. Then for every $E \in \mathcal{M}$ with $\bar{\mu}(E) < \infty$ and every $\varepsilon > 0$, $\exists A \in \mathcal{A} \ni \bar{\mu}(E \triangle A) < \varepsilon$.

<u>Pf:</u> Let $E \in \mathcal{M}$ with $\bar{\mu}(E) < \infty$, and $\varepsilon > 0$. Since $\bar{\mu}(E) = \mu^*(E) = \inf \sum_{n=1}^{\infty} \mu_0(A_n)$, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ that covers E and satisfies $\sum_{n=1}^{\infty} \mu_0(A_n) < \bar{\mu}(E) + \frac{\varepsilon}{2}$. Choose

$N \in \mathbb{N} \ni \sum_{n=N+1}^{\infty} \mu_0(A_n) < \frac{\varepsilon}{2}$, and define $A = \bigcup_{n=1}^{N} A_n \in \mathcal{A}$.

Then $E \setminus A \subset \bigcup_{n=N+1}^{\infty} A_n$ and $A \setminus E \subset \left(\bigcup_1^{\infty} A_n\right) \setminus E$. Thus,

$$\begin{aligned} \bar{\mu}(E \Delta A) &= \bar{\mu}(E \setminus A) + \bar{\mu}(A \setminus E) \\ &\leq \bar{\mu}\left(\bigcup_{N+1}^{\infty} A_n\right) + \bar{\mu}\left(\bigcup_1^{\infty} A_n\right) - \bar{\mu}(E) \\ &\leq \sum_{N+1}^{\infty} \bar{\mu}(A_n) + \left(\sum_1^{\infty} \bar{\mu}(A_n)\right) - \bar{\mu}(E) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}. \quad \square \end{aligned}$$

<u>Corollary L.89</u> If E is Lebesgue m'ble with $m(E) < \infty$ and $\varepsilon > 0$, then $\exists$ a set A which is a finite union of half-open intervals such that $m(E \Delta A) < \varepsilon$.

<u>Remark:</u> The above intervals may be taken to be open or closed also, since $m(\{x\}) = 0 \ \forall x \in \mathbb{R}$.

Cor. L.89 makes precise the heuristic idea

Let us now recall an old result:

<u>Lemma L.88</u> Let $\mathcal{A}$ be an algebra on a set X and μ_0 a premeasure on $(X, \mathcal{A})$. Let $(X, \mathcal{M}, \bar{\mu})$ be the extension of μ_0 given in Thm L.86. Then for every $E \in \mathcal{M}$ with $\bar{\mu}(E) < \infty$ and every $\varepsilon > 0$, $\exists A \in \mathcal{A} \ni \bar{\mu}(E \Delta A) < \varepsilon$.

When we applied this to Lebesgue meas. on $\mathbb{R}$, we got:

<u>Corollary L.89</u> If E is Lebesgue m'ble with $m(E) < \infty$ and $\varepsilon > 0$, then $\exists$ a set A which is a finite union of half-open intervals such that $m(E \Delta A) < \varepsilon$.

If we apply it to Lebesgue meas. on $\mathbb{R}^k$, we get something similar.

First, recall that a k-cell in $\mathbb{R}^k$ is a set of the form

$$I^k = (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_k, b_k].$$

It is the k-dimensional analogue of a rectangle.

<u>Thm L.120</u> Let $E \subset \mathbb{R}^k$ be Lebesgue m'ble with $m(E) < \infty$. Then, $\forall \varepsilon > 0$, $\exists$ a set $A \subset \mathbb{R}^k$ which is a finite, p.w. disj. union of k-cells, $\ni$ $m(E \triangle A) < \varepsilon$, where m is Leb. meas. on $\mathbb{R}^k$.

<u>Pf</u>: This is an immediate consequence of Lemma L.88. $\square$

Proofs of the following three theorems are omitted. See me for references. In these theorems, m is Leb. meas. on $\mathbb{R}^k$