From this point forward, we will be working out of the book, Applied Analysis. Please read and review, on your own, Chapter 1 and its exercises. It is mostly a review of what we have already covered on metric spaces and normed spaces. We will begin in Ch. 4.

TOPOLOGICAL SPACES

Def'n 4.1 Let X be a nonempty set. Suppose $\mathcal{T} \subset \mathcal{P}(X)$ satisfies:

(a) $\phi \in \mathcal{T}$ and $X \in \mathcal{T}$.

(b) $\mathcal{T}$ is closed under arbitrary unions.

That is, if $\mathcal{a} \subset \mathcal{T}$, then $\bigcup_{G \in \mathcal{a}} G \in \mathcal{T}$.

(c) $\mathcal{T}$ is closed under finite intersections.

That is, if $\{G_1, \dots, G_n\} \subset \mathcal{T}$, then $\bigcap_{j=1}^{n} G_n \in \mathcal{T}$.

Then $\mathcal{T}$ is a <u>topology</u> on X, and $(X, \mathcal{T})$ is a <u>topological space</u>.

If (X, d) is a metric space, then $\mathcal{T} = \{U \subset X : U \text{ is open}\}$ is a topology on X. It is called the metric <u>topology</u>.

If $(X, \mathcal{T})$ is a topological space, then $A \subset X$ is <u>open</u> if $A \in \mathcal{T}$, and $A \subset X$ is <u>closed</u> if $A^c \in \mathcal{T}$.

<u>Expl 4.2</u> Let X be any nonempty set. Then $\mathcal{T} = \{\phi, X\}$ is a topology, called the <u>trivial topology</u>. Also, $\mathcal{T} = \mathcal{P}(X)$ is a topology, called the <u>discrete topology</u>.

__Expl 4.4__ Let $(X, \mathcal{T})$ be a top. sp. Let $Y \subset X$. Define

$$\mathcal{S} = \{G \cap Y : G \in \mathcal{T}\}.$$

Then $(Y, \mathcal{S})$ is a top. sp. We call this a __topological subspace__ of X, and $\mathcal{S}$ the __relative topology__ of Y in X.

Note that if (X, d) is a metric space with metric topology $\mathcal{T}$, and $(Y, d|_{Y \times Y})$ is a metric subspace with metric topology $\mathcal{S}$, then $\mathcal{S}$ is, in fact, the relative topology of Y in X.

Let $(X, \mathcal{T})$ be a top. sp. Suppose $x \in V \subset X$. Then V is a <u>neighborhood of x</u> if $\exists G \in \mathcal{T} \ni x \in G \subset V$.

Suppose that, $\forall x, y \in X \ni x \neq y$, $\exists$ a nbhd V_x of x and a nbhd V_y of y $\ni V_x \cap V_y = \phi$. Then $\mathcal{T}$ is a called a <u>Hausdorff topology</u> and $(X, \mathcal{T})$ is a <u>Hausdorff space</u>.

All metric topologies are Hausdorff. (Why?) If X has more than one elt, then the trivial topology is not Hausdorff.

A sequence $\{x_n\}$ in X <u>converges to $x \in X$</u> if, $\forall$ nbhds V of x, $\exists N \in \mathbb{N} \ni$ $\forall n \geq N$, $x_n \in V$.

Note that if (X,d) is a metric sp., then $x_n \to x$ in the metric top. iff $d(x_n,x) \to 0$. In other words, this notion of convergence is a generalization of our notion for metric spaces. The same will be true for continuity and compactness, which we will see below.

Let $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ be top. spaces. A func $f: X \to Y$ is <u>continuous at $x \in X$</u> if, $\forall$ nbhd W of $f(x)$, $\exists$ a nbhd V of $x \ni f(V) \subset W$. If f is cont. at x, $\forall x \in X$, then f is <u>continuous</u>.

<u>Thm 4.7</u> The func. $f: X \to Y$ is cont. iff $f^{-1}(G) \in \mathcal{T}$ $\forall G \in \mathcal{S}$.

Pf: Exer. 4.4. $\square$

If $f: X \to Y$ is a cont. bijection with f^{-1} also cont., then f is a <u>homeomorphism</u>. If $\exists$ a homeo. $f: X \to Y$, then $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are <u>homeomorphic</u> topological spaces.

<u>Expl 4.9</u> Let $X = [0, 2\pi)$, with the metric topology coming from the Euclidean metric. (Let us call this the "Euclidean topology", for short.)

Let $Y = \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, also with the Euclidean topology (in $\mathbb{R}^2$).

Define $f: X \to Y$ by $f(\theta) = e^{i\theta}$. Then f is a continuous bijection, but f^{-1} is not continuous. To see this, let $g = f^{-1}$.

Then $g: Y \to X$. We must find an open set $G \subset X \ni g^{-1}(G) = f(G)$ is not open in Y.

Let us try $G = [0, \pi)$. First of all, $G = X \cap (-\infty, \pi)$, and $(-\infty, \pi)$ is open in $\mathbb{R}$. Therefore G is open in X.

(Recall that "open" is a relative property. In this case, $[0, 2\pi)$ is a top. subsp. of $\mathbb{R}$, and $[0, \pi)$ is open in $[0, 2\pi)$, but not open in $\mathbb{R}$.)

It remains to check that $f(G) = \{ e^{i\theta} : 0 \le \theta < \pi \}$ is not open in $Y = \mathbb{T}$. This is left as an exercise.

Let $(X, \mathcal{T})$ be a top. space, and $K \subset X$. An <u>open cover</u> of K is a collection of open sets $\mathcal{C} \subset \mathcal{T} \ni K \subset \bigcup_{G \in \mathcal{C}} G$. The set K is <u>compact</u> if every open cover has a finite subcover.

Note K is compact in the space $(X, \mathcal{T})$ iff K is compact in $(K, \mathcal{S})$, where $\mathcal{S}$ is the relative top. of K in X. In other words, compactness is <u>not</u> a relative property.

Let $\mathcal{B} \subset \mathcal{T}$. Suppose that, $\forall G \in \mathcal{T}$, $\exists \{B_\alpha\}_{\alpha \in \Lambda} \subset \mathcal{B} \ni G = \bigcup_{\alpha \in \Lambda} B_\alpha$. Then $\mathcal{B}$ is called a <u>base</u> for the topology $\mathcal{T}$. For example, the collection of open balls in a metric space is a base for the metric topology.

Fix $x \in X$. Suppose $\mathcal{N}$ is a collection of nbhds of $x \ni \forall$ nbhds V of x, $\exists\, W \in \mathcal{N} \ni W \subset V$. Then $\mathcal{N}$ is a <u>neighborhood base</u> for x.

Note that elts of $\mathcal{N}$ (or nbhds in general, for that matter), need not be open.

For example, if X is a metric space, then the collection of all open (or closed) balls with rational radius, centered at x, is a (ct'ble) nbhd base for x.

A top. sp. $(X, \mathcal{T})$ is <u>first countable</u> if every $x \in X$ has a ct'ble nbhd base; it is <u>second countable</u> if X has a ct'ble base.

As we saw above, every metric space is first countable.

Expl Let X be a metric sp. and $A \subset X$ dense. Then $\mathcal{B} = \{B_r(x) : x \in A, r \in \mathbb{Q} \cap (0,\infty)\}$ is a base for the metric top. on X (check). It follows that every separable metric sp. is second ct'ble.

Let $(X, \mathcal{T})$ be a top. sp. and $\mathcal{B}$ a base. Fix $x \in X$. Then $\mathcal{N} = \{B \in \mathcal{B} : x \in B\}$ is a nbhd base for x (check). It follows that if X is second ct'ble, then X is first ct'ble.

Thm 4.15 Let $(X, \mathcal{T})$ be a top. sp. and $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a base iff $\forall x \in X$, $\exists\, \eta \subset \mathcal{B} \ni \eta$ is a nbhd base for x.

Pf: We noted above that if $\mathcal{B}$ is a base, then $\eta = \{B \in \mathcal{B} : x \in B\} \subset \mathcal{B}$ is a nbhd base for x.

Conversely, suppose that $\forall x \in X$, $\exists\, \eta_x \subset \mathcal{B} \ni \eta_x$ is a nbhd base for x. Let $G \in \mathcal{T}$ be arbitrary. Fix $x \in G$. Since η_x is a nbhd base for x, $\exists\, B_x \in \eta_x \subset \mathcal{B} \ni x \in B_x \subset G$. Thus, $\{B_x\}_{x \in G} \subset \mathcal{B}$ and $G = \bigcup_{x \in G} B_x$, which shows that $\mathcal{B}$ is a base. $\square$

Expl 4.14 Let X have the discrete topology, $\mathcal{T} = \mathcal{P}(X)$. Then $\eta = \{\{x\}\}$ is a nbhd base for x, and every nbhd base for x must contain η. Thus, $\mathcal{B} = \{\{x\} : x \in X\}$ is a base for $\mathcal{T}$, and every base must contain $\mathcal{B}$. It follows that the discrete top. is always first ct'ble, and it is second ct'ble iff X is ct'ble.

If $\{\mathcal{T}_\alpha\}_{\alpha \in \Delta}$ is any collection of topologies on a set X, then $\bigcap_{\alpha \in \Delta} \mathcal{T}_\alpha$ is a topology on X (check).

Let $\mathcal{E} \subset \mathcal{P}(X)$ be arbitrary. Then

$$\mathcal{T} = \bigcap \{\mathcal{S} : \mathcal{S} \text{ is a top. on } X \ni \mathcal{E} \subset \mathcal{S}\}$$

is the smallest topology on X that contains $\mathcal{E}$. It is called the <u>topology</u>

generated by $\mathcal{E}$. Let $\mathcal{E}'$ denote the collection of all finite intersections of sets in $\mathcal{E}$. It can be shown that $\mathcal{E}'$ is a base for the topology generated by $\mathcal{E}$.

Expl 4.16 Let I be any set. (For concreteness, you may imagine $I = [a,b]$.) Let $X = \mathbb{R}^I$, the set of all functions $f: I \to \mathbb{R}$. A set $B \subset X$ is called a cylinder set if $\exists$ finite sets $\{x_1, \dots, x_n\} \subset I$ and $\{y_1, \dots, y_n\} \subset \mathbb{R}$, and $\varepsilon > 0 \ni$

$$B = \{ f \in X : |f(x_j) - y_j| < \varepsilon \ \forall\, 1 \le j \le n \}.$$

Let $\mathcal{E}$ be the collection of all cylinder sets, and let $\mathcal{T}$ be the topology generated by $\mathcal{E}$, with base $\mathcal{E}'$.

Note that a set $B' \in \mathcal{E}'$ will have the form

$$B' = \{ f \in X : |f(x_j) - y_j| < \varepsilon_j \ \forall\, 1 \le j \le n \}.$$

Let $\{f_n\} \subset X$ and $f_n \in X$. It is an exercise to show that $f_n \to f$ ptwise on I iff $f_n \to f$ in the topology $\mathcal{T}$. Therefore, the topology generated by the cylinder sets is called the <u>topology of pointwise convergence</u> on $\mathbb{R}^I$.

A top. sp. $(X, \mathcal{T})$ is <u>metrizable</u> if there is a metric on X whose metric topology is $\mathcal{T}$. Every metrizable top. is first ct'ble. Every metrizable top. is Hausdorff. (Thus, if X has more than one elt, then the trivial top. is not metrizable.)

If $(X, \mathcal{T})$ is metrizable, then all thms about metric spaces become available. For example, $K \subset X$ is cpct iff every seq. in K has a subseq. converging to some $x \in X$ (Thm 1.62, Defn 1.54); a set $F \subset X$ is closed iff the limit of every convergent seq. in F belongs to F (Prop. 1.41); and a func. f is cont. iff $x_n \to x$ implies $f(x_n) \to f(x)$ (Prop. 1.34, Defn 1.33). All of these results may fail if $(X, \mathcal{T})$ is not metrizable.

The topology of ptwise convergence on $\mathbb{R}^{[0,1]}$ is not metrizable. (See Expl

4.17 for a proof sketch; making Expl 4.17 rigorous requires the "Baire characterization thm".)

Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be top. spaces. Let

$$\mathcal{E} = \{ U \times Y : U \in \mathcal{T} \} \cup \{ X \times V : V \in \mathcal{S} \}.$$

The <u>product topology</u> $\mathcal{T} \otimes \mathcal{S}$ on $X \times Y$ is the topology generated by $\mathcal{E}$. A base for $\mathcal{T} \otimes \mathcal{S}$ is

$$\mathcal{E}' = \{ U \times V : U \in \mathcal{T}, V \in \mathcal{S} \}.$$

The product topology is the smallest topology under which the projection function $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ (given by $p_X(x,y) = x$ and $p_Y(x,y) = y$) are cont.

Let V be a vector space over a field $F \in \{\mathbb{R}, \mathbb{C}\}$. Let $\mathcal{T}$ be a topology on V. Equip F with the Euclidean top., and $F \times V$ and $V \times V$ with the prod. top. If the funcs. $(\alpha, v) \mapsto \alpha v$ from $F \times V \to V$ and $(u, v) \mapsto u + v$ from $V \times V \to V$ are cont., then $(V, \mathcal{T})$ is a <u>topological vector space</u>.

It can be shown that $\mathbb{R}^I$, with the top. of ptwise conv., is a top. vector sp.

Let $\mathcal{T}_1, \mathcal{T}_2$ be top. on X with $\mathcal{T}_1 \subset \mathcal{T}_2$. Then we say $\mathcal{T}_1$ is <u>weaker</u> (or <u>coarser</u>) than $\mathcal{T}_2$, and $\mathcal{T}_2$ is <u>stronger</u> (or <u>finer</u>) than $\mathcal{T}_1$.

The weakest topology is the trivial topology, and every seq. converges under the trivial top. The strongest top. is the discrete top., and only seq. which are eventually constant are convergent.

If $\mathcal{T}_2$ is stronger than $\mathcal{T}_1$ and $x_n \to x$ under $\mathcal{T}_2$, then $x_n \to x$ under $\mathcal{T}_1$ (check).

Review, on your own, the remainder of Section 4.3.

BANACH SPACES

A <u>Banach space</u> is a normed vector sp. that is complete (as a metric sp., with metric induced by the norm).